Communications in Mathematical Analysis
Volume 17, Number 2, pp. 231–238 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

On Toeplitz Operators with Piecewise Continuous and Slowly Oscillating Radial Symbols

Erick Lee[*]
Departamento de Matemáticas
Cinvestav
México D.F., México.

Maribel Loaiza[†]
Departamento de Matemáticas
Cinvestav
México D.F., México

Nikolai Vasilevski[‡]
Departamento de Matemáticas
Cinvestav
México D.F., México

(Communicated by Vladimir Rabinovich)

Abstract

We describe the symbol (Calkin) algebra for the algebra generated by Toeplitz operators with piecewise continuous and slowly oscillating radial symbols that act on the Bergman space on the unit disk.

AMS Subject Classification: 47B35.

Keywords: Toeplitz operator, piecewise continuous symbol, slowly oscillating symbol, Calkin algebra.

1 Introduction

Let $\mathbb{D}$ be the open unit disk with the normalized Lebesgue measure $dA = \frac{1}{\pi}dxdy$. Let $\mathcal{A}^2(\mathbb{D})$ be the Bergman space of all functions analytic and square integrable in $\mathbb{D}$ and let

[*]E-mail address: elee@math.cinvestav.mx
[†]E-mail address: mloaiza@math.cinvestav.mx
[‡]E-mail address: nvasilev@math.cinvestav.mx

$B_{\mathbb{D}} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ be the (orthogonal) Bergman projection. Denote by $T_f\varphi = B_{\mathbb{D}}(f\varphi)$, $\varphi \in \mathcal{A}^2(\mathbb{D})$, the Toeplitz operator with symbol $f \in L_\infty(\mathbb{D})$.

For an algebra of functions $\mathcal{B} \subset L_\infty(\mathbb{D})$ denote by $\mathcal{T}(\mathcal{B})$ the C^*-algebra generated by all Toeplitz operators T_f with symbols $f \in \mathcal{B}$. Unfortunately for a generic $\mathcal{B} \subset L_\infty(\mathbb{D})$ practically nothing can be said about the properties of the operators from $\mathcal{T}(\mathcal{B})$. The natural approach thus is to select specific class $\mathcal{B}$ that make it possible to describe the properties of the corresponding operators. The first step in this direction was made by L. Coburn [2], who described the algebra generated by Toeplitz operators with continuous symbols. The case of piecewise continuous symbols (with a fixed finite set of discontinuity points on the boundary) was considered in [9, 10]. In this note we extend the setting and results of [9, 10] in two directions. First, we relax the above condition on the boundary discontinuity points allowing non fixed countable sets of discontinuity that depend on a symbol, and second, we admit a slowly oscillating behavior of symbols in the radial direction. The main result, Theorem 5.3, describes the corresponding symbol (Calkin) algebra.

2 On a symbol class

We introduce first two C^*-algebras of functions. The first one is defined on the interval $[0,1)$ and is the Sarason class $\mathrm{SO}[0,1)$, see [7]. We recall its definition.

For a function a, defined in an interval I, let $\omega(a,I)$ denote the oscillation of a in I

$$\omega(a,I) = \sup_{r,s\in I} |a(r) - a(s)|.$$

Given a half-open interval $[0,1)$, the class $\mathrm{SO}[0,1)$ consists of all functions $a \in C([0,1))$ satisfying the condition: for each $\sigma \in (0,1)$,

$$\lim_{\delta\to 0} \omega(a,[1-\delta, 1-\sigma\delta]) = 0.$$

To introduce the algebra of functions defined on the unit circle $\mathbb{T}$, we denote by $PC_0(\mathbb{T})$ the set of all functions $f \in L_\infty(\mathbb{T})$ with the following properties:

1. f is continuous except for a finite set of points.

2. If f is not continuous at t_0, then there exist both lateral limits

$$f(t_0^-) = \lim_{t\to t_0^-} f(t), \quad f(t_0^+) = \lim_{t\to t_0^+} f(t).$$

The closure of $PC_0(\mathbb{T})$ in $L_\infty(\mathbb{T})$ is denoted by $PC(\mathbb{T})$ and is called the algebra of piecewise continuous functions. The following theorem summarizes the main characteristics of $PC(\mathbb{T})$ (see [1]). For the reader convenience we include its proof here.

