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On Regularization of Mellin PDO's with Slowly Oscillating Symbols of Limited Smoothness

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Abstract

We study Mellin pseudodifferential operators (shortly, Mellin PDO's) with symbols in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating functions of limited smoothness introduced in [12]. We show that if $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ does not degenerate on the "boundary" of $\mathbb{R}_+ \times \mathbb{R}$ in a certain sense, then the Mellin PDO $\mathrm{Op}(\mathfrak{a})$ is Fredholm on the space L^p for $p \in (1,\infty)$ and each its regularizer is of the form $\mathrm{Op}(\mathfrak{b}) + K$ where K is a compact operator on L^p and $\mathfrak{b}$ is a certain explicitly constructed function in the same algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that $\mathfrak{b} = 1/\mathfrak{a}$ on the "boundary" of $\mathbb{R}_+ \times \mathbb{R}$. This result complements the known Fredholm criterion from [12] for Mellin PDO's with symbols in the closure of $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and extends the corresponding result by V.S. Rabinovich (see [16]) on Mellin PDO's with slowly oscillating symbols in $C^\infty(\mathbb{R}_+ \times \mathbb{R})$.

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1 Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X, and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\operatorname{Ind} A := \dim\ker A - \dim\ker A^*$$

is referred to as the *index* of A (see, e.g., [1, Sections 1.11–1.12], [3, Chap. 4]). For bounded linear operators A and B, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$.

Recall that an operator $B_r \in \mathcal{B}(X)$ (resp. $B_l \in \mathcal{B}(X)$) is said to be a right (resp. left) regularizer for A if

$$AB_r \simeq I \quad (\text{resp.} \quad B_l A \simeq I).$$

It is well known that the operator A is Fredholm on X if and only if it admits simultaneously a right and a left regularizers. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [3, Chap. 4, Section 7]). Therefore we may speak of a regularizer $B = B_r = B_l$ of A and two different regularizers of A differ from each other by a compact operator.

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathcal{M} : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}), \quad (\mathcal{M}f)(x) := \int_{\mathbb{R}_+} f(t) t^{-ix} \frac{dt}{t}.$$

It is an invertible operator, with inverse given by

$$\mathcal{M}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu), \quad (\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x) t^{ix}\, dx.$$

For $1 < p < \infty$, let $\mathcal{M}_p$ denote the Banach algebra of all Mellin multipliers, that is, the set of all functions $a \in L^\infty(\mathbb{R})$ such that $\mathcal{M}^{-1} a \mathcal{M} f \in L^p(\mathbb{R}_+, d\mu)$ and

$$\|\mathcal{M}^{-1} a \mathcal{M} f\|_{L^p(\mathbb{R}_+, d\mu)} \le c_p \|f\|_{L^p(\mathbb{R}_+, d\mu)} \quad \text{for all} \quad f \in L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu).$$

If $a \in \mathcal{M}_p$, then the operator $f \mapsto \mathcal{M}^{-1} a \mathcal{M} f$ defined initially on $L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu)$ extends to a bounded operator on $L^p(\mathbb{R}_+, d\mu)$. This operator is called the Mellin convolution operator with symbol a.

Mellin pseudodifferential operators are generalizations of Mellin convolution operators. Let $\mathfrak{a}$ be a sufficiently smooth function defined on $\mathbb{R}_+ \times \mathbb{R}$. The Mellin pseudodifferential operator (shortly, Mellin PDO) with symbol $\mathfrak{a}$ is initially defined for smooth functions f of compact support by the iterated integral

$$[\operatorname{Op}(\mathfrak{a}) f](t) = [\mathcal{M}^{-1} \mathfrak{a}(t, \cdot) \mathcal{M} f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathfrak{a}(t, x) \left(\frac{t}{\tau}\right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+.$$

In 1991 Rabinovich [14] proposed to use Mellin pseudodifferential operators techniques to study singular integral operators on slowly oscillating Carleson curves. This idea was exploited in a series of papers by Rabinovich and coauthors (see, e.g., [15, 16] and [17, Sections 4.5–4.6] and the references therein). Rabinovich stated in [16, Theorem 2.6] a Fredholm criterion for Mellin PDO's with C^∞ slowly oscillating (or slowly varying) symbols on the spaces $L^p(\mathbb{R}_+, d\mu)$ for $1 < p < \infty$. Namely, he considered symbols $\mathfrak{a} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} \left|(t\partial_t)^j \partial_x^k \mathfrak{a}(t,x)\right|(1+x^2)^{k/2} < \infty \quad \text{for all} \quad j,k \in \mathbb{Z}_+ \tag{1.1}$$

and

$$\lim_{t\to s}\sup_{x\in\mathbb{R}} \left|(t\partial_t)^j \partial_x^k \mathfrak{a}(t,x)\right|(1+x^2)^{k/2} = 0 \quad \text{for all} \quad j\in\mathbb{N}, \quad k\in\mathbb{Z}_+, \quad s\in\{0,\infty\}. \tag{1.2}$$

