

## **EVOLUTION OF ENERGY OF PERTURBATIONS IN BAROTROPIC ATMOSPHERE**

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### **Abstract**

The barotropic vorticity equation describing the vortex dynamics of viscous and forced incompressible fluid on a rotating sphere is considered. This equation is also used for studying the large-scale dynamics of barotropic atmosphere. Operators of orthogonal projection on the subspaces of homogeneous spherical polynomials and derivatives of real order for functions are introduced. A family of Hilbert spaces of generalized functions having fractional derivatives of real order  $s$  is introduced, and a few embedding theorems are given. An equation for the evolution of kinetic energy of perturbations to a basic flow is analyzed. A relationship between the rate of generation of kinetic energy perturbations and the eigenfunctions of the symmetric part of the operator linearized about the basic flow is shown. As an illustrative example, the numerical solution of the spectral problem for such operator is discussed in the case when the basic flow is the climatic January circulation.

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### **Introduction**

At present the widely used hypothesis is that the existence of permanent fluctuations of the different time-space scales in the circulation of barotropic atmosphere can be explained by the instability of the atmosphere with respect to small perturbations [2]. Since the real atmospheric state is not known exactly it is interesting to study the stability properties of some approximate idealized states. The choice of a suitable basic state is generally determined by objectives of the study, and in particular, by time-space scales of the atmospheric dynamics under consideration. Comparison of observed and calculated time-space structures of the

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most unstable perturbations enables us to select such idealized basic state of the atmosphere that is more close to the observed state and hence is appropriate for the instability study.

In recent decades, the interest in the problem of barotropic instability of the atmosphere has considerably increased due to the establishment of the following facts, namely:

(1) the remote response of the atmosphere to the sea surface temperature anomalies has an equivalent barotropic structure;

(2) The conversion of the barotropic energy of perturbations plays an important role in the low frequency variability of the atmospheric circulation [5,6,17,18].

In this work, the turbulent viscosity term in the model is considered of common form  $\nu(-\Delta)^{s+1}\psi$ , where  $s \geq 1$  is arbitrary real number [12]. In this connection, we first define in section 2 the operators of orthogonal projection on the subspaces of homogeneous spherical polynomials of degree  $n$ , and then use them to introduce fractional derivatives of smooth functions on the sphere. A family of Hilbert spaces  $\mathbb{H}^s$  of functions of real degree of smoothness on a sphere is introduced in section 3. The nonlinear barotropic vorticity equation (BVE) describing the motion of incompressible viscous and forced fluid on a rotating sphere is given in section 4. The equation for the evolution of the kinetic energy of perturbations of a steady BVE solution  $\psi(\lambda, \mu)$  is derived in section 5. It is also shown that the time derivative of the kinetic energy of a perturbation  $\psi'$  is determined by the inner product  $\langle B\psi', \psi' \rangle$  where  $B$  is the symmetric part of the problem operator linearized about the steady solution  $\psi(\lambda, \mu)$ . The spectral problem for operator  $B$  is considered in section 6. Since the eigenfunctions of operator  $B$  form an orthogonal basis in the space of perturbations, each eigenfunction represents a basic disturbance and the sign of the corresponding eigenvalue determines the growth or decay of the kinetic energy of this disturbance. The geometric structure of the unstable set of perturbations is also analyzed. The application of the spectral problem for operator  $B$  to the analysis of the stability of a climatic January atmospheric flow at 300 mb is given in section 7. In particular, it is shown that the most unstable eigenfunctions (perturbations) are localized near the two strong jets in the basic flow. The two main mechanisms of the instability related with the existence of jets in the basic flow are given and discussed.

## Orthogonal Projection operators and Fractional Derivatives

Let  $S = \{x \in \mathbb{R}^3 : |x| = 1\}$  be a unit sphere in the three-dimensional Euclidean space; we denote by  $\mathbb{C}^\infty(S)$  the set of infinitely differentiable functions on  $S$  and by

$$\langle f, g \rangle = \int_S f(x) \overline{g(x)} dS \quad \text{and} \quad \|f\| = \langle f, f \rangle^{1/2} \quad (1)$$

the inner product and norm in  $\mathbb{C}^\infty(S)$ , respectively. Here  $x = (\lambda, \mu)$  is a point on the sphere,  $dS = d\lambda d\mu$  is an element of sphere surface,  $\mu = \sin \phi$ ;  $\mu \in [-1, 1]$ ,  $\phi$  is the latitude,  $\lambda \in [0, 2\pi)$  is the longitude and  $\overline{g(x)}$  is the complex conjugate of  $g(x)$ . It is known that the spherical harmonics

$$Y_n^m(\lambda, \mu) = \left[ \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(\mu) e^{im\lambda}, \quad n \geq 0, \quad |m| \leq n$$

form the orthonormal basis in  $\mathbb{C}^\infty(S)$  (here  $P_n^m(\mu)$  is the associated Legendre function of degree  $n$  and zonal wavenumber  $m$ ) [9]. For each integer  $n \geq 0$ , the span of  $2n + 1$  spherical harmonics  $Y_n^m(\lambda, \mu)$  ( $|m| \leq n$ ) forms a  $(2n + 1)$ -dimensional eigen-subspace

$$\mathbf{H}_n = \{\psi : \Delta\psi = \left[ (1 - \mu^2)\psi_\mu \right]_\mu + \frac{1}{1 - \mu^2}\psi_{\lambda\lambda} = -\chi_n\psi \quad (2)$$

of homogeneous spherical polynomials of degree  $n$  which are the eigenfunctions of Laplace operator on  $S$  corresponding to the eigenvalue  $-\chi_n$  where

$$\chi_n = n(n + 1) \quad (3)$$

and  $\psi_\lambda$  and  $\psi_\mu$  denote partial derivatives of  $\psi$  with respect to  $\lambda$  and  $\mu$ , respectively. The subspace  $\mathbf{H}_n$  is invariant not only with respect to the Laplace operator but also to each element of the group  $SO(3)$  of rotations of sphere about any its axis [3].

