Communications in **M**athematical **A**nalysis
Volume 17, Number 2, pp. 253–262 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

THE STRUCTURE OF THE CHERENKOV FIELD EMISSION IN METAMATERIALS

ERIKA MARTÍNEZ-SÁNCHEZ*
Centro de Investigación en Ingeniería y Ciencias Aplicadas
Universidad Autónoma del Estado de Morelos,
Cuernavaca, Mor. 62209, México.

GENNADIY BURLAK†
Centro de Investigación en Ingeniería y Ciencias Aplicadas
Universidad Autónoma del Estado de Morelos,
Cuernavaca, Mor. 62209, México.

VLADIMIR RABINOVICH‡
National Polytechnic Institute, ESIME Zacatenco, D.F. Mexico

(Communicated by Vladimir Rabinovich)

Abstract

We report results of systematic analytic and numerical study of the electromagnetic field generated by a moving with the uniform velocity source through a dispersive lossy metamaterial. The Cherenkov radiation in a far zone is considered with the use of 2D stationary phase method. In our simulations the Drude model is implemented for a metamaterial. As a result of passage of modulated charged source in metamaterial a double branching of the Cherenkov cone with negative refractive index is registered.

AMS Subject Classification: 65N25; 74J05; 78A25; 78A40.

Keywords: Cherenkov radiation, dispersive metamaterial, stationary phase method.

1 Introduction

Cherenkov radiation generated by charged particles moving through various materials has been studied in numerous papers [1]-[21]. To expand the already completed studies, it is of great interest to investigate the electromagnetic (EM) field from a charged particle

*E-mail address: erika.martinez@uaem.mx
†E-mail address: gburlak@uaem.mx
‡E-mail address: vladimir.rabinovich@gmail.com

moving through dispersive lossy metamaterials. With the help of FDTD method [7] it is possible to study the structure of the emission field in the near zone. However the study of the field properties in the far zone already requires the knowledge of the field asymptotic for large values of the parameter a/λ, where a is the distance to the observer and λ is the field wavelength. In Ref. [8] was developed the approach based on the two-dimensional stationary phase method that allows to study the details of the field dynamics in the far zone. Such technique was applied to obtain the retarded time and the Doppler shift for modulated moving source in dispersive metamaterial [9] that shows the splitting of the Cherenkov cone into two branches. In this paper the amplitude and phase of the field for metamaterials with the negative refraction index it is studied with details.

2 Basic equations

The electromagnetic effects governed by Maxwell's equations in a dispersive medium, read [8]

$$\begin{aligned} \nabla\times\mathbf{H} &= \epsilon(D_t)\frac{\partial\mathbf{E}}{\partial t}+j,\\ \nabla\times\mathbf{E} &= \mu(D_t)\frac{\partial\mathbf{H}}{\partial t},\\ \epsilon(D_t)\nabla\cdot\mathbf{E} &= \rho,\\ \nabla\cdot\mathbf{H} &= 0. \end{aligned} \tag{2.1}$$

If in the system (2.1) the Fourier transformation is applied and after some calculations it is possible to obtain representation for the electric and magnetic fields in the following form [8]

$$\mathbf{H}(t,\mathbf{x}) = \frac{1}{8\pi^2}\int_{-\infty}^{t}\int_{-\infty}^{\infty}A(\tau)\nabla_x\times\left(\frac{e^{iS(t,x,\omega,\tau)}}{|\mathbf{x}-\mathbf{x}_0(\tau)|}\mathbf{v}(\tau)\right)d\omega d\tau, \tag{2.2}$$

$$\mathbf{E}(t,\mathbf{x}) = \frac{1}{8\pi^2 i}\int_{-\infty}^{t}\int_{-\infty}^{\infty}A(\tau)\omega\mu_{d_\mu}(\omega)(I+k_d^{-2}(\omega)\nabla_x\nabla_x\cdot)\frac{e^{iS(t,x,\omega,\tau)}}{|\mathbf{x}-\mathbf{x}_0(\tau)|}\mathbf{v}(\tau)d\omega d\tau, \tag{2.3}$$

where it is considered a moving source of the form

$$\mathbf{j}(t,\mathbf{x}) = A(t)\mathbf{v}(t)\delta(\mathbf{x}-\mathbf{x}_0(t)),$$

here δ is the Dirac delta function, $\mathbf{x}_0(t)=(x_{01}(t),x_{02}(t),x_{03}(t))$, is the vector defining the motion of the source, and $\mathbf{v}(t)=\dot{\mathbf{x}}_0(t)$ is the velocity of the source.