Here and in what follows ∂_t and ∂_x denote the operators of partial differentiation with respect to t and to x. Notice that (1.1) defines nothing but the Mellin version of the Hörmander class $S^0_{1,0}(\mathbb{R})$ (see, e.g., [6], [13, Chap. 2, Section 1] for the definition of the Hörmander classes $S^m_{\varrho,\delta}(\mathbb{R}^n)$). If $\mathfrak{a}$ satisfies (1.1), then the Mellin PDO $\mathrm{Op}(\mathfrak{a})$ is bounded on the spaces $L^p(\mathbb{R}_+, d\mu)$ for $1 < p < \infty$ (see, e.g., [21, Chap. VI, Proposition 4] for the corresponding Fourier PDO's). Condition (1.2) is the Mellin version of Grushin's definition of slowly varying symbols in the first variable (see, e.g., [4], [13, Chap. 3, Defintion 5.11]).

The above mentioned results have a disadvantage that the smoothness conditions imposed on slowly oscillating symbols are very strong. In this paper we will use a much weaker notion of slow oscillation, which goes back to Sarason [19]. A bounded continuous function f on $\mathbb{R}_+ = (0,\infty)$ is called slowly oscillating at 0 and ∞ if

$$\lim_{r\to s}\max_{t,\tau\in[r,2r]} |f(t) - f(\tau)| = 0 \quad \text{for} \quad s\in\{0,\infty\}.$$

This definition can be extended to the case of bounded continuous functions on $\mathbb{R}_+$ with values in a Banach space X.

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$. For a unital commutative Banach algebra $\mathfrak{A}$, let $M(\mathfrak{A})$ denote its maximal ideal space. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \{\xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0,\infty\}$. By [12, Proposition 2.1], the set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $(\mathrm{clos}_{SO^*}\mathbb{R}_+) \setminus \mathbb{R}_+$ where $\mathrm{clos}_{SO^*}\mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$.

The second author [10] developed a Fredholm theory for Fourier pseudodifferential operators with slowly oscillating $V(\mathbb{R})$-valued symbols where $V(\mathbb{R})$ is the Banach algebra of absolutely continuous functions of bounded total variation on $\mathbb{R}$. Those results were translated to the Mellin setting in [12]. In particular, the important algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating $V(\mathbb{R})$-valued functions was introduced and a Fredholm criterion for Mellin PDO's with symbols in the closure of $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ in the norm of the Banach algebra

$C_b(\mathbb{R}_+, C_p(\mathbb{R}))$ of bounded continuous $C_p(\mathbb{R})$-valued functions was obtained on the space $L^p(\mathbb{R}, d\mu)$ for all $p \in (1,\infty)$ [12, Theorem 4.3]. Here $C_p(\mathbb{R})$ is the smallest closed subalgebra of the algebra $\mathcal{M}_p(\mathbb{R})$ that contains the algebra $V(\mathbb{R})$. We refer, e.g., to [1, Sections 9.1–9.7], [2, Chap. 1], [5, Section 2.1], [18, Section 4.2], and [20] for properties of the algebras $V(\mathbb{R})$, $C_p(\mathbb{R})$, and $\mathcal{M}_p(\mathbb{R})$.

For symbols in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ the above mentioned Fredholm criterion has a simpler form [8, Theorem 3.6]. That result was already used in [7] (see also [8]) to prove that the simplest weighted singular integral operator with two shifts

$$U_\alpha P_\gamma^+ + U_\beta P_\gamma^- \tag{1.3}$$

is Fredholm of index zero on the space $L^p(\mathbb{R}_+)$ with $p \in (1,\infty)$, where $\alpha,\beta : \mathbb{R}_+ \to \mathbb{R}_+$ are orientation preserving diffeomorphisms with the only fixed points 0 and ∞ such that $\log\alpha', \log\beta'$ are bounded, $\alpha', \beta' \in SO(\mathbb{R}_+)$,