We now introduce the operators of orthogonal projection on subspaces  $\mathbf{H}_n$  and fractional derivatives of functions on  $S$  [12].

**Definition 1** Denote by  $\mathbb{L}^2(S)$  the completion of  $\mathbb{C}^\infty(S)$  in the norm (1). It is the Hilbert space with inner product (1). Besides,  $\mathbb{L}^2(S)$  is the direct orthogonal sum of subspaces  $\mathbf{H}_n$  of  $\mathbb{L}^2(S) = \bigoplus_{n=0}^\infty \mathbf{H}_n$ .

**Definition 2** [3] Let  $\omega$  be an angle between two unit radius-vectors  $\vec{x}_1, \vec{x}_2$  corresponding to the points  $x_1, x_2 \in S$ . Then  $\vec{x}_1 \cdot \vec{x}_2 = \cos \omega$  is the scalar product of vectors  $\vec{x}_1$  and  $\vec{x}_2$ . A function  $z(\vec{x} \cdot \vec{y})$  depending only on the distance  $\rho(x, y) = \arccos(\vec{x} \cdot \vec{y}) = \omega$  between two points  $x$  and  $y$  of sphere is called the zonal function. The convolution of a function  $\psi \in \mathbb{L}^2(S)$  with a zonal function  $Z(\vec{x} \cdot \vec{y}) \in \mathbb{L}^2(S)$  is defined by

$$(\psi * z)(x) = \frac{1}{4\pi} \int_S \psi(y) Z(\vec{x} \cdot \vec{y}) dS(y) \quad (4)$$

**Definition 3** [15] Let  $n \geq 0$ . The operator of orthogonal projection  $Y_n : \mathbb{L}^2(S) \mapsto \mathbf{H}_n$  of  $\mathbb{L}^2(S)$  on subspace  $\mathbf{H}_n$  is introduced by

$$Y_n(\psi; x) = (2n + 1)(\psi * P_n)(x) \quad (5)$$

For brevity we also write  $Y_n(\psi)$  instead of  $Y_n(\psi; x)$ . Obviously, any function of subspace  $\mathbf{H}_0$  is constant:

$$Y_0(\psi) = \frac{1}{4\pi} \int_S \psi(y) dS(y) = Const \quad (6)$$

Note that the Parseval-Steklov identities

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \|Y_n(\psi)\|^2 \quad \text{and} \quad \langle \psi, h \rangle = \sum_{n=0}^{\infty} \langle Y_n(\psi), Y_n(h) \rangle \quad (7)$$

hold for any functions  $\psi, h \in \mathbb{L}^2(S)$ . Due to (7), each function  $\psi(x) \in \mathbb{L}^2(S)$  is represented by the Fourier-Laplace series

$$\psi(x) = \sum_{n=0}^{\infty} Y_n(\psi; x) \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^n \psi_n^m Y_n^m(x) \quad (8)$$

**Definition 4** Let  $s > 0$ ,  $\chi_n = n(n+1)$ , and  $\psi(x) \in \mathbb{C}^\infty(S)$ . A derivative  $\Lambda^s = (-\Delta)^{s/2}$  of real order  $s$  of functions on a sphere is defined by means of equations

$$Y_n(\Lambda^s \psi) = \chi_n^{s/2} Y_n(\psi), \quad n \geq 0 \quad (9)$$

Thus,  $\Lambda^s$  is a multiplier operator which is completely defined by infinite set of multipliers  $\{\chi_n^{s/2}\}_{n=0}^{\infty}$ , besides,

$$\Lambda^s \psi(x) = \sum_{n=1}^{\infty} \chi_n^{s/2} Y_n(\psi; x) \quad (10)$$

In particular,  $\Lambda^{2n} = (-\Delta)^n$  for any natural  $n$ , and operator  $\Lambda$  can be interpreted as the square root of Laplace operator (2). It is well known that the main disadvantage of local derivatives  $\partial^n / \partial \lambda^n$  and  $\partial^n / \partial \mu^n$  is that they depend on the choice of coordinate system (i.e., on sphere rotation). The new derivatives  $\Lambda^s$  and orthogonal projections  $Y_n$  are invariant with respect to any element of the group  $SO(3)$  of sphere rotations [12].

Let  $\mathbb{C}_0^\infty(S) = \{\psi \in \mathbb{C}^\infty(S) : Y_0(\psi) = 0\}$  denote the subspace of functions of  $\mathbb{C}^\infty(S)$  which are orthogonal to any constant on the sphere. Note that operator  $\Lambda^s$  may be defined on functions from  $\mathbb{C}_0^\infty(S)$  by means of (9) or (10) for every real order  $s$ .

**Definition 5** For any real  $s$ , we introduce in  $\mathbb{C}_0^\infty(S)$  the inner product  $\langle \cdot, \cdot \rangle_s$  and norm  $\|\cdot\|_s$  in the following way:

$$\langle \psi, h \rangle_s = \langle \Lambda^s \psi, \Lambda^s h \rangle = \sum_{n=1}^{\infty} \chi_n^s \langle Y_n(\psi), Y_n(h) \rangle \quad (11)$$

$$\|\psi\|_s = \|\Lambda^s \psi\| = \langle \psi, \psi \rangle_s^{1/2} = \left\{ \sum_{n=1}^{\infty} \chi_n^s \|Y_n(\psi)\|^2 \right\}^{1/2} \quad (12)$$

**Definition 6** Let  $s$  be a real. Denote as  $\mathbb{H}^s$  the Hilbert space obtained by closing the space  $\mathbb{C}_0^\infty(S)$  in the norm (12).

Thus, a function  $\psi \in \mathbb{H}^s$  for some  $s$  if its  $s$ th fractional derivative belongs to  $\mathbb{L}^2(S)$  [12].