The integration with respect to τ from $-\infty$ to τ in (2.2), (2.3) is due to the analyticity of the integrands in (2.2), (2.3) for ω in $\mathbb{C}_+$. That is, τ can not exceed t, that agrees with the causality principle.

Based on the main idea of the stationary phase method, there is a cancellation of the sinusoids in the main term of the integral in Eqs. (2.2)-(2.3) for very fast variations ($\omega\to$

∞). However, in a neighborhood near x_0, where $\partial S/\partial\omega = 0$ and $\partial S/\partial\tau = 0$ (stationary point), the component of the EM field become non-zero. The contribution of the stationary points (ω_S, τ_S) in the asymptotic of $\mathbf{H}$ and $\mathbf{E}$, simplifies the calculations and allows one to obtain the following equations for the magnetic and electric fields

$$\mathbf{H}(t,\mathbf{x}) \sim \frac{1}{4\pi}\nabla_x \times \left(\frac{e^{iS(t,x,\omega,\tau)}}{|\mathbf{x}-\mathbf{x}_0(\tau)|}\mathbf{v}(\tau)\right)\frac{a(\tau)}{(det(-S''(t,\mathbf{x},\omega,\tau)))^{1/2}}, \tag{2.4}$$

$$\mathbf{E}(t,\mathbf{x}) \sim \frac{1}{4\pi i}a(\tau)\omega\mu(\omega)\left(I + \frac{1}{k^2(\omega)}\nabla_x\nabla_x \cdot \frac{e^{iS(t,x,\omega,\tau)}}{|\mathbf{x}-\mathbf{x}_0(\tau)|}\mathbf{v}(\tau)\right)\times \frac{1}{(det(-S''(t,\mathbf{x},\omega,\tau)))^{1/2}}, \tag{2.5}$$

where $t = \frac{T}{\lambda}$, $|\mathbf{x}-\mathbf{x}_0(t)| = \frac{|\mathbf{X}-\mathbf{X}_0(T)|}{\lambda}$, see more details in [8]. In the formulas (2.4), (2.5) the phase is:

$$S(t,\mathbf{x},\omega,\tau) = k(\omega)|\mathbf{x}-\mathbf{x}_0(\tau)| - \omega(t-\tau) - \omega_0\tau,$$

Here $k(\omega) = \omega\sqrt{\epsilon(\omega)\mu(\omega)}$ is the wave number, and $\epsilon(\omega)$, $\mu(\omega)$ are the frequency dependent electric permittivity and magnetic permeability.

We note that the contributions in the main term of the asymptotic behavior of the integral (2.2) and (2.3) are given by the stationary points $(\omega = \omega(t,\mathbf{x}), \tau = \tau(t,\mathbf{x}))$ in the phase S that are solutions of the following system

$$\begin{aligned} \frac{\partial S(t,\mathbf{x},\omega,\tau)}{\partial\omega} &= \frac{|\mathbf{x}-\mathbf{x}_0(\tau)|}{v_g(\omega)} - (t-\tau) = 0, \\ \frac{\partial S(t,\mathbf{x},\omega,\tau)}{\partial\tau} &= -k(\omega)v(\mathbf{x},\tau) + (\omega-\omega_0) = 0. \end{aligned} \tag{2.6}$$

After simplification the system (2.6) in the stationary phase points $(\omega(t,\mathbf{x}), \tau(t,\mathbf{x}))$ reads

$$\begin{aligned} &\frac{(x_1^2 + (x_2 - v_0\tau)^2 + (x_3)^2)^{1/2}}{v_g(\omega)} - (t-\tau) = 0, \\ -k(\omega)&\frac{v_0(x_2 - v\tau)}{(x_1^2 + (x_2 - v\tau)^2 + x_3{}^2)^{1/2}} + (\omega-\omega_0) = 0. \end{aligned} \tag{2.7}$$

It is readily to see that system (2.7) already agrees with the causality principle $t > \tau$.