$$U_\alpha f = (\alpha')^{1/p}(f \circ \alpha), \quad U_\beta f = (\beta')^{1/p}(f \circ \beta), \quad P_\gamma^\pm := (I \pm S_\gamma)/2,$$

and S_γ is the weighted Cauchy singular integral operator given by

$$(S_\gamma f)(t) := \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t}d\tau$$

with $\gamma \in \mathbb{R}$ satisfying $0 < 1/p + \gamma < 1$ (for $\gamma = 0$ this result was obtained in [8]). To study more general operators than (1.3) in the forthcoming paper [9], we need not only a Fredholm criterion for $\mathrm{Op}(\mathfrak{a})$ with $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ given in [8, Theorem 3.6], but also an information on the regularizers of $\mathrm{Op}(\mathfrak{a})$. Note that a full description of the regularizers of a Fredholm Mellin PDO $\mathrm{Op}(\mathfrak{a})$ is available if $\mathfrak{a} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfies (1.1)–(1.2), see [16, Theorem 2.6]), however such a description is missing for the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

The aim of this paper is to fill in this gap and to complement the Fredholm criterion for Mellin PDO's with symbols in $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Here we provide an explicit description of all regularizers of a Fredholm operator $\mathrm{Op}(\mathfrak{a})$ with $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. Namely, we prove that if $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ does not degenerate on the "boundary" of $\mathbb{R}_+ \times \mathbb{R}$ in a certain sense, then the Mellin PDO $\mathrm{Op}(\mathfrak{a})$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ for $p \in (1,\infty)$ and each its regularizer is of the form $\mathrm{Op}(\mathfrak{b}) + K$ where K is a compact operator on $L^p(\mathbb{R}_+, d\mu)$ and $\mathfrak{b}$ is a certain explicitly constructed function in the same algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that $\mathfrak{b} = 1/\mathfrak{a}$ on the "boundary" of $\mathbb{R}_+ \times \mathbb{R}$. By the "boundary" of $\mathbb{R}_+ \times \mathbb{R}$ we mean the set

$$(\mathbb{R}_+ \times \{\pm\infty\}) \cup (\Delta \times \overline{\mathbb{R}}). \tag{1.4}$$

The paper is organized as follows. In Section 2 we define the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ of all bounded continuous $V(\mathbb{R})$-valued functions and state that if $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then $\mathrm{Op}(\mathfrak{a})$ is bounded on $L^p(\mathbb{R}_+, d\mu)$. In Section 3 we introduce the algebra $SO(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating $V(\mathbb{R})$-valued functions (a generalization of $SO(\mathbb{R}_+)$) and its subalgebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Further we explain how the values of a function $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ on the boundary (1.4) are defined and recall that

$$\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b}) \simeq \mathrm{Op}(\mathfrak{a}\mathfrak{b}) \quad \text{whenever} \quad \mathfrak{a}, \mathfrak{b} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})). \tag{1.5}$$

In Section 4 we define our main algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \subset \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and show that all algebras $C_b(\mathbb{R}_+, V(\mathbb{R}))$, $SO(\mathbb{R}_+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ are inverse closed in $C_b(\mathbb{R}_+ \times \mathbb{R})$, the algebra of all bounded continuous functions on $\mathbb{R}_+ \times \mathbb{R}$. Combining the inverse closedness of the algebras $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ (resp. $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$) with (1.5), we get a description of all regularizers for $\mathrm{Op}(\mathfrak{a})$ with $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ (resp. $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$) bounded away from zero on $\mathbb{R}_+ \times \mathbb{R}$. In Section 5 we show that the latter strong hypothesis can be essentially relaxed in the case of the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. We show that if $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ does not degenerate on the "boundary" (1.4), then there exists $\mathfrak{b} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that $\mathfrak{b} = 1/\mathfrak{a}$ on the "boundary" (1.4). This construction becomes possible for $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ because the limiting values of $\mathfrak{a}(t, \cdot)$ on Δ are attained uniformly in the norm of $V(\mathbb{R})$ (see Lemma 5.2). Finally we recall that if $\mathfrak{c} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, then $\mathrm{Op}(\mathfrak{c})$ is compact if and only if its symbol $\mathfrak{c}$ degenerates on the "boundary" (1.4). Combining this result with our construction, we arrive at the main result of the paper.