Hereinafter we will keep for the inner product and norm in  $\mathbb{H}^0 \equiv \mathbb{L}_0^2(S)$  the symbols  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  introduced in (1). It is shown in [15] that the embeddings  $\mathbb{C}_0^\infty(S) \subset \mathbb{H}^r \subset \mathbb{H}^s \subset \mathbb{H}^0 \subset \mathbb{H}^{-s} \subset \mathbb{H}^{-r}$  are continuous if  $0 < s < r$ , and the dual space  $(\mathbb{H}^s)^*$  coincides with  $\mathbb{H}^{-s}$  for all  $s > 0$ . Also note that for any real numbers  $s$  and  $r$ , the operator  $\Lambda^r : \mathbb{C}_0^\infty(S) \mapsto \mathbb{C}_0^\infty(S)$  is symmetric:  $\langle \Lambda^r \psi, h \rangle_s = \langle \psi, \Lambda^r h \rangle_s$ , and hence closable, that is, it can be extended to the whole space  $\mathbb{H}^s$ .

**Definition 7** An element  $z \in \mathbb{H}^s$  is called the  $r$ th derivative  $\Lambda^r \psi$  of a function  $\psi \in \mathbb{H}^s$  if the equality  $\langle z, h \rangle_s = \langle \psi, \Lambda^r h \rangle_s$  holds for all  $h \in \mathbb{C}_0^\infty(S)$ , where  $\Lambda^r h$  is defined by (10).

The following assertion establishes embedding estimates for the functions of subspaces  $\mathbb{H}^s$ .

**Lemma 1** [15]. Let  $s$  be real,  $r > 0$ , and  $\psi \in \mathbb{H}^{s+r}$ . Then  $\psi \in \mathbb{H}^s$  and

$$\|\psi\|_s \leq 2^{-r/2} \|\psi\|_{s+r} \quad (13)$$

$$\|\psi\|_{s+r} = \|\Lambda^r \psi\|_s \quad (14)$$

**Corollary 1** [15]. Let  $s$  and  $r$  be real numbers. The mapping  $\Lambda^r : \mathbb{H}^{s+r} \mapsto \mathbb{H}^s$  is isometry and isomorphism. In particular, at  $r = -2s$ , the operator  $\Lambda^{-2s} : \mathbb{H}^{-s} \mapsto \mathbb{H}^s$  is isometric isomorphism.

**Lemma 2** [15]. Let  $r, s$  and  $t$  be real numbers,  $r < t$ , and  $a = \sqrt{2}$ . Then for any  $\psi \in \mathbb{H}^{s+t}$ ,

$$\|\Lambda^r \psi\|_s \leq a^{r-t} \|\Lambda^t \psi\|_s \quad (15)$$

## Barotropic Vorticity Equation

Let us consider the nonlinear problem

$$\Delta \psi_t + J(\psi, \Delta \psi + 2\mu) = -\sigma \Delta \psi + \nu(-\Delta)^{s+1} \psi + F \quad (16)$$

$$\Delta \psi(0, x) = \Delta \psi_0(x) \quad (17)$$

where (16) is a dimensionless form of the barotropic vorticity equation describing the evolution of relative vorticity  $\Delta \psi(t, x)$  in a viscous and forced 2D incompressible rotating fluid on sphere  $S$  using the geographical coordinate system  $(\lambda, \mu)$  whose pole  $N$  is on the axis of rotation of unit sphere [12]. The problem takes into account an external vorticity source  $F(t, x)$  and Rayleigh friction  $\sigma \Delta \psi$  in the planetary boundary layer. Here  $\Delta$  is the spherical Laplace operator (2),  $\nabla$  is the gradient,  $\psi$  is the stream function,  $\Delta \psi + 2\mu$  is the absolute vorticity,

$$J(\psi, h) = \psi_\lambda h_\mu - \psi_\mu h_\lambda = (\vec{n} \times \nabla \psi) \cdot \nabla h \quad (18)$$

is the Jacobian,  $J(\psi, 2\mu) = 2\psi_\lambda$  is the sphere rotation term,  $\vec{n}$  is the outward unit normal to  $S$ , “ $\cdot$ ” and “ $\times$ ” denote the scalar and vector products, and  $\psi_t$ ,  $\psi_\lambda$  and  $\psi_\mu$  denote partial derivatives of  $\psi$  with respect to  $t$ ,  $\lambda$  and  $\mu$ , respectively. The velocity vector  $\vec{u} = \vec{n} \times \nabla \psi$  with components

$$u = -\sqrt{1-\mu^2} \psi_\mu, \quad v = \frac{1}{\sqrt{1-\mu^2}} \psi_\lambda \quad (19)$$

is solenoidal:  $\nabla \cdot \vec{u} = 0$ .

We consider the turbulent viscosity term of common form  $\nu(-\Delta)^{s+1}\psi$ , where  $s \geq 1$  is arbitrary real number [12]. The case  $s = 1$  corresponds to classical viscosity term in Navier-Stokes equations [7,16], while the case  $s = 2$  was considered in [1,11,12]. The turbulent term of this form for natural numbers  $s$  is also used in [8] to prove the solvability of Navier-Stokes equations in a limited area by using the method of artificial viscosity. Equation (16) is obtained by applying the curl operator to the 2D Navier-Stokes equations. Since the sphere is a smooth manifold without boundary, such a transformation results in the fact that the solution of problem (16), (17) is determined up to a constant. In order to eliminate this constant, the problem (16),(17) is considered in classes of the functions which are orthogonal to a constant on the sphere:  $Y_0(\psi) = 0$ ,  $Y_0(F) = 0$ .