Theorem 2.1. *A function $f : \mathbb{T} \to \mathbb{C}$ is in $PC(\mathbb{T})$ if and only if f has finite one-sided limits at every point of $\mathbb{T}$.*

Proof. Since every function in $PC_0(\mathbb{T})$ has one-sided limits at every point of $\mathbb{T}$, the uniform limit of a convergent sequence of functions $f_n \in PC_0(\mathbb{T})$ also possesses such a property. Moreover, the set of discontinuities for the uniform limit of the sequence $\{f_n\}$ is at most countable.

Consider now a function $f : \mathbb{T} \to \mathbb{C}$ such that f has one-sided limits at every point of $\mathbb{T}$. Let $r > 0$, define the set

$$A_r = \{t \in \mathbb{T} : |f(t^+) - f(t^-)| > r\}.$$

It is easy to prove that, for every $r > 0$, the set A_r is finite. In particular, the set $A_{\frac{1}{2^n}}$ is finite. For $z \in A_{\frac{1}{2^n}}$, let $\delta_z > 0$ such that, if $w \in (ze^{i\delta_z}, z)$ then $|f(w) - f(z^-)| < \frac{1}{2^n}$ and, if $w \in (z, ze^{-i\delta_z})$, then $|f(w) - f(z^+)| < \frac{1}{2^n}$. Denote by U_z the open set $(ze^{i\delta_z}, ze^{-i\delta_z})$.

If f is not continuous at the point $z \in \mathbb{T}$ and $z \notin A_{\frac{1}{2^n}}$ choose $\delta_z > 0$ such that, if $w \in (ze^{i\delta_z}, z)$ then, $|f(w) - f(z^-)| < \frac{1}{2^{n+1}}$ and if $w \in (z, ze^{-i\delta_z})$ then, $|f(w) - f(z^+)| < \frac{1}{2^{n+1}}$. Then, for each $w \in U_z = (ze^{i\delta_z}, ze^{-i\delta_z})$,

$$|f(w) - f_z| < \frac{1}{2^n}, \quad \text{where } f_z = \frac{f(z^+) + f(z^-)}{2}.$$

Finally, if f is continuous at the point $z \in \mathbb{T}$, let $\delta_z > 0$ such that, for every $w \in U_z = (ze^{i\delta_z}, ze^{-i\delta_z})$

$$|f(w) - f(z)| < \frac{1}{2^n}.$$

The family $\{U_z : z \in \mathbb{T}\}$ is an open cover of $\mathbb{T}$. Then, there exists a finite set of points $z_1, \ldots, z_m$ such that

$$\mathbb{T} = \bigcup_{i=1}^{m} U_{z_i}.$$

Define the following function, let $z \in \mathbb{T}$ and select U_{z_k} such that $z \in U_{z_k}$

$$f_n(z) = \begin{cases} f(z_k), & \text{if } f \text{ is continuous at } z_k, \\ \frac{f(z_k^+)+f(z_k^-)}{2}, & \text{if } f \text{ is not continuous at } z_k \text{ and } z_k \in \mathbb{T} \setminus A_{\frac{1}{2^n}}, \\ f(z_k^-), & \text{if } z_k \in A_{\frac{1}{2^n}} \text{ and } z \in (z_k e^{i\delta_{z_k}}, z_k], \\ f(z_k^+), & \text{if } z_k \in A_{\frac{1}{2^n}} \text{ and } z \in (z_k, z_k e^{-i\delta_{z_k}}). \end{cases}$$

Then, $|f_n(z) - f(z)| < \frac{1}{2^n}$ $\forall z \in \mathbb{T} \setminus A_{\frac{1}{2^n}}$. Since $A_{\frac{1}{2^n}}$ is finite for each $n \in \mathbb{N}$ we get

$$f_n \to f \quad \text{in } L_\infty.$$

Since each $f_n \in PC_0(\mathbb{T})$ the function f belongs to $PC(\mathbb{T})$.