2 Algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$ and Boundedness of Mellin PDO's

2.1 Definition of the Algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$

Let a be an absolutely continuous function of finite total variation

$$V(a) := \int_{\mathbb{R}} |a'(x)| dx$$

on $\mathbb{R}$. The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on $\mathbb{R}$ becomes a Banach algebra equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a). \tag{2.1}$$

Following [10, 11], let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm

$$\|\mathfrak{a}(\cdot, \cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} = \sup_{t \in \mathbb{R}_+} \|\mathfrak{a}(t, \cdot)\|_V.$$

2.2 Boundedness of Mellin PDO's

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

The following boundedness result for Mellin pseudodifferential operators can be extracted from [11, Theorem 6.1] (see also [10, Theorem 3.1]).

Theorem 2.1. *If* $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, *then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$, *defined for functions* $f \in C_0^\infty(\mathbb{R}_+)$ *by the iterated integral*

$$[\mathrm{Op}(\mathfrak{a}) f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathfrak{a}(t, x) \left(\frac{t}{\tau}\right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+,$$

extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ *and there is a positive constant* C_p *depending only on* p *such that*

$$\|\mathrm{Op}(\mathfrak{a})\|_{\mathcal{B}(L^p(\mathbb{R}_+, d\mu))} \le C_p \|\mathfrak{a}\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}.$$

3 Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and Compactness of Semi-Commutators of Mellin PDO's

3.1 Definitions of the Algebras $SO(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$

Let $SO(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ on $\mathbb{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{r\to 0} \mathrm{cm}_r^C(\mathfrak{a}) = \lim_{r\to\infty} \mathrm{cm}_r^C(\mathfrak{a}) = 0,$$

where

$$\mathrm{cm}_r^C(\mathfrak{a}) := \max\{\|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\|_{L^\infty(\mathbb{R})} : t,\tau \in [r,2r]\}. \tag{3.1}$$

Let $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions $\mathfrak{a} \in SO(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$\lim_{|h|\to 0} \sup_{t\in\mathbb{R}_+} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}^h(t,\cdot)\|_V = 0 \tag{3.2}$$

where $\mathfrak{a}^h(t,x) := \mathfrak{a}(t,x+h)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

Remark 3.1. Replacing the $L^\infty(\mathbb{R})$ norm in (3.1) by the stronger $V(\mathbb{R})$ norm, one can define smaller algebras $SO^V(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}^V(\mathbb{R}_+, V(\mathbb{R})) \subset SO^V(\mathbb{R}_+, V(\mathbb{R}))$ instead of the algebras $SO(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, respectively. This was done in [12, p. 86], where the algebras $SO^V(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}^V(\mathbb{R}_+, V(\mathbb{R}))$ were denoted, respectively, by the same symbols $SO(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ (see also Remark 4.1 below).

3.2 Limiting Values of Functions in the Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$

Let $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $\mathfrak{a}(t,\cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm\infty$, which will be denoted by $\mathfrak{a}(t,\pm\infty)$. Now we explain how to extend the function $\mathfrak{a}$ to $\Delta \times \overline{\mathbb{R}}$. By analogy with [10, Lemma 2.7] one can prove the following.

Lemma 3.2. *Let $s \in \{0,\infty\}$ and $\{\mathfrak{a}_k\}_{k=1}^\infty$ be a countable subset of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ and functions $\mathfrak{a}_k(\xi,\cdot) \in V(\mathbb{R})$ such that $t_j \to s$ as $j \to \infty$ and*

$$\mathfrak{a}_k(\xi,x) = \lim_{j\to\infty} \mathfrak{a}_k(t_j,x)$$

for every $x \in \overline{\mathbb{R}}$ and every $k \in \mathbb{N}$.

The following lemma will be of some importance in applications we have in mind [9] (although it will not be used in the current paper).

Lemma 3.3. *Let $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that the series $\sum_{n=1}^\infty \mathfrak{a}_n$ converges in the norm of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ to a function $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Then*

$$\mathfrak{a}(t,\pm\infty) = \sum_{n=1}^\infty \mathfrak{a}_n(t,\pm\infty) \ \textit{for all}\ \ t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi,x) = \sum_{n=1}^\infty \mathfrak{a}_n(\xi,x) \ \textit{for all}\ \ (\xi,x) \in \Delta \times \mathbb{R}. \tag{3.3}$$

Proof. Fix $\varepsilon > 0$. For $N \in \mathbb{N}$, put

$$\mathfrak{s}_N := \sum_{n=1}^{N} \mathfrak{a}_n.$$

By the hypothesis, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\mathfrak{a}(t,x) - \mathfrak{s}_N(t,x)| \le \|\mathfrak{a} - \mathfrak{s}_N\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} < \varepsilon/3. \tag{3.4}$$