3 Numerical results

In this section, the solution of the field equations (2.4)-(2.5) in the far zone is numerical studied. The following sequence of calculations is applied: first we calculate the solutions τ and ω to the nonlinear system (2.7) by the Newton-Raphson method. Next, we calculate the phase S for found τ and ω that allows to study the structure of the electric and magnetic field in equations (2.4) -(2.5).

As fixed time t and the position of an observer, the Eq. (2.7) are simplified if to take the axis x_2 parallel to the particle velocity v_0

$$r(\tau) - (t-\tau)v_g(\omega) = 0 \tag{3.1}$$

$$-v_0 k(\omega)(x_2 - v_0\tau) + (\omega - \omega_0) r(\tau) = 0 \tag{3.2}$$

where $r(\tau) = \sqrt{x_1^2 + (x_2 - \tau v_0)^2 + x_3^2}$, v_0 is the particle velocity, $k = n(\omega)\omega/c$ is the wave number, $n(\omega)$ is the refractive index of the metamaterial, and $c = (\epsilon_0\mu_0)^{-0.5}$ is the light velocity in free space and v_g is the group velocity $v_g(\omega) = d\omega/dk = c/(n'(\omega)\omega + n)$. The simplest case of the system (3.1)-(3.2) corresponds to a dispersiveless medium with $n = const$. In this situation, the solutions to Eq. (3.1)-(3.2) for τ and ω can be explicitly written as follows

$$\tau_{1,2} = \frac{tc^2 - x_2 n^2 v_0 \pm cn\sqrt{r_1}}{(1-\beta^2)c^2}, \tag{3.3}$$

$$\omega_{1,2} = \frac{c\omega_0 r(\tau)}{n v_0^2 \tau_{1,2} - x_2 n v_0 + c r(\tau)}, \tag{3.4}$$

where $r_1 = (x_2 - v_0 t)^2 + (x_1^2 + x_3^2)(1 - n^2 v_0^2/c^2)$, $\beta = nv_0/c$ and inequality $t > \tau$ has to be applied in Eqs.(3.3)-(3.4).

More interesting is the dispersive medium case with $n = n(\omega)$. However for a dispersive material it is difficult to solve analytically the equations (3.1)-(3.2). Therefore in what follows the numerical solutions to the system (3.1)-(3.2) will be analyzed .

In our approach, the properties of the EM field in dispersive metamaterials, where Re(n) for some range of frequencies becomes negative Re(n) < 0 are studied. The deep insight of the EM waves structure in the dispersive already case requires the explicit formulation the model for the dispersive material. In recent literature the Drude or Lorenz models normally are used as basic models for metamaterials, see [2, 13] and references therein. Since the metamaterials has a metal ingredient, here the Drude model is applied for our simulations. In what follows we consider a medium that is characterized by the electrical permittivity $\epsilon(\omega)$ and magnetic permeability $\mu(\omega)$ that both are complex functions of frequency ω and read

$$\epsilon(\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{Te}^2 - \omega^2 - i\omega\gamma_e}, \tag{3.5}$$

$$\mu(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{Tm}^2 - \omega^2 - i\omega\gamma_m}. \tag{3.6}$$

For such $\epsilon(\omega)$ and $\mu(\omega)$, the complex refractive index $n(\omega)$ of a metamaterial can be written as follows [16], [17]:

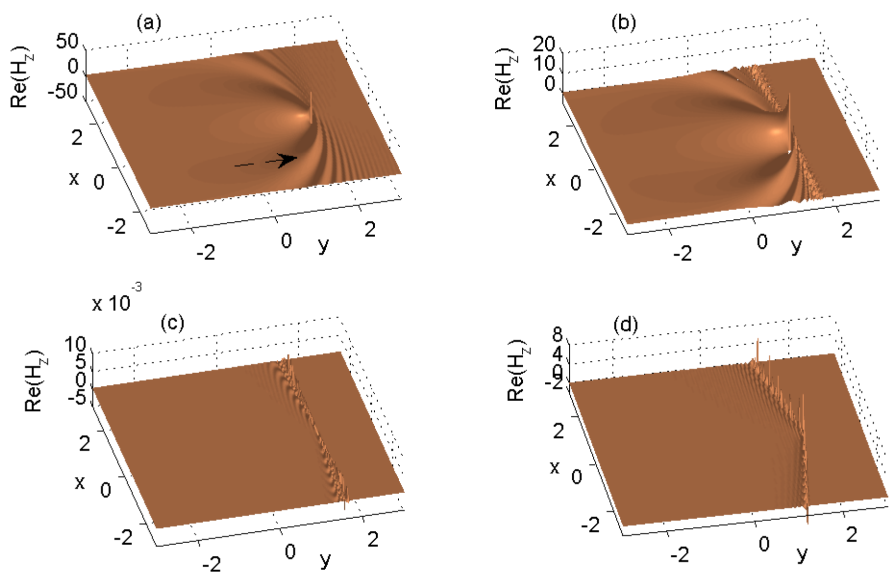

Figure 1. (Color on line.) The spatial structure of the magnetic field component Re(H_z) for the dispersiveless medium at time $t = 2$ for various particle velocity v_0. (a) $v_0 = 0.81 < v_c$. (b) $v_0 = 0.8333 \simeq v_c$; (c) $v_0 = 0.833334$ overlapping wavefronts; (d) Cherenkov cone for $v_0 = 0.845 > v_c$, where the critical velocity is $v_c = 0.83$. See details in the text.

$$n(\omega) = \sqrt{|\epsilon(\omega)\mu(\omega)|}e^{i[\phi_\epsilon(\omega)+\phi_\mu(\omega)]/2} \tag{3.7}$$

The results of our calculations are shown in Figs.1-5, for magnetic Re(H_z) and electric Re(E_x). For numerical simulations we use the dimensionless variables, where for renormalization are applied the vacuum light c and the plasma wavelength $l_0 = c/\omega_{pe}$. The electrical and magnetic fields are renormalized with the electrical scale $E_0 = ql_0\epsilon_0$ and magnetic scale $H_0 = (\epsilon_0/\mu_0)^{0.5}$. Fig.1 shows the spatial field distribution in a dispersionless medium ($n = 1.2$). Only field components H_z are displayed, that is calculated for various particle velocities at fixed time $t = 2$. From Fig.1(a) for case of the velocity $v_0 = 0.81 < v_c = 0.833$, it is seen that the wave fronts are expanded away from the source. At larger velocity $v_0 = 0.8333$ and $v_0 = 0.833334 \simeq v_c$ (see Fig. 1(b-c) respectively), we observe decrease of the field amplitude, and generations some field perturbation in direction perpendicular to v_0. For v_0 slightly more than v_c (Fig. 1(d)) the oscillations occur inside the cone and field amplitude greatly grows. In this case ($v_0 = 0.845$) the field perturbation radiates like a shock wave that giving rise to the Cherenkov cone.

Fig. 2 shows the structure of the electric field E_x in dispersionless system. In this case, the E_x component is calculated for various velocities of the particle at time $t = 2$. From Fig. 1 and Fig. 2 we observe that the component E_x has similar shape to H_z, but with considerable reduction of the amplitude.