It is clear that  $J(\psi, h) = -J(h, \psi)$  and  $J(\psi, \psi) = 0$ . Let  $n$  be a natural number, and  $s$  be a real number. Since  $\Lambda^s Y_n(\psi) = \chi_n^{s/2} Y_n(\psi)$  then  $J(\psi, \Lambda^s \psi) = 0$  for any homogeneous spherical polynomial  $\psi$  of degree  $n$  ( $\psi \in \mathbf{H}_n$ ). Note that a smooth vector field  $\vec{n} \times \nabla \psi$  is solenoidal, and due to (18),  $J(\psi, h) = \nabla \cdot [h (\vec{n} \times \nabla \psi)]$ . If  $\vec{X}$  is a smooth vector-function defined on  $S$  with a compact support  $K \subset S$  then  $\int_S \nabla \cdot \vec{X} dS = 0$ . Using the theorem on the partition of unity, we obtain

$$\int_S J(\psi, h) dS = 0$$

**Lemma 3** [15] *Let  $r$  be a real number, and let  $\psi$ ,  $g$  and  $h$  be continuously differentiable complex-valued functions on  $S$ . Then*

$$\langle J(\psi, g), h \rangle = \langle J(g, \bar{h}), \bar{\psi} \rangle = -\langle J(\psi, \bar{h}), \bar{g} \rangle \quad (20)$$

and

$$\langle J(\psi, h\bar{\psi}), \bar{\psi}^r \rangle = 0, \quad \langle J(\psi, \mu), \bar{\Lambda}^r \psi \rangle = 0 \quad (21)$$

## Evolution of kinetic energy of perturbations

Let  $\tilde{\psi}(\lambda, \mu)$  be a steady solution of (16). We now consider the evolution of kinetic energy of perturbations to flow  $\tilde{\psi}(\lambda, \mu)$ . Let  $\tilde{\psi}(\lambda, \mu) + \psi'(t, \lambda, \mu)$  is another solution of (16). Then  $\psi'(t, \lambda, \mu)$  can be considered as a perturbation of  $\tilde{\psi}(\lambda, \mu)$  which satisfies the equation

$$\Delta\psi'_t = L\psi' - J(\psi', \Delta\psi') \quad (22)$$

where

$$L\psi' = J(\tilde{\Omega}, \psi') - J(\tilde{\psi}, \Delta\psi') - [\sigma + \nu\Lambda^{2s}] \Delta\psi' \quad (23)$$

is the linear operator,  $\tilde{\Omega} = \Delta\tilde{\psi} + 2\mu$  is the absolute vorticity of flow  $\tilde{\psi}$ ,  $s \geq 1$ ,  $\nu > 0$  and  $\sigma \geq 0$ .

Taking the inner product (1) of equation (22) with  $\psi'$  and using (20) and (21), we obtain equation

$$K_t(t) = -\langle L\psi', \psi' \rangle = -\langle J(\psi', \Delta\psi'), \tilde{\psi} \rangle - 2\sigma K(t) - \nu \|\Lambda^{s+1}\psi'\|^2 \quad (24)$$

for the evolution of kinetic energy  $K(t) = \frac{1}{2} \|\nabla\psi'\|^2 = \frac{1}{2} \|\psi'\|_1^2$  of perturbation  $\psi'$ .

Denote by  $\vec{u} = (\vec{u}, \vec{v}) = \vec{n} \times \nabla\tilde{\psi}$  the basic flow and by  $\vec{u}' = (u', v') = \vec{n} \times \nabla\psi'$  the perturbation velocity. Defining the new vectors  $\vec{U} = (U, V) = \sqrt{1-\mu^2}\vec{u}'$  and  $\vec{\tilde{U}} = (\tilde{U}, \tilde{V}) = \sqrt{1-\mu^2}\vec{u}$  and using the fact that  $\nabla \cdot \vec{u} = 0$  and  $\nabla \cdot \vec{u}' = 0$  we get

$$U = -(1-\mu^2)\psi'_\mu, \quad V = \psi'_\lambda, \quad \tilde{U} = -(1-\mu^2)\tilde{\psi}_\mu, \quad \tilde{V} = \tilde{\psi}_\lambda \quad (25)$$

$$\frac{1}{1-\mu^2}U_\lambda = -V_\mu, \quad \frac{1}{1-\mu^2}\tilde{U}_\lambda = -\tilde{V}_\mu \quad (26)$$

$$\Delta\psi' = (1-\mu^2)^{-1}V_\lambda - U_\mu \quad (27)$$

*Ideal fluid.* Only the first term in the right-hand side of equation (24) can generate the perturbation energy. Therefore we now consider the case when the dissipation is absent ( $\sigma = \nu = 0$ ).

**Theorem** Let  $\sigma = \nu = 0$ . Then equation (24) can be written in terms of  $U$  and  $V$  as

$$K_t(t) = -\frac{1}{2} \left\langle U^2 - V^2, \frac{\tilde{U}_\lambda}{(1-\mu^2)^2} - \left( \frac{\tilde{V}}{1-\mu^2} \right)_\mu \right\rangle - \left\langle UV, \left( \frac{\tilde{U}}{1-\mu^2} \right)_\mu + \frac{\tilde{V}_\lambda}{(1-\mu^2)^2} \right\rangle \quad (28)$$

**Proof.** Using (25) and (27), we get

$$\begin{aligned}
 K_t(t) &= \int_S \frac{\tilde{V}}{(1-\mu^2)^2} U V_\lambda dS - \int_S \frac{\tilde{V}}{1-\mu^2} U U_\mu dS \\
 &\quad - \int_S \frac{\tilde{U}}{(1-\mu^2)^2} V V_\lambda dS + \int_S \frac{\tilde{U}}{1-\mu^2} V U_\mu dS
 \end{aligned} \tag{29}$$

Integrating by parts the first integral in (29) and using (26) we obtain

$$\int_S \frac{\tilde{V}}{(1-\mu^2)^2} U V_\lambda dS = - \int_S \frac{\tilde{V}_\lambda}{(1-\mu^2)^2} U V dS - \frac{1}{2} \int_S V^2 \left( \frac{\tilde{V}}{1-\mu^2} \right)_\mu dS$$

The second and third integrals in (29) can be written as

$$\begin{aligned}
 - \int_S \frac{\tilde{V}}{1-\mu^2} U U_\mu dS &= \frac{1}{2} \int_S U^2 \left( \frac{\tilde{V}}{1-\mu^2} \right)_\mu dS \\
 - \int_S \frac{\tilde{U}}{(1-\mu^2)^2} V V_\lambda dS &= \frac{1}{2} \int_S V^2 \frac{\tilde{U}_\lambda}{(1-\mu^2)^2} dS
 \end{aligned}$$

Finally, the last integral in (29) is

$$\int_S \frac{\tilde{U}}{1-\mu^2} V U_\mu dS = - \int_S \left( \frac{\tilde{U}}{1-\mu^2} \right)_\mu U V dS - \frac{1}{2} \int_S U^2 \frac{\tilde{U}_\lambda}{(1-\mu^2)^2} dS$$

The theorem is proved.