Fix some $t \in \mathbb{R}_+$. For every $N > N_0$ there exists $x(t,N) \in \mathbb{R}_+$ such that for all $x \in (x(t,N),+\infty)$,

$$|\mathfrak{a}(t,+\infty) - \mathfrak{a}(t,x)| < \varepsilon/3, \quad |\mathfrak{s}_N(t,+\infty) - \mathfrak{s}_N(t,x)| < \varepsilon/3. \tag{3.5}$$

From (3.4) and (3.5) it follows that for every $N > N_0$ and $x \in (x(t,N),+\infty)$,

$$|\mathfrak{a}(t,+\infty) - \mathfrak{s}_N(t,+\infty)| \le |\mathfrak{a}(t,+\infty) - \mathfrak{a}(t,x)| + |\mathfrak{a}(t,x) - \mathfrak{s}_N(t,x)| + |\mathfrak{s}_N(t,x) - \mathfrak{s}_N(t,+\infty)| < \varepsilon.$$

This implies the first equality in (3.3) for the sign "+". The proof for the sign "−" is analogous.

Fix $s \in \{0,\infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$. In view of Lemma 3.2, there exists a sequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_j \to s$ as $j \to \infty$ and functions $\mathfrak{a}(\xi,\cdot) \in V(\mathbb{R}_+)$ and $\mathfrak{s}_N(\xi,\cdot) \in V(\mathbb{R}_+)$, $N \in \mathbb{N}$, such that

$$\mathfrak{a}(\xi,x) = \lim_{j\to\infty} \mathfrak{a}(t_j,x), \quad \mathfrak{s}_N(\xi,x) = \lim_{j\to\infty} \mathfrak{s}_N(t_j,x)$$

for all $x \in \overline{\mathbb{R}}$ and all $N \in \mathbb{N}$.

Fix $x \in \mathbb{R}$. For every $N > N_0$ there exists $j_0(x,N) \in \mathbb{N}$ such that for $j > j_0(x,N)$,

$$|\mathfrak{a}(\xi,x) - \mathfrak{a}(t_j,x)| < \varepsilon/3, \quad |\mathfrak{s}_N(\xi,x) - \mathfrak{s}_N(t_j,x)| < \varepsilon/3. \tag{3.6}$$

From (3.4) and (3.6) we obtain that for $N > N_0$ and $j > j_0(x,N)$,

$$|\mathfrak{a}(\xi,x) - \mathfrak{s}_N(\xi,x)| \le |\mathfrak{a}(\xi,x) - \mathfrak{a}(t_j,x)| + |\mathfrak{a}(t_j,x) - \mathfrak{s}_N(t_j,x)| + |\mathfrak{s}_N(t_j,x) - \mathfrak{s}_N(\xi,x)| < \varepsilon,$$

which concludes the proof of the second equality in (3.3). □

3.3 Compactness of Semi-Commutators of Mellin PDO's

Let E be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \tag{3.7}$$

Applying the relation

$$\mathrm{Op}(\mathfrak{a}) = E^{-1} a(x,D) E \tag{3.8}$$

between the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ and the Fourier pseudodifferential operator $a(x,D)$ considered in [10], where

$$\mathfrak{a}(t,x) = a(\ln t, x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.9}$$

and taking into account the fact that $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ if and only if $a \in \mathcal{E}$, where the algebra $\mathcal{E}$ is defined on p. 719 of [10], we infer from [10, Theorem 8.3] the following compactness result.

Theorem 3.4. *If* $\mathfrak{a}, \mathfrak{b} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$*, then* $\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b}) \simeq \mathrm{Op}(\mathfrak{a}\mathfrak{b})$.

4 Regularization of Mellin PDO's with Symbols Globally Bounded Away from Zero

4.1 Definition of the Algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

We denote by $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ the Banach algebra consisting of all functions $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ that satisfy the condition

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} |\partial_x \mathfrak{a}(t,x)|\,dx = 0. \tag{4.1}$$

This algebra plays a crucial role in the paper.

Remark 4.1. Analogously to Remark 3.1, replacing the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ by the smaller algebra $\mathcal{E}^V(\mathbb{R}_+, V(\mathbb{R}))$ in the above definition, one can define the algebra $\widetilde{\mathcal{E}}^V(\mathbb{R}_+, V(\mathbb{R})) \subset \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. But, actually, the algebras $\widetilde{\mathcal{E}}^V(\mathbb{R}_+, V(\mathbb{R}))$ and $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ coincide, which follows from [10, formula (2.34) and Theorem 2.8] with $\mathbb{R}_+$ in place of $\mathbb{R}$. Thus, both definitions of $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, given here and by formula (3.4) in [12, p. 86], are equivalent.