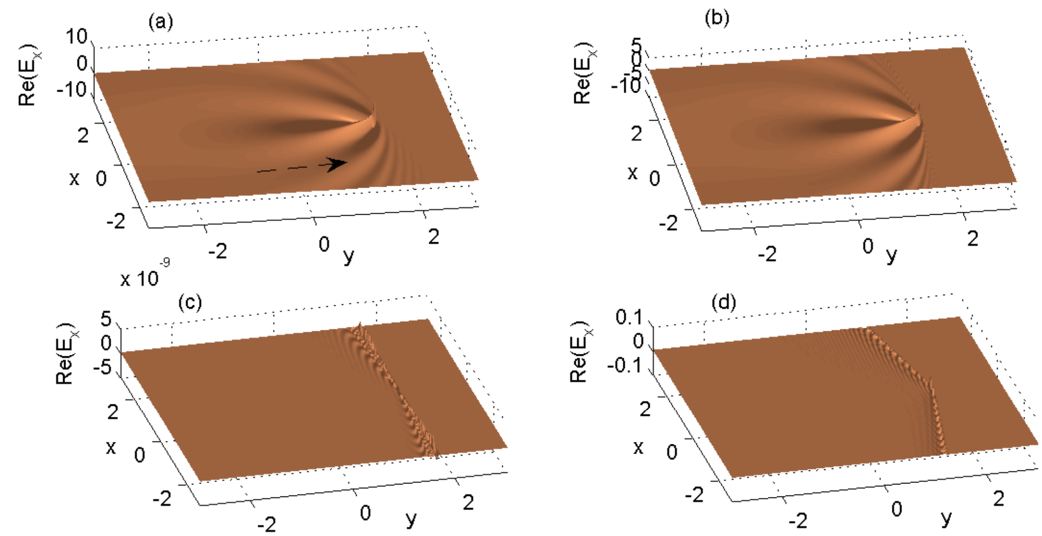

Figure 2. (Color on line.) The spatial structure of the electric field component $\mathrm{Re}(E_x)$ for the dispersiveless medium at $t = 2$ for various particle velocity v_0. (a) $v_0 = 0.81$. (b) $v_0 = 0.8333$; (c) $v_0 = 0.833334$ overlapping wavefronts; (d) Cherenkov cone for $v_0 = 0.845$. See details in the text.

In Figs. 3-4 the results of numerical simulations for the case of dispersive metamaterial is shown. Fig. 3 shows the distribution of the EM field for the $\mathrm{Re}(H_z)$ component. An interesting situation occurs at $v_0 = 0.81$ (Fig. 3(a)), when the field E_x splits in two cones. For this case we have calculated that the refractive index for both branches is different $\mathrm{Re}(n_1) = -1.77$ and $\mathrm{Re}(n_2) = 1.2$ for the internal (n_1) and external (n_2) branches respectively. Here the normalized frequencies are $\omega_{1,2} = 0.117$ and 8.0; group velocities are $v_{g1,2} = 0.22$ and 0.833 respectively. The internal branch corresponds to the double negative metamaterial, while the external branch looks like conventional medium with $vg \simeq vc$. At larger velocity of the particle $v_0 = 0.8333$, (Fig. 3(b)) we see that the waves leave the external branches, reach the velocity limit in the medium and are overlapped. For a dispersive medium, the critical velocity is not well defined due to the dependence $n(\omega)$. However, using the same fixed parameters in both simulations, v_c in Drude medium (with $n > 0$) is very close to the critical velocity in the dispersionless medium. There fore the Fig. 3(c), is rather similar to Fig. 1(c). In Fig. 3(d) we observed the Cherenkov conventional cone, in this case the medium behaves as a dispersionless one with $n = 1.2$.

Fig. 4 shows the results of the spatial field simulations in metamaterial. In this figure, the $\mathrm{Re}(E_x)$ component is shown, that is calculated for various velocities of the particle at time $t = 2$. Compared to Fig. 3 the field Ex shows rather similar shape to H_z, but with considerable reduction of amplitude.

It is instructively to compare the structure of the field in dispersiveless dielectric and dispersive (Drude) metamaterial for the same of the particle velocity $v_0 < v_c$, see Fig. 5. From Fig. 5(a,b) we observe that for small v_0 the shape of field has typical structure of the moving source (panel (a)) with the increase of amplitude of radiated field at for $v_0 \to v_c$