□

Corollary 2.2. *The set of all piecewise constant (step) functions is dense in $PC(\mathbb{T})$.*

Our aim is to study the algebra generated by Toeplitz operators whose symbols belong to the tensor product $\mathrm{SO}[0,1) \otimes PC(\mathbb{T})$. The set of finite products of the form

$$\sum_{i=1}^{n} a_i \otimes b_i, \qquad a_i \in \mathrm{SO}[0,1), \quad b_i \in PC(\mathbb{T})$$

is dense in $\mathrm{SO}[0,1)\otimes PC(\mathbb{T})$. Then each summand $a_i\otimes b_i$ can be decomposed as $a_i\otimes b_i = (a_i\otimes 1)(1\otimes b_i)$. The properties (2.3) and (2.4) imply thus that, to study the algebra generated by Toeplitz operators with symbols from $\mathrm{SO}[0,1)\otimes PC(\mathbb{T})$, it is sufficient to consider symbols of the form $a\otimes 1$ and $1\otimes b$ only, where $a\in \mathrm{SO}[0,1)$ and $b\in PC(\mathbb{T})$.

Note that $a\otimes 1$ is the radial extension of the function $a\in\mathrm{SO}[0,1)$, i.e., $a\otimes 1(z) = a(|z|)$ and $1\otimes b$ is the homogeneous of zero order extension of the function $b\in PC(\mathbb{T})$, i.e., $1\otimes b(z) = b\left(\frac{z}{|z|}\right)$. In what follows we will identify the functions $a\in\mathrm{SO}[0,1)$ and $b\in PC(\mathbb{T})$ with their extensions to the unit disk, and will write just $a\in\mathrm{SO}$ and $b\in PC$, where, in case when the domains of a and b are not explicitly specified, their common domain is the unit disk.

2.1 Compactness properties

Denote by $\mathcal{K}$ the ideal of all compact operators acting on $\mathcal{A}^2(\mathbb{D})$. Following [11], introduce

$$\Lambda = \{f\in L_\infty(\mathbb{D}):\ T_fT_g - T_{fg}\in\mathcal{K}\ \ \forall g\in L_\infty(\mathbb{D})\}. \tag{2.1}$$

It is known (see [11]) that the intersection $Q = \Lambda\cap\overline{\Lambda}$ is the largest C^*-algebra in $L_\infty(\mathbb{D})$ such that the mapping

$$\phi : Q\to\mathcal{T}(Q)/\mathcal{K}, \tag{2.2}$$

defined by

$$\phi(f) = T_f + K,$$

is a C^*-homomorphism.

Let B denote the set of all functions $f\in L_\infty(\mathbb{D})$ such that the corresponding Toeplitz operator T_f is compact. The kernel of the homomorphism (2.2) coincides with $Q\cap B$, and thus

$$Q/Q\cap B\cong\mathcal{T}(Q)/\mathcal{K}.$$

The algebra Q admits another independent description (see [11]),

$$Q = ESV(\mathbb{D}) + Q\cap B,$$

where the space $ESV(\mathbb{D})$ is set of all $L_\infty(\mathbb{D})$-functions f satisfying the following property: for any $\varepsilon > 0$ and $\sigma\in(0,1)$, there is $\delta_0 > 0$, such that $|f(z) - f(w)| < \varepsilon$, whenever $|z|,|w|\in[1-\delta, 1-\sigma\delta]$, $\delta<\delta_0$, and $|\arg w - \arg z|\le\max(1-|z|, 1-|w|)$.

It is straightforward to check that $\mathrm{SO}\subset ESV(\mathbb{D})\,(\subset Q)$, which in turn implies, see (2.1), that

$$\begin{aligned}
[T_{a_1}, T_{a_2}) &= T_{a_1}T_{a_2} - T_{a_1a_2}\in\mathcal{K}, \quad \text{for all}\quad a_1, a_2\in\mathrm{SO},\\
[T_{a_1}, T_{a_2}] &= T_{a_1}T_{a_2} - T_{a_2}T_{a_1}\in\mathcal{K}, \quad \text{for all}\quad a_1, a_2\in\mathrm{SO},\\
[T_a, T_b) &= T_aT_b - T_{ab}\in\mathcal{K}, \quad \text{for all}\quad a\in\mathrm{SO},\ b\in PC, \qquad (2.3)\\
[T_a, T_b] &= T_aT_b - T_bT_a\in\mathcal{K}, \quad \text{for all}\quad a\in\mathrm{SO},\ b\in PC. \qquad (2.4)
\end{aligned}$$

Furthermore, it is well known that, for arbitrary functions $b_1, b_2\in PC$, the semi-commutator $[T_{b_1}, T_{b_2})$ is not compact, in general, while the commutator $[T_{b_1}, T_{b_2}]$ remains to be compact (see for example [6]).