4.2 Inverse Closedness of the Algebras $C_b(\mathbb{R}_+, V(\mathbb{R}))$, $SO(\mathbb{R}_+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ in the Algebra $C_b(\mathbb{R}_+ \times \mathbb{R})$

Let $\mathfrak{B}$ be a unital Banach algebra and $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$, which contains the identity element of $\mathfrak{B}$. The algebra $\mathfrak{A}$ is said to be inverse closed in the algebra $\mathfrak{B}$ if every element $a \in \mathfrak{A}$, invertible in $\mathfrak{B}$, is invertible in $\mathfrak{A}$ as well.

Lemma 4.2. *The algebras $C_b(\mathbb{R}_+, V(\mathbb{R}))$, $SO(\mathbb{R}_+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ are inverse closed in the Banach algebra $C_b(\mathbb{R}_+ \times \mathbb{R})$ of all bounded continuous functions on the half-plane $\mathbb{R}_+ \times \mathbb{R}$.*

Proof. The proof is developed by analogy with [10, pp. 755–756]. Let $\mathfrak{a} \in C_b(\mathbb{R}_+, V(\mathbb{R}))$ be invertible in $C_b(\mathbb{R}_+ \times \mathbb{R})$. Then

$$\|\mathfrak{a}^{-1}\|_{C_b(\mathbb{R}_+\times\mathbb{R})} = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\mathfrak{a}^{-1}(t,x)| = \left(\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\mathfrak{a}(t,x)|\right)^{-1} < \infty.$$

Therefore, for every $t \in \mathbb{R}_+$,

$$\begin{aligned} \|\mathfrak{a}^{-1}(t,\cdot)\|_V &= \|\mathfrak{a}^{-1}(t,\cdot)\|_{L^\infty(\mathbb{R}_+)} + V(\mathfrak{a}^{-1}(t,\cdot)) = \sup_{x\in\mathbb{R}} \left|\frac{\mathfrak{a}(t,x)}{\mathfrak{a}^2(t,x)}\right| + \int_{\mathbb{R}} \left|\frac{\partial_x \mathfrak{a}(t,x)}{\mathfrak{a}^2(t,x)}\right| dx \\ &\le \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \big(\|\mathfrak{a}(t,\cdot)\|_{L^\infty(\mathbb{R})} + V(\mathfrak{a}(t,\cdot))\big) = \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \|\mathfrak{a}(t,\cdot)\|_V. \end{aligned} \tag{4.2}$$

Hence

$$\|\mathfrak{a}^{-1}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \le \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \|\mathfrak{a}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \tag{4.3}$$

and for every $t, \tau \in \mathbb{R}_+$,

$$\begin{aligned} \|\mathfrak{a}^{-1}(t,\cdot) - \mathfrak{a}^{-1}(\tau,\cdot)\|_V &\le \|\mathfrak{a}^{-1}(t,\cdot)\|_V \|\mathfrak{a}^{-1}(\tau,\cdot)\|_V \|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\|_V \\ &\le \|\mathfrak{a}^{-1}\|^4_{C_b(\mathbb{R}_+\times\mathbb{R})} \|\mathfrak{a}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\|_V. \end{aligned} \tag{4.4}$$

From inequalities (4.3)–(4.4) it follows that the function $\mathfrak{a}^{-1}$ is a bounded and continuous $V(\mathbb{R})$-valued function. Thus, $C_b(\mathbb{R}_+, V(\mathbb{R}))$ is inverse closed in $C_b(\mathbb{R}_+ \times \mathbb{R})$.

Suppose $\mathfrak{a} \in SO(\mathbb{R}_+, V(\mathbb{R}))$ is invertible in $C_b(\mathbb{R}_+ \times \mathbb{R})$. If $t, \tau \in \mathbb{R}_+$, then

$$\|\mathfrak{a}^{-1}(t,\cdot) - \mathfrak{a}^{-1}(\tau,\cdot)\|_{L^\infty(\mathbb{R})} \le \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}(\tau,\cdot)\|_{L^\infty(\mathbb{R})}. \tag{4.5}$$

Therefore

$$\mathrm{cm}_r^C(\mathfrak{a}^{-1}) \le \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \mathrm{cm}_r^C(\mathfrak{a}), \quad r \in \mathbb{R}_+.$$

From the above inequality we conclude that $\mathfrak{a}^{-1} \in SO(\mathbb{R}_+, V(\mathbb{R}))$. Thus, $SO(\mathbb{R}_+, V(\mathbb{R}))$ is inverse closed in $C_b(\mathbb{R}_+ \times \mathbb{R})$.