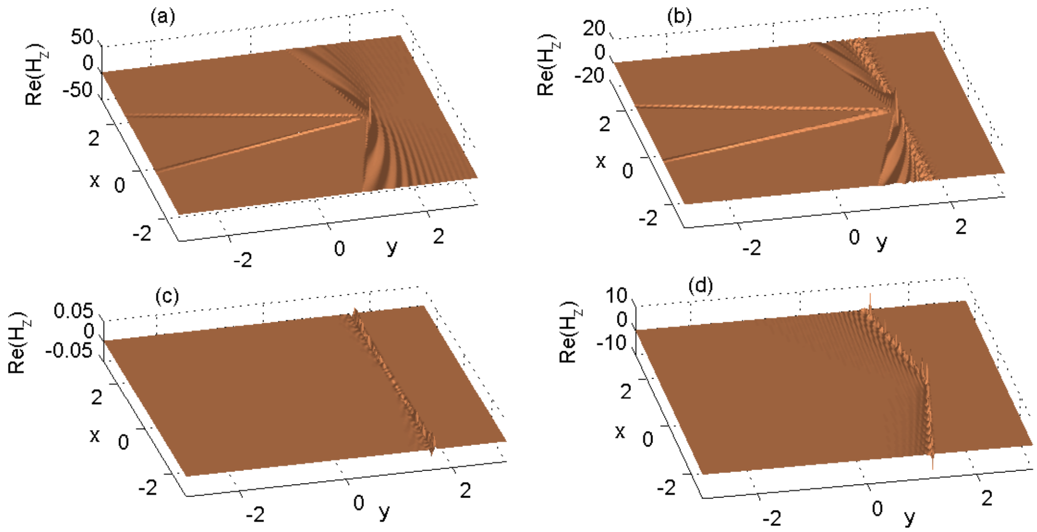

Figure 3. (Color on line.) Field component Re(H_z) for a dispersive lossy medium in time $t = 2$. (a) $v_0 = 0.81 < v_c$, $n_1 = -1.77$ and $n_2 = 1.2$. (b) $v_0 = 0.8333 \simeq v_c$, $n_1 = -1.83$ and $n_2 = 1.2$. (c) $v_0 = 0.833334$, $n = 1.2$. (d) Cherenkov cone with $v_0 = 0.845 > v_c$.

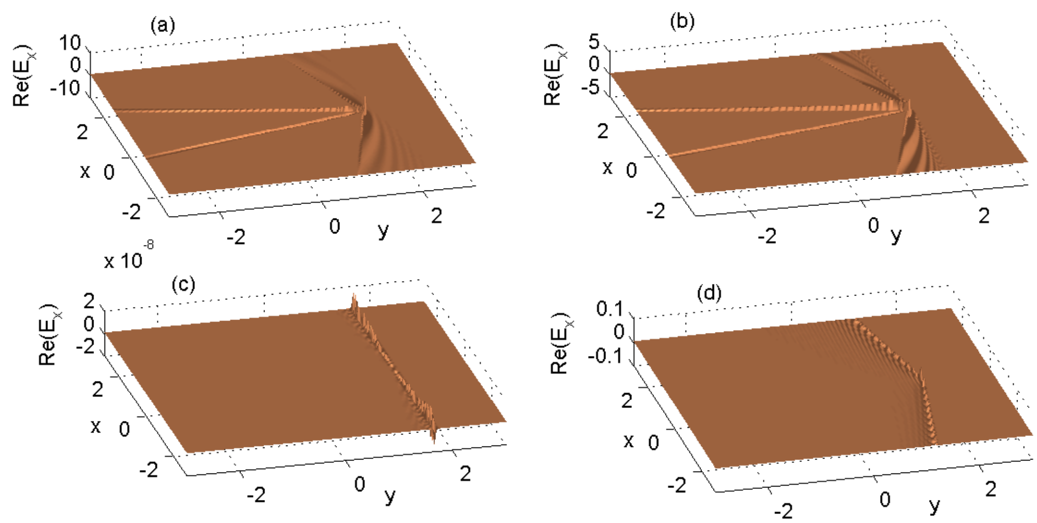

Figure 4. (Color on line.) The structure of the electric field Re(E_x) for dispersive lossy metamaterial at $t = 2$. (a) $v_0 = 0.81$, $n_1 = -1.77$ and $n_2 = 1.2$, (b) $v_0 = 0.8333$, $n_1 = -1.83$ and $n_2 = 1.2$, (c)$v_0 = 0.833334$, $n = 1.2$ and, (d) Cherenkov cone for $v_0 = 0.84$.