Let $\mathcal{T}(PC)$ be the C^*-algebra generated by all Toeplitz operators T_b acting on $\mathcal{A}^2(\mathbb{D})$ and with symbol $b \in PC(\mathbb{T})$, and let $\mathcal{T}(\mathrm{SO})$ be the C^*-algebra generated by all Toeplitz operators T_a acting on $\mathcal{A}^2(\mathbb{D})$ and with symbols $a \in \mathrm{SO}$. By the above discussion, the Calkin algebra of the algebra $\mathcal{T}(\mathrm{SO}[0,1) \otimes PC(\mathbb{T}))$ is generated by the Calkin algebras of the algebras $\mathcal{T}(PC)$ and $\mathcal{T}(\mathrm{SO})$. In the next two sections we describe each of these two algebras.

3 Algebra $\mathcal{T}(PC)/\mathcal{K}$

The description of the algebra $\mathcal{T}(PC)/\mathcal{K}$ is quite similar to the one of the Calkin algebra $\mathcal{T}(PC(\mathbb{T}))/\mathcal{K}$ for the Hardy space case [1, 4].

To describe the symbol (Calkin) algebra $\mathrm{Sym}\,\mathcal{T}(PC) = \mathcal{T}(PC)/\mathcal{K}$ we use the Douglas-Varela local principle [3, 8]. As a central commutative subalgebra of $\mathcal{T}(PC)/\mathcal{K}$ we take $\mathcal{T}(1 \otimes C(\mathbb{T}))/\mathcal{K}$, which is isomorphic to $C(\mathbb{T})$; its maximal ideal space is isomorphic to $\mathbb{T}$.

For each point $t \in \mathbb{T}$ consider the diameter starting at t. This diameter divide the unit disk into two regions R_+, R_-. Introduce the operator $P_+ = \chi_{R_+} I_{\mathcal{A}^2(\mathbb{D})}$, and note that the Toeplitz operator T_a is locally equivalent at the point t to the operator

$$a(t^+)B_{\mathbb{D}}P_+ + a(t^-)B_{\mathbb{D}}(I - P_+),$$

where $a(t^+) = \lim\limits_{\substack{w \to t \\ w \in R_+}} a(w)$, $a(t^-) = \lim\limits_{\substack{w \to t \\ w \in R_-}} a(w)$. Thus we have

Theorem 3.1. *The local algebra $\mathcal{T}(PC)(t)$ of $\mathcal{T}(PC)/\mathcal{K}$ at the point $t \in \mathbb{T}$ is generated by the operator $B_{\mathbb{D}}P_+B_{\mathbb{D}}$ and the identity $I_{\mathcal{A}^2(\mathbb{D})}$.*

It is well known (see, for example, [9, 10]) that $\mathrm{sp}\, B_{\mathbb{D}}P_+B_{\mathbb{D}} = [0,1]$, which implies the next corollary

Corollary 3.2. *The local algebra $\mathcal{T}(PC)(t)$ of $\mathcal{T}(PC)/\mathcal{K}$ at the point t is isomorphic to the algebra of all continuous functions on* $[0,1]$. *The homomorphism*

$$\pi_t : \mathcal{T}(PC) \longrightarrow \mathcal{T}(PC)/\mathcal{K} \longrightarrow \mathcal{T}(PC)(t)$$

is given by the following mapping of the generators of $\mathcal{T}(PC)$

$$\pi_t : T_a \longmapsto a(t^+)x + (1-x)a(t^-), \quad x \in [0,1].$$

Gluing together the descriptions of all local algebras $\mathcal{T}(PC)(t)$, $t \in \mathbb{T}$ we come to the next description of the algebra $\mathcal{T}(PC)/\mathcal{K}$.