Let $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ be invertible in $C_b(\mathbb{R}_+ \times \mathbb{R})$. Taking into account inequality (4.2) and that the norm in $V(\mathbb{R})$ is translation-invariant, we get for $h \in \mathbb{R}$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}\|\mathfrak{a}^{-1}(t,\cdot) - (\mathfrak{a}^{-1})^h(t,\cdot)\|_V &\le \|\mathfrak{a}^{-1}(t,\cdot)\|_V \|(\mathfrak{a}^{-1})^h(t,\cdot)\|_V \|\mathfrak{a}(t,\cdot) - (\mathfrak{a})^h(t,\cdot)\|_V \\ &\le \|\mathfrak{a}^{-1}\|^4_{C_b(\mathbb{R}_+\times\mathbb{R})} \|\mathfrak{a}(\cdot,\cdot)\|^2_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}^h(t,\cdot)\|_V.\end{aligned} \tag{4.6}$$

From the above inequality and $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ it follows that

$$\lim_{|h|\to 0} \sup_{t\in\mathbb{R}_+} \|\mathfrak{a}^{-1}(t,\cdot) - (\mathfrak{a}^{-1})^h(t,\cdot)\|_V = 0.$$

This means that $\mathfrak{a}^{-1} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$, whence the proof of the inverse closedness of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ in the algebra $C_b(\mathbb{R}_+ \times \mathbb{R})$ is completed.

Finally, if $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is invertible in $C_b(\mathbb{R}_+ \times \mathbb{R})$, then

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} |\partial_x \mathfrak{a}^{-1}(t,x)|\,dx \le \|\mathfrak{a}^{-1}\|^2_{C_b(\mathbb{R}_+\times\mathbb{R})} \lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} |\partial_x \mathfrak{a}(t,x)|\,dx = 0.$$

Therefore, $\mathfrak{a}^{-1} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and thus the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ is inverse closed in the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$. □

4.3 First Result on the Regularization of Mellin PDO's

Lemma 4.3. *If* $\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ *(resp.* $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *is such that*

$$\inf_{(t,x)\in\mathbb{R}_+\times\mathbb{R}} |\mathfrak{a}(t,x)| > 0, \tag{4.7}$$

then the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ *is Fredholm on the space* $L^p(\mathbb{R}_+, d\mu)$ *and each its regularizer is of the form* $\mathrm{Op}(1/\mathfrak{a}) + K$ *where* K *is a compact operator on the space* $L^p(\mathbb{R}_+, d\mu)$ *and* $1/\mathfrak{a} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ *(resp.* $1/\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$*).*

Proof. If $\mathfrak{a}$ satisfies (4.7) and belongs to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ (resp. to $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$), then $1/\mathfrak{a}$ belongs to $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ (resp. to $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$) in view of Lemma 4.2. Then in both cases from Theorem 3.4 we obtain $\mathrm{Op}(\mathfrak{a})\mathrm{Op}(1/\mathfrak{a}) \simeq \mathrm{Op}(1) = I$ and $\mathrm{Op}(1/\mathfrak{a})\mathrm{Op}(\mathfrak{a}) \simeq \mathrm{Op}(1) = I$, which completes the proof. □

As it happens, the very strong hypothesis (4.7) can be essentially relaxed for Mellin PDO's with symbols in the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. This issue will be discussed in the next section.

5 Algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and Fredholmness of Mellin PDO's

5.1 Elementary Properties of Two Important Functions in $V(\mathbb{R})$

We prelude our main construction with properties of two important functions in $V(\mathbb{R})$.

Lemma 5.1. (a) *For $x \in \mathbb{R}$, put*

$$p_-(x) := (1 - \tanh(\pi x))/2, \quad p_+(x) := (1 + \tanh(\pi x))/2. \tag{5.1}$$

Then $\|p_-\|_V = \|p_+\|_V = 2$.