In the particular case of a zonal flow  $\tilde{\psi}(\mu)$ ,  $\tilde{U}_\lambda = 0$  and  $\tilde{V} = 0$ , and equation (28) is reduced to

$$\begin{aligned}
 K_t(t) &= - \langle UV, [(1-\mu^2)^{-1} \tilde{U}]_\mu \rangle \\
 &= \int_S u' v' \left[ \frac{\tilde{u}}{\sqrt{1-\mu^2}} \right]_\mu dS = - \int_S u' v' \tilde{\psi}_{\mu\mu} dS
 \end{aligned} \tag{30}$$

Let us represent the operator (23) as the sum of its symmetric and skew-symmetric parts,  $L = 0.5(L + L^*) + 0.5(L - L^*)$  where  $L^*$  is the operator adjoint to  $L$ :  $\langle Lf, h \rangle \equiv \langle f, L^*h \rangle$ . Then equation (24) for the kinetic energy of a real perturbation  $\psi'(t, \lambda, \mu)$  of flow  $\tilde{\psi}(\lambda, \mu)$  can be written as

$$K_t(t) = \langle B\psi', \psi' \rangle \tag{31}$$

where

$$B\psi' = -0.5(L + L^*)\psi' = [\sigma + \nu\Lambda^{2s}] \Delta\psi' - 0.5[\Delta J(\tilde{\psi}, \psi') - J(\tilde{\psi}, \Delta\psi')] \tag{32}$$

is the symmetric part of operator (23).



**Example 1.** *Solid-body rotation flow.* If  $\tilde{\psi}(\mu) = C\mu$  and the dissipation is absent ( $\sigma = \nu = 0$ ) then  $B = 0$  and  $K_t(t) = 0$ , that is, the perturbation kinetic energy does not change (the solid-body rotation is stable). In the presence of dissipation the perturbation kinetic energy decreases:  $K_t(t) < 0$  (the solid-body rotation is asymptotically stable).

**Example 2.** *A flow in the form of a homogeneous spherical polynomial of degree one.* Let  $\tilde{\psi}(\lambda, \mu) \in \mathbf{H}_1$ . Then  $K_t(t) = 0$  if  $\sigma = \nu = 0$  (flow  $\tilde{\psi}$  is stable), and

$$K_t(t) = -2\sigma K(t) - \nu \|\Lambda^{s+1} \psi'\|^2 < 0 \quad (33)$$

if  $\sigma > 0$  or/and  $\nu > 0$  (flow  $\tilde{\psi}$  is asymptotically stable). Indeed, it is sufficient to prove that  $\langle J(\psi, \Delta\psi), \tilde{\psi} \rangle = 0$  for any differentiable function  $\psi$ . To this end we recall that  $\tilde{\psi}$  can be considered as the projection (5) on subspace  $\mathbf{H}_1$ :  $\tilde{\psi} = Y_1(\tilde{\psi}; x) = 3(\tilde{\psi} * P_1)(x)$ . Therefore

$$\begin{aligned} \langle J(\psi, \Delta\psi), \tilde{\psi} \rangle &= 3 \langle J(\psi(x), \Delta\psi(x)), (\tilde{\psi} * P_1)(x) \rangle \\ &= \frac{3}{4\pi} \left\langle J(\psi(x), \Delta\psi(x)), \int_S \tilde{\psi}(y) P_1(\vec{x} \cdot \vec{y}) dy \right\rangle \\ &= 3 \langle (J(\psi, \Delta\psi) * P_1)(y), \tilde{\psi}(y) \rangle \end{aligned}$$

Here we changed the order of integration over  $x$  and  $y$  and used formula (4). Since  $P_1(\vec{x} \cdot \vec{y}) = \mu$  we obtain  $\langle J(\psi, \Delta\psi), \tilde{\psi} \rangle = 0$  due to the second equality (21). Obviously, (24) implies (33).

**Example 3.** *A Legendre polynomial flow  $P_2(\mu)$ .* Let  $\sigma = \nu = 0$  and  $\tilde{\psi}(\mu) = aP_2(\mu) = \frac{a}{2}(3\mu^2 - 1)$ . It was shown in [13] that this flow is linearly stable. However, the kinetic energy of small but finite perturbations satisfies the equation (30). Since  $\tilde{\psi}_{\mu\mu} = 3a$ , we obtain  $K_t(t) = -3a \int_S u'v' dS$ . Let  $a > 0$ . Then the domains of sphere where  $u'v' < 0$  will contribute to the growth of kinetic energy of perturbations  $K(t)$ . For example, the growth of energy will take place in the regions in which the perturbation represents localized vortex structures, the principal axes of which are directed from north-west to south-east (as it is shown, for example, in Fig.2a).

**Example 4.** *A flow in the form of a homogeneous spherical polynomial of degree  $n$ ,  $n \geq 2$ .* Let  $\tilde{\psi}(\lambda, \mu) \in \mathbf{H}_n$ ,  $n \geq 2$ , and initial perturbation  $\psi'(t_0, x) \in \mathbf{H}_n$ . Then, due to (22) and (23),  $\mathbf{H}_n$  is invariant subspace of perturbations, and a perturbation  $\psi'(t, x)$  will never leave  $\mathbf{H}_n$ . Besides, it follows from (24) and (21) that inequality (33) is valid for any  $\psi'(t, x) \in \mathbf{H}_n$ . Thus for any  $\psi' \in \mathbf{H}_n$ ,  $K_t(t) = 0$  if  $\sigma = \nu = 0$  ( $\mathbf{H}_n$  is the subset of stable perturbations of flow  $\tilde{\psi}$ ), and  $K_t(t) \leq 0$  if  $\sigma > 0$  or/and  $\nu > 0$  ( $\mathbf{H}_n$  is the subset of asymptotically stable perturbations of flow  $\tilde{\psi}$ ).