Theorem 3.3. *The algebra $\widehat{\mathcal{T}}(PC) = \mathcal{T}(PC)/\mathcal{K}$ is isomorphic and isometric to the algebra of all continuous functions on the cylinder* $\mathbb{T} \times [0,1]$. *The Gelfand isomorphism*

$$\Gamma : \widehat{\mathcal{T}}(PC) \longrightarrow C(\mathbb{T} \times [0,1])$$

is defined on the set of generators of $\mathcal{T}(PC)$ as follows

$$\Gamma : \widehat{T}_a \longmapsto a(t^+)x + a(t^-)(1-x), \quad x \in [0,1],\ t \in \mathbb{T}.$$

We note that the cylinder $\mathbb{T}\times[0,1]$, like in [1, 4], is equipped with the special topology. A local base of neighborhoods at the point (t,x) consists of the sets $\{t\}\times(x-\epsilon,x+\epsilon)$, $0<\epsilon<min\{x,1-x\}$ if $x\notin\{0,1\}$. For the point $(t,1)$ a local base of neighborhoods is $\left(\left[t,te^{i\epsilon}\right)\times(1-\epsilon,1]\right)\cup\left(\left(t,te^{i\epsilon}\right)\times[0,1-\epsilon]\right)$ with $0<\epsilon<1$. Finally for the point $(t,0)$ a local base of neighborhoods is

$$\left(\left(te^{-i\epsilon},t\right]\times[0,\epsilon)\right)\bigcup\left(\left(te^{-i\epsilon},t\right)\times[\epsilon,1]\right),\quad \text{with } 0<\epsilon<1.$$

Note as well that the topology of the cylinder, described above, is the weakest topology that makes $\Gamma(\widehat{T_a})$ continuous for every $a\in PC$.

4 Algebra $\mathcal{T}(\mathrm{SO})/\mathcal{K}$

The algebra $\mathcal{T}(\mathrm{SO})$ is a subalgebra of $\mathcal{T}(Q)$, considered in [11], and, at the same time, a subalgebra of $\mathcal{T}(L_\infty[0,1))$, which in turn is a subalgebra of the algebra considered in [5].

Being a subalgebra of $\mathcal{T}(Q)$, the algebra $\mathcal{T}(\mathrm{SO})$ admits the homomorphism (see (2.2))

$$\phi:\ \mathrm{SO}\longrightarrow \mathrm{Sym}\,\mathcal{T}(\mathrm{SO})=\mathcal{T}(\mathrm{SO})/\mathcal{K},$$

defined by $\phi:\ a\longmapsto T_a+\mathcal{K}$. The kernel of the homomorphism ϕ is $\mathrm{SO}\cap B=\{a\in\mathrm{SO}:\lim_{r\to1}a(r)=0\}$, and thus

$$\mathrm{Sym}\,\mathcal{T}(\mathrm{SO})=\mathcal{T}(\mathrm{SO})/\mathcal{K}\cong\mathrm{SO}/\mathrm{SO}\cap B.$$

Let $M(\mathrm{SO}[0,1))$ be the maximal ideal space of $\mathrm{SO}[0,1)$. For each $r\in[0,1)$, let $\delta_r\in M(\mathrm{SO}[0,1))$ be the evaluation functional

$$\delta_r(f)=f(r).$$

Identifying δ_r with the point r, we can consider $[0,1)$ as a subset of $M(\mathrm{SO}[0,1))$. That is

$$M(\mathrm{SO}[0,1))=[0,1)\cup M_1(\mathrm{SO}[0,1)),$$

where $M_1(\mathrm{SO}[0,1))$ is the fiber of $M(\mathrm{SO}[0,1))$ consisting of all multiplicative functionals g of $M(\mathrm{SO}[0,1))$ such that $g(a)=0$ whenever $\lim_{r\to1}a(r)=0$.