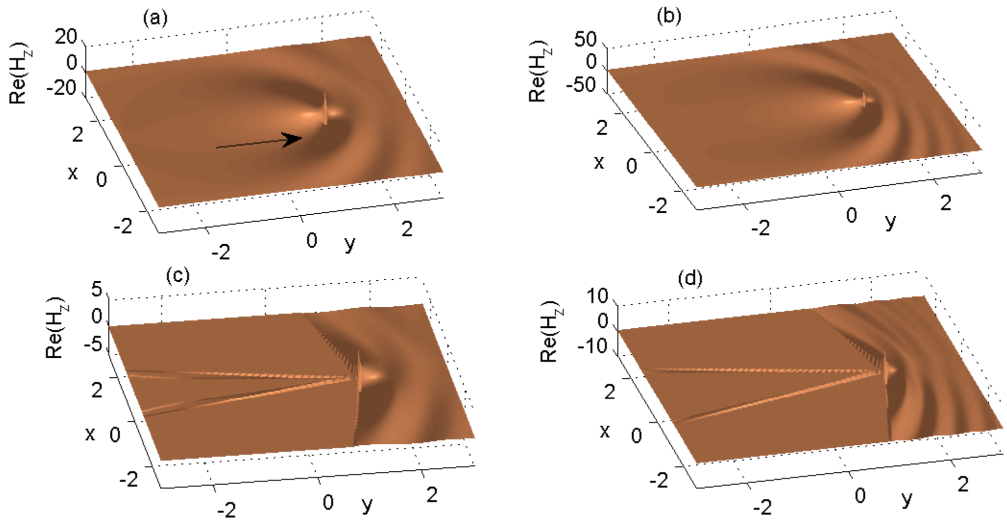

Figure 5. (Color on line.) The structure of the field for dispersiveless medium (d.m.) (a), (b) and metamaterial (m.m.) (c), (d): (a) d.m. for $v_0 = 0.65$, $n = 1.2$ at $f = 0.163$, $\tau = -2.1$, $v_{ph} = 0.833 = v_g$; (b) Same as in panel (a) but for $v_0 = 0.74$ and $f = 0.16$, $\tau = -2.03$, (c) m.m. for $v_0 = 0.65$, $n_1(f) = -1.27$ at $f = 0.128$, $\tau = 1.37$, $v_{ph} = -0.79$, $v_g = 0.238$, $n_2 = 1.16$ at $f = 1.02$, $\tau = -9.03$, $v_{PhMax} = 0.865$, $v_{GrMax} = 0.804$; (d) m.m. for $v_0 = 0.74$, $n_1 = -1.48$ at $f = 0.123$, $\tau = 1.41$, $v_{PhMin} = -0.677$, $v_{GrMin} = 0.231$ and $n_2 = 1.19$ at $f = 2$, $\tau = -15.8$, $v_{PhMax} = 0.841$, $v_{gMax} = 0.826$ (see details in text).

(panel (b)). However the situation becomes more involved for dispersive metamaterial, see Fig. 5(c,d). In this case the refractive index $n = n(f), f = \omega/2\pi$ (and the critical velocity) already depends on the field frequency f that is the solution to the system (3.1),(3.2) and (3.5), (3.6) for dispersive metamaterial. Such a complex dependence leads to that the Cherenkov cone splits in two branches, see Fig. 5 (c,d). Besides, as we observe from Fig. 5 (c,d), the shape of the wave's phase acquires more sharp structure at the increase of the particle velocity v_0.

4 Conclusion

In this paper the electromagnetic field generated by source moving through a dispersive lossy metamaterial is analytically and numerically studied. The Cherenkov radiation in far zone is considered with the use of 2D stationary phase method. In our simulations the Drude model is implemented for metamaterial. It is found that the spatial electromagnetic phase dependencies contain the important information on the field details in the far zone. As result of passage of modulated charged source moving with the uniform velocity in metamaterial a double branching (splitting) of the Cherenkov cone with negative refractive index is registered.

5 Acknowledgments

This work was partially supported by CONACyT (México) grants No. 169496, No. 168104, and project Redes de PROMEP (México).

References

[1] Yu. O. Averkov and V. M. Yakovenko, Cherenkov radiation by an electron particle that moves in a vacuum above a left-handed material. *Phys. Rev. B.* **79** (2005), pp 193402-193412.

[2] Z. Y. Duan, B. I. Wu, S. Xi, H. S. Chen, and M. Chen, Research progress in reversed Cherenkov radiation in double-negative metamaterials. *Progress In Electromagnetics Research* **90** (2009), pp 75-87.