## Geometric structure of unstable set of perturbations

Let now  $\tilde{\psi}(\lambda, \mu)$  be a steady flow on  $S$ , and initial perturbation  $\psi'(t_0, x) \in \mathbf{H}_n$  for some  $n$ . Then due to (24),  $K_t(t_0) = 0$  if  $\sigma = \nu = 0$ , or by (33),  $K_t(t_0) = -2\sigma K(t) - \nu \|\Lambda^{s+1} \psi'\|^2 < 0$  if  $\sigma > 0$  or/and  $\nu > 0$ . However, since  $\tilde{\psi}(\lambda, \mu)$  is an arbitrary flow, the subspace  $\mathbf{H}_n$  of

perturbations is not invariant, and the perturbation  $\psi'(t, x)$  of  $\mathbf{H}_n$  can leave it at any moment  $t > t_0$ . Let

$$BG_n(x) = \rho_n G_n(x) \quad (34)$$

be the spectral program for the symmetric operator  $B$  defined by (32). Assume that  $\|G_n(x)\| = 1$ , i.e. the eigenfunctions  $\{G_n(x)\}$  form the orthonormal basis in real space  $\mathbb{H}^0$ . It was shown in [14] that if  $\nu > 0$  then operator  $B : \mathbb{H}^0 \rightarrow \mathbb{H}^0$  with the domain  $D(B) = \mathbb{H}^{2s}$  has a compact resolvent and real isolated eigenvalues of unit geometrical multiplicity. The only limit point of its spectrum is  $-\infty$ , and the number of positive eigenvalues  $\rho_n$  of operator  $B$  is finite.

Let us renumber the eigenvalues in such a way that  $\rho_n > \rho_{n+1}$ , and assume that the first  $N$  eigenvalues  $\rho_1, \dots, \rho_N$  are positive. A perturbation  $\psi'(t, x)$  can be represented by its Fourier series as

$$\psi'(t, x) = \sum_{n=1}^{\infty} a_n(t) G_n(x) \quad (35)$$

and due to (31),

$$K_t(t) = \sum_{n=1}^{\infty} \rho_n a_n^2(t) \quad (36)$$

Therefore if at initial moment  $t_0$ , a perturbation has the form of a single eigenfunction:  $\psi'(t_0, x) = a_n(t_0) G_n(x)$  then

$$K_t(t_0) = \rho_n a_n^2(t_0) \quad (37)$$

and the kinetic energy of such initial perturbation will grow if  $n \leq N$  and decrease if  $n > N$ . Besides, the growth (or decay) rate of energy  $K(t)$  at moment  $t_0$  is determined by the values of eigenvalue  $|\rho_n|$  and amplitude  $a_n(t_0)$ .

Let us consider the sequence  $\{a_n\}_{n=1}^{\infty}$  of the Fourier coefficients of perturbation (35) as a point in the phase space of solutions to problem (22). Due to (36), the condition

$$\sum_{n=N+1}^{\infty} |\rho_n| a_n^2 < \sum_{n=1}^N \rho_n a_n^2 \quad (38)$$

defines a set  $\mathbf{M}$  of unstable perturbations, since every point  $\{a_n\} \in \mathbf{M}$  represents a perturbation  $\psi'$  whose kinetic energy  $K$  grows with time. It is easy to see that for the basic flows considered in examples 1 and 2, all the eigenvalues are negative, and hence, the set  $\mathbf{M}$  is empty.

It is interesting to study the geometric structure of unstable set of perturbations  $\mathbf{M}$ . First, the set  $\mathbf{M}$  is unbounded because it includes  $N$ -dimensional Euclidean space  $E_N$  of vectors  $\{a_1, a_2, \dots, a_N\}$  except for its origin  $\{0, \dots, 0\}$ . Second, this set is of infinite dimension, and it is not invariant with respect to applying the nonlinear operator of equation (22), that is the

trajectory of a solution to equation (22) can enter and leave the set  $\mathbf{M}$ . Third, among all the points  $\{a_n\}$  belonging to the surface  $\sum_{n=1}^{\infty} a_n^2 = C_0 = \text{const}$ , the maximum growth of energy is achieved when  $a_1 = \sqrt{C_0}$  and  $a_n = 0$  for all  $n > 1$ . Also, it follows from (38) that if  $\sum_{n=1}^N \rho_n a_n^2$  is bounded then  $\sum_{n=N+1}^{\infty} |\rho_n| a_n^2$  is bounded too. Since  $|\rho_n| \rightarrow \infty$  as  $n \rightarrow \infty$  (operator  $B$  has a compact resolvent), the inequality  $\sum_{n=N+1}^{\infty} |\rho_n| a_n^2 < \text{const}$  defines a compact set in the coordinate space of sequences  $\{a_n\}_{n=N+1}^{\infty}$  which is orthogonal to the  $N$ -dimensional space  $E_N$ .

Thus, the nonlinear process of evolution of perturbations can be described as follows. Assume that the initial perturbation  $\psi'(t_0, x)$  is such that  $a_n(t_0) = 0$  for all  $n > N$ , i.e. the point  $\{a_n(t_0)\} \subset E_N$ . Then the perturbation energy  $K(t)$  will grow. Since  $E_N$  is not invariant, non-zero coefficients  $a_n(t)$  for  $n > N$  will eventually appear. Their growth will destroy the inequality (38), and the point  $\{a_n(t)\}$  will leave the set  $\mathbf{M}$ . From this moment the energy  $K(t)$  will decrease. Note that the larger the number  $n$  ( $n > N$ ) of nonzero coefficient  $a_n(t)$ , the higher is the possibility for the point  $\{a_n(t)\}$  to leave the set  $\mathbf{M}$ .