The maximal ideal space of $\mathrm{SO}[0,1)/\mathrm{SO}[0,1)\cap B$ is thus isomorphic to $M_1(\mathrm{SO}[0,1))$, which implies

$$\mathcal{T}(\mathrm{SO})/\mathcal{K}=\mathrm{Sym}\,\mathcal{T}(\mathrm{SO})\cong C(M_1(\mathrm{SO}[0,1))).\tag{4.1}$$

5 Toeplitz algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes PC(\mathbb{T}))$

Denote by $\mathcal{T}=\mathcal{T}(\mathrm{SO}[0,1)\otimes PC(\mathbb{T}))$ the C^*-algebra generated by all Toeplitz operators T_f with symbols $f\in\mathrm{SO}[0,1)\otimes PC(\mathbb{T})$. It is easy to see that $C(\overline{\mathbb{D}})\subset\mathrm{SO}[0,1)\otimes PC(\mathbb{T})$, and thus $\mathcal{T}(C(\overline{\mathbb{D}}))\subset\mathcal{T}(\mathrm{SO}[0,1)\otimes PC(\mathbb{T}))$, which in turn implies that the C^*-algebra is irreducible and contains the ideal $\mathcal{K}$ of all compact operators on $\mathcal{A}^2(\mathbb{D})$. Furthermore, by the commutator properties, given in Section 2.1, the quotient algebra $\widehat{\mathcal{T}}=\mathcal{T}/\mathcal{K}$ is commutative.

To describe $\widehat{\mathcal{T}} = \mathcal{T}/\mathcal{K}$ we will start by describing the Calkin algebra of the C^*-algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))$ generated by all Toeplitz operators whose symbols are of the form $a(r)b(t)$ with $a(r)\in \mathrm{SO}[0,1)$, $b(t)\in C(\mathbb{T})$. To do so, we will use the Douglas-Varela local principle ([3, 8]) with $\mathcal{T}(1\otimes C(\mathbb{T}))/\mathcal{K}$ as a central commutative subalgebra of $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))/\mathcal{K}$. Once we have done the description of $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))/\mathcal{K}$, we will use this algebra as a central commutative subalgebra of $\widehat{\mathcal{T}} = \mathcal{T}/\mathcal{K}$.

5.1 Algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))/\mathcal{K}$

This section is devoted to the description of the Calkin algebra of the C^*-algebra generated by all Toeplitz operators whose symbols are of the form $a(r)b(t)$, where $a(r)\in \mathrm{SO}[0,1)$, $b(t)\in C(\mathbb{T})$. The main tool we use here is the Douglas-Varela local principle, and we use $\mathcal{T}(1\otimes C(\mathbb{T}))/\mathcal{K}$ as a central commutative subalgebra of $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))/\mathcal{K}$. It is well known that $\mathcal{T}(1\otimes C(\mathbb{T}))/\mathcal{K}$ is isomorphic to the algebra $C(\mathbb{T})$. Thus, its maximal ideal space is the set $\mathbb{T}$, with the usual topology.

Theorem 5.1. *Let t_0 be a point in $\mathbb{T}$. Then, the local algebra of $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))/\mathcal{K}$ at t_0 is isomorphic to $C(M_1(\mathrm{SO}[0,1)))$. The isomorphism is given by the following transforms of the generators*

$$T_{ab}+\mathcal{K}\mapsto b(t_0)\widehat{a},$$

where $\widehat{a}$ denotes the Gelfand transform of a.

Proof. It is easy to see that two operators T_{a_1} and T_{a_2}, with $a_1, a_2\in \mathrm{SO}$ are locally equivalent, at the point t_0, if and only if $T_{a_1-a_2}\in\mathcal{K}$, or if and only if $\widehat{a}_1(\varphi)=\widehat{a}_2(\varphi)$ for all $\varphi\in M_1(\mathrm{SO}[0,1))$. On the other hand, T_b is locally equivalent, at the point t_0, to $b(t_0)I$. These facts imply that the local algebra, at t_0, is isomorphic to the C^*-algebra generated by all $T_a+\mathcal{K}$, where $a\in\mathrm{SO}$. From (4.1) the last algebra is isomorphic to $C(M_1(\mathrm{SO}[0,1)))$. □

Corollary 5.2. *The Calkin algebra of the C^*-algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))$ is isomorphic and isometric to $C(M_1(\mathrm{SO}[0,1))\times\mathbb{T})$. The isomorphism is given by the following mapping of the generators*

$$T_{ab}+\mathcal{K}\mapsto b(t)\widehat{a}(\varphi),\quad (\varphi,t)\in M_1(\mathrm{SO}[0,1))\times\mathbb{T}.$$

5.2 Description of the algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes PC(\mathbb{T}))/\mathcal{K}$

In this section we give the description of the Calkin algebra of the C^*-algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes PC(\mathbb{T}))$. We use the Douglas-Varela local principle using as a central commutative algebra the Calkin algebra of the C^*-algebra $\mathcal{T}(\mathrm{SO}[0,1)\otimes C(\mathbb{T}))$. By Corollary 5.2 the maximal ideal space of the last algebra is $M_1(\mathrm{SO}[0,1))\times\mathbb{T}$. Therefore, we localize by the points of $M_1(\mathrm{SO}[0,1))\times\mathbb{T}$.