[3] Sheng Xi, Hongsheng Chen, Tao Jiang, Lixin Ran, Jiangtao Huangfu, Bae-Ian Wu, Jin Au Kong, and Min Chen, Experimental verification of reversed Cherenkov radiation in left-handed Metamaterial. *Phys. Rev. Lett.* **103** (2009), pp 194801.

[4] Yu. O. Averkov, A. V. Kats, and V. M. Yakovenko, Electron beam excitation of left-handed surface electromagnetic waves at artificial interfaces. *Phys. Rev. B* **72** (2005), pp 205110-205114.

[5] Jun Zhou, Zhaoyun Duan, Yaxin Zhang, Min Hu, Weihao Liu, Ping Zhang, and Sheng-gang Liu, Numerical investigation of Cherenkov radiations emitted by an electron beam particle in isotropic double-negative metamaterials. *Nuclear Instruments and Methods in Physics Research Section A.* **654** (2011), pp 475-480.

[6] Z. Y. Duan, Y. S. Wang, X. T. Mao, W. X. Wang, and M. Chen, Experimental demonstration of double-negative metamaterials partially filled in a circular waveguide. *Progress In Electromagnetics Research* **121** (2011), pp 215-224.

[7] A. Taflove, S. C. Hagness, Computational Electrodynamics: The Finite-Difference Time-Domain Method, Artech House, 3D edition (2005).

[8] G. Burlak and V. Ravinovich, Time-Frequency Integrals and the Stationary Phase Method in Problems of Waves Propagation from Moving Sources. SIGMA 8 **096** (2012), pp 21.

[9] G. Burlak and V. Ravinovich, Propagation of electromagnetic waves generated by modulated moving sources in dispersive lossy media. *Math. Meth. Appl. Sci.* (2014).

[10] V. M Babich, V. S. Buldirev, and I. A. Molotkov, Space-time Ray Methods, *Linear and and Non-linear Waves*. Leningrad Univ, 1985.

[11] L. D. Landau and E. M. Lifshitz, *Field Theory, Theoretical Physics V.II*, Pergamon Press, 1971.

[12] G. N. Afanasiev, *Vavilov-Cherenkov and Synchrotron Radiation*, Ed. Alwin Van Der Merwe, University of Denver, U.S.A., 2005.

[13] J. Lu, T. Grzegorczyk, Y. Zhang, J. Pacheco, B.I. Wu, J. Kong, and M. Chen, Cherenkov radiation in materials with negative permittivity and permeability. *Opt. Express* **11** (2003), pp 723-734.

[14] P.A. Cherenkov, Visible radiation produced by electrons moving in a medium with velocities exceeding that of light. *Phys. Rev.* **54** (1937), pp 449.

[15] Shumin Xiao, V. P. Drachev, A. V. Kildishev, Xingjie Ni, K. Uday Chettiar, Hsiao-Kuan Yuan, and V. M. Shalaev, Loss-free and active optical negative-index metamaterials. *Nature* **466** (2010), pp 735-738.

[16] Ho Trung Dung, Stefan Yoshi Buhmann, and Ludwig Knoll, Electromagnetic-field quantization and spontaneous decay in left-handed media. *Phys. Rev. A.* **68** (2003), pp 043816.

[17] R. W. Ziolkowski and E. Heyman, Wave propagation in media having negative permittivity and permeability. *Phys. Rev.E* **64** (2001) pp 056625.

[18] J. D. Jackson, *Classical electrodynamics*, John Willey and Sons, 1975.

[19] A.V. Tyukhtin and S.N. Galyamin, Vavilov-Cherenkov radiation in passive and active media with complex resonant dispersion. *Phys. Rev. E.* **77** (2008), pp 066606.

[20] S.N. Galyamin and A.V. Tyukhtin, Electromagnetic field of a moving charge in the presence of a left-handed medium. *Phys. Rev. B.* **81** (2010), pp 235134.

[21] E.S. Belonogaya, S.N. Galyamin, and A.V. Tyukhtin, Properties of Vavilov-Cherenkov radiation in an anisotropic medium with a resonant dispersion of permittivity. *Opt. Soc. Am. B.* **28** (2011), pp 2871.