## Numerical experiment

It was mentioned in [11] that in the case when the basic state  $\bar{\psi}(\lambda, \mu)$  is the climatic January flow of atmosphere at 300 mb (Fig.1a), equation (28) could be written with a rather good accuracy as

$$K_t(t) = \int_S \vec{E} \cdot \nabla \bar{u} dS = \int_S \frac{(u')^2 - (v')^2}{\sqrt{1-\mu^2}} \bar{u}_\lambda dS + \int_S u'v' \sqrt{1-\mu^2} \bar{u}_\mu dS \quad (39)$$

Here  $\vec{E} = \langle (u')^2 - (v')^2, u'v' \rangle$  is the generalized Eliassen-Palm flux and  $\nabla \bar{u}$  is the gradient of the zonal velocity component  $\bar{u}$  of the basic flow. Thus, the parts of the sphere where  $\vec{E} \cdot \nabla \bar{u} > 0$  give a contribution into the generation of the perturbation kinetic energy  $K(t)$

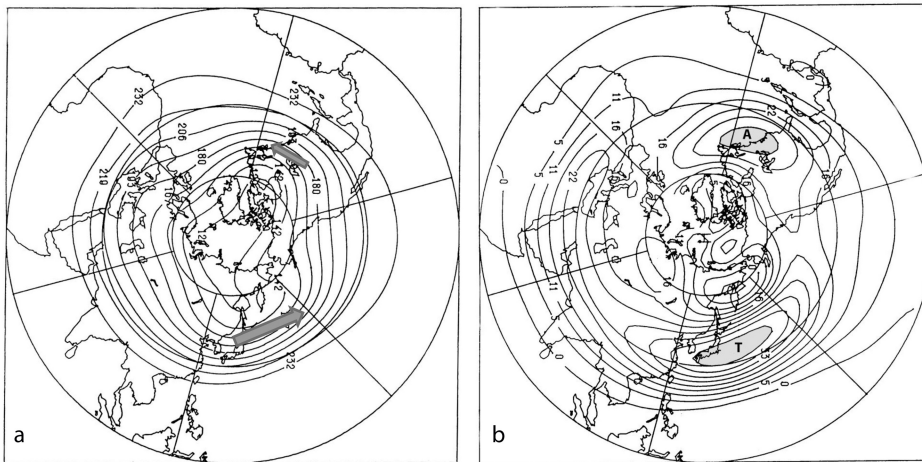


Figure 1. The climatic January circulation of atmosphere at 300 mb. Streamfunction (a) and zonal velocity component (b).

The zonal velocity component  $\tilde{u}$  of the basic flow is shown in Fig.1b. One can see two strong westerly jets (two local maxima in  $\tilde{u}$  field) situated near the western coast of the North Pacific (T-jet) and near the eastern coast of the North America (A-jet). Obviously, the term  $\tilde{u}_\mu$  is large at the lateral sides of the jets. Therefore, the integral  $\int_S u'v' \sqrt{1-\mu^2} \tilde{u}_\mu dS$  in (39) discloses one of the instability mechanisms (the growth of energy of perturbations). Indeed, by this integral, the principal axis of a localized vortex structure of perturbation and the zonal velocity profile of the basic flow must be tilted in opposite sides (or in the same side) in the areas of generation (dissipation) of the perturbation kinetic energy [2,10]. For example, if  $\tilde{u}_\mu > 0$  in a limited region of a zonal jet on a sphere, then localized vortex structures of perturbation in this region such that  $u'v' < 0$  will result in the generation of perturbation kinetic energy  $K(t)$  (Fig.2a). And conversely, localized vortex structures of a perturbation such that  $u'v' > 0$  will lead to the dissipation of perturbation energy  $K(t)$ . This mechanism is especially important in the case of a zonal basic flow  $\tilde{\psi}(\mu)$  (see (30) and example 3).

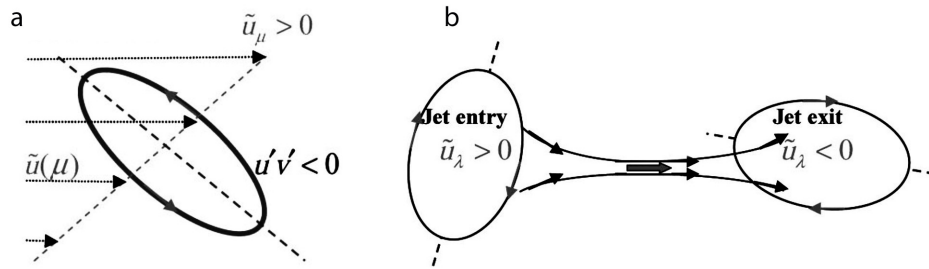


Figure 2. Two instability mechanisms that cause the growth of kinetic energy of perturbations.

Unlike the zonal flow, the additional term

$$- \int_S \frac{(u')^2 - (v')^2}{\sqrt{1-\mu^2}} \tilde{u}_\lambda dS \quad (40)$$

of equation (39) is also of great significance in the evolution of perturbation energy of the January climatic flow  $\tilde{\psi}(\lambda, \mu)$ . Indeed, it follows from (40) that the perturbation kinetic energy is increased if the main axes of localized vortex structures of the perturbation have a meridional orientation ( $(v')^2 \gg (u')^2$ ) at the entry of the jets T and A where  $\tilde{u}_\lambda > 0$ , or zonal orientation ( $(u')^2 \gg (v')^2$ ) at the exit of the jets where  $\tilde{u}_\lambda < 0$  (Fig.2b) [4,11]. The opposite orientation of the main axes of localized vortex structures of the perturbation leads to decreasing the perturbation energy.

The two mechanisms of the growth and decay of the kinetic energy of perturbations (i.e. the two integrals in formula (39)) were described by using the generalized Eliassen-Palm flux analysis, valid for the case when the basic flow is the climatic January circulation of atmosphere at 300 mb. Now, in order to analyze the evolution of energy of perturbation, we apply the new method based on the numerical solution of spectral problem (34). We consider the same basic flow  $\tilde{\psi}(\lambda, \mu)$  (Fig.1), since it allows us to interpret the results by using the instability mechanisms (39). We take  $s = 2$  in the operator (32). The discretization

of the spectral problem (34) is performed by the Galerkin method, with the use of the spherical harmonics as the basic orthonormal functions for the approximation of the main flow and perturbations by spherical polynomials of the subspaces  $\mathbf{P}_M$  and  $\mathbf{P}_N$ , respectively, where  $\mathbf{P}_K = \oplus_{n=1}^K \mathbf{H}_n$ .