Theorem 5.3. *The Calkin algebra of the C^*-algebra generated by Toeplitz operators with symbols in $\mathrm{SO}[0,1)\otimes PC(\mathbb{T})$ is isomorphic and isometric to the algebra of all continuous functions on the compact set $M_1(\mathrm{SO}[0,1))\times(\mathbb{T}\times[0,1])$. The isomorphism is given by the following mapping of the generators*

$$T_{ab}+\mathcal{K}\longmapsto \widehat{a}(\varphi)[b(t^-)(1-x)+b(t^+)x],\quad \text{where } \varphi\in M_1(\mathrm{SO}[0,1)),\ (t,x)\in\mathbb{T}\times[0,1],$$

where $a(r) \in \mathrm{SO}[0,1)$, $b(t) \in C(\mathbb{T})$, $\widehat{a}$ *is the Gelfand transform of the function a, and the topology of the cylinder* $[0,1] \times \mathbb{T}$ *is as described in Section 3.*

Proof. Let (φ, t_0) be a point of the set $M_1(\mathrm{SO}[0,1)) \times \mathbb{T}$. Then, the operator T_{ab} is locally equivalent, at the point (φ, t_0) to $\widehat{a}(\varphi)[b(t_0^+)B_{\mathbb{D}}P_+B_{\mathbb{D}} + b(t_0^-)B_{\mathbb{D}}(I-P_+)B_{\mathbb{D}}]$, where P_+ and P_- are defined in Section 3. The result then follows from Theorem 3.3. □

Acknowledgments

The authors thank the referees for their careful reading of the manuscript and insightful comments. The work was partially supported by PROMEP (México) via ”Proyecto de Redes” and by CONACYT project 180049.

References

[1] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, second edition, Springer Verlag, Berlin 2006.

[2] L. A. Coburn, Singular Integral Operators and Toeplitz Operators on Odd Spheres. *Indiana Univ. Math. J.* **23** (1973), no. 5, pp 433-439.

[3] R. Douglas, *Banach Algebra Techniques in Operator Theory*, Springer Verlag, 1998.

[4] I. Gohberg and N. Krupnik, On the algebra generated by the Toeplitz matrices. *Funct. Anal. Appl.* **3** (1969), pp 119-127.

[5] S. Grudsky and N.L. Vasilevski, Bergman-Toeplitz Operators: Radial Component Influence. *Integ. Equ. Oper. Theory* **40** (2001), no. 1, pp 16-33.

[6] M. Loaiza, On an algebra of Toeplitz operators with piecewise continuous symbols. *Integr. Equ. Oper. Theory* **51** (2005), no.1, pp 141-153.

[7] D. Sarason, Toeplitz operators with piecewise quasicontinuous symbols. *Indiana University Mathematics Journal* **26** (1977), no. 5, pp 817-838.

[8] J. Varela, Duality of C^*-algebras. In: *Recent advances in the representation theory of rings and* C^**-algebras by continuous sections (Sem., Tulane Univ., New Orleans, La., 1973)*, Mem. Amer. Math. Soc., no. 148, Amer. Math. Soc., Providence, R.I., 1974, pp 97-108.

[9] N. L. Vasilevski, Banach algebras produced by two-dimensional integral operators with a Bergman kernel and piecewise continuous coefficients II. *Investiya VUZ, matematika* **30** (1986), no. 3, pp 33-38.

[10] N. L. Vasilevski, On Toeplitz operators with piecewise continuous symbols on the Bergman space. In: *Modern Operator Theory and Applications*, Operator Theory: Advances and Applications **170** (2007), pp 229-248.

[11] K. Zhu, VMO, ESV, and Toeplitz operators on the Bergman space. *Transactions of the American Mathematical Society* **302**, (1987), no. 2, pp 617-646.