The contours of the eigenfunction  $G_1(x)$  corresponding to the largest eigenvalue  $\rho_1 > 0$  are presented in Fig.3a. A perturbation in the form of  $G_1(x)$  causes the fastest growth of perturbation energy  $K(t)$ . The remarkable property of this perturbation is a group of localized vortex structures of alternating-sign situated only near the strongest westerly T-jet of the basic flow. It is seen that eddies have meridional orientation at the entry of the T-jet. Moreover, the contours of eddies at the lateral sides of the T-jet are tilted opposite to the zonal velocity profile shear of the basic flow. This result is in full accordance with the conclusions of the generalized Eliassen-Palm flux analysis (see (39)).

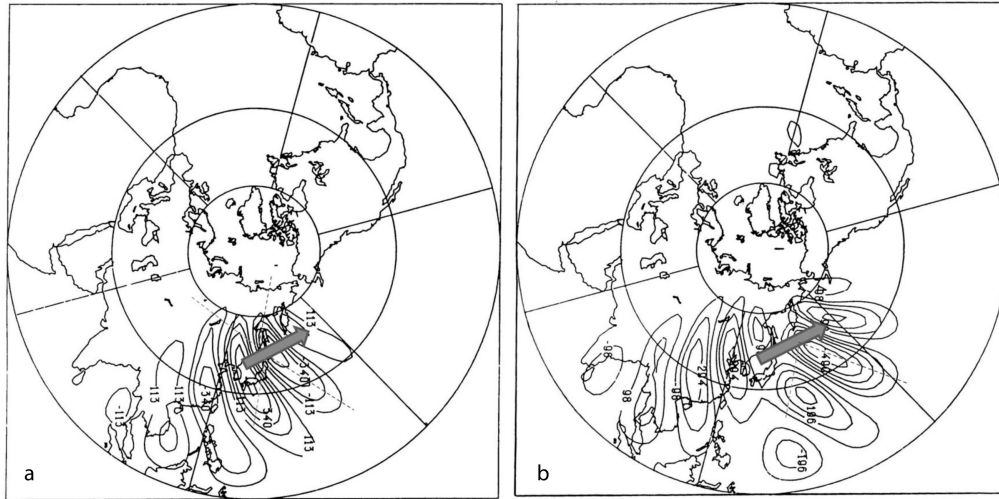


Figure 3. Eigenfunctions of problem (34) corresponding to eigenvalues  $\rho_1$  (a) and  $\rho_2$  (b).

The contours of the eigenfunction  $G_2(x)$  corresponding to the second largest eigenvalue  $\rho_2 > 0$  of the operator  $B$  are presented in Fig.3b. Note that the meridional orientation of eddies at the entry of the T-jet and their tilt at the southern side of the jet lead to the growth of perturbation energy  $K(t)$ . However, one can see that there are no perturbations at the northern side of T-jet, and the eddies at the exit of T-jet have meridional orientation. These two facts lead to decreasing the energy  $K(t)$ . Therefore, the growth rate of  $K(t)$  for the perturbations in the form of  $G_2(x)$  is smaller than that for the perturbations in the form of  $G_1(x)$ .

It is interesting to note that there is also a subset  $\Omega$  of the eigenvalues  $\rho_n$  of operator  $B$  such that the nonzero values of the corresponding eigenfunctions  $G_n(x)$  are localized only near the A-jet. The eigenfunctions presented in Fig.4 correspond to the two largest eigenvalues from  $\Omega$  which are almost twice less than  $\rho_1$  and  $\rho_2$  respectively. It is seen that the location of eddies near the A-jet and their orientation are quite similar to the location and orientation of eddies near the T-jet (see Fig.3). Note that for all eigenvalues  $\rho_n$  such

that  $|\rho_n| \ll 1$ , the corresponding eigenfunctions have a global structure on a sphere.

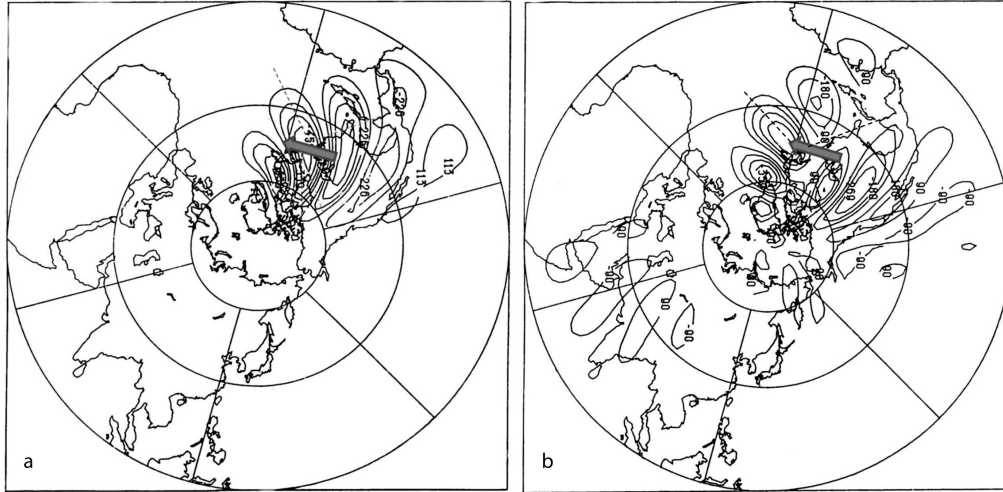


Figure 4. Eigenfunctions corresponding to the two largest eigenvalues from set  $\Omega$

It should be stressed that unlike the Eliassen-Palm flux method suitable for analyzing the climatic January flow, the new approach, based on the solution of spectral problem (34), can be applied to analyze arbitrary steady basic state and takes into account all dissipative processes. Therefore, unlike the Eliassen-Palm flux technique, the new method based on the solution of spectral problem (33) will be a convenient and effective tool for the instability study of stationary flows in the barotropic atmosphere.

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