(b) *For every $h \in \mathbb{R}$, put $p_\pm^h(x) := p_\pm(x + h)$. Then*

$$\left\|p_\pm - p_\pm^h\right\|_V \le 5\pi|h|/2. \tag{5.2}$$

(c) *For every $m > 0$,*

$$\int_{\mathbb{R}\setminus[-m,m]} |(p_\pm)'(x)|\,dx < 2e^{-2\pi m}. \tag{5.3}$$

Proof. (a) Since the function p_+ (resp. p_-) is monotonically increasing (resp. decreasing), $p_\pm(\mp\infty) = 0$ and $p_\pm(\pm\infty) = 1$, we have $\|p_\pm\|_{L^\infty(\mathbb{R})} = 1$ and $V(p_\pm) = |p_\pm(+\infty) - p_\pm(-\infty)| = 1$. Thus $\|p_\pm\|_V = \|p_\pm\|_{L^\infty(\mathbb{R})} + V(p_\pm) = 2$. Part (a) is proved.

(b) From (5.1) it follows that

$$(p_\pm)'(x) = \pm\frac{\pi}{2\cosh^2(\pi x)}, \quad (p_\mp)''(x) = \mp\frac{\pi^2 \tanh(\pi x)}{\cosh^2(\pi x)}, \quad x \in \mathbb{R}. \tag{5.4}$$

Hence $|(p_\pm)'(x)| \le \pi/2$ for all $x \in \mathbb{R}$. From here, by the mean value theorem, we obtain

$$|p_\pm(\pi x) - p_\pm[\pi(x + h)]| \le \pi|h|/2, \quad x, h \in \mathbb{R},$$

whence

$$\|p_\pm - p_\pm^h\|_{L^\infty(\mathbb{R})} \le \pi|h|/2. \tag{5.5}$$

Taking into account identities (5.4), we obtain

$$|p_\pm''(x)| \le 2\pi p_+'(x), \quad x \in \mathbb{R}.$$

Then for $h \in \mathbb{R}$,

$$\begin{aligned} V(p_\pm - p_\pm^h) &= \int_{\mathbb{R}} |p_\pm'(x) - p_\pm'(x + h)|\,dx = \int_{\mathbb{R}} \left| \int_x^{x+h} p_\pm''(y)dy \right| dx \\ &\le \int_{\mathbb{R}} dx \int_x^{x+|h|} |p_\pm''(y)|\,dy \le 2\pi \int_{\mathbb{R}} dx \int_x^{x+|h|} p_+'(y)\,dy \\ &= 2\pi \int_{\mathbb{R}} p_+'(y)\,dy \int_{y-|h|}^{y} dx = 2\pi|h|\big(p_+(+\infty) - p_+(-\infty)\big) = 2\pi|h|. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we arrive at (5.2).

(c) From (5.1) it follows that for $m > 0$,

$$\int_{\mathbb{R}\setminus[-m,m]} |p_\pm'(x)|\,dx = \pi \int_m^{+\infty} \frac{dx}{\cosh^2(\pi x)} = 1 - \tanh(\pi m) = \frac{2}{e^{2\pi m} + 1} < 2e^{-2\pi m},$$

which completes the proof. □

5.2 Limiting Values of Elements of $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

For functions in the algebra $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, we have a stronger result than Lemma 3.2, which follows from [10, Lemma 2.9] with the aid of the diagonal process.

Lemma 5.2. *Let $s \in \{0,\infty\}$ and $\{\mathfrak{a}_k\}_{k=1}^{\infty}$ be a countable subset of the algebra $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ and functions $\mathfrak{a}_k(\xi,\cdot) \in V(\mathbb{R})$ such that $t_j \to s$ as $j \to \infty$ and*

$$\lim_{j\to\infty} \|\mathfrak{a}_k(t_j,\cdot) - \mathfrak{a}_k(\xi,\cdot)\|_V = 0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.7}$$

Conversely, every sequence $\{\tau_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\tau_j \to s$ as $j \to \infty$ contains a subsequence $\{t_j\}_{j\in\mathbb{N}}$ such that (5.7) holds for some $\xi \in M_s(SO(\mathbb{R}_+))$.

As usual, the maximal ideal space $M(SO(\mathbb{R}_+))$ is equipped with the Gelfand topology. Then, in view of [1, Section 1.24], the set Δ is a compact Haudorff subspace of $M(SO(\mathbb{R}_+))$. It is equipped with the induced topology. Finally, the compact Hausdorff space $\Delta \times \overline{\mathbb{R}}$ is equipped with the product topology generated by the topologies of Δ and $\overline{\mathbb{R}}$.

Lemma 5.3. *For every $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$, the function $(\xi, x) \mapsto \mathfrak{a}(\xi, x)$ is continuous on the compact Hausdorff space $\Delta \times \overline{\mathbb{R}}$.*

Proof. Fix $\varepsilon > 0$. It follows from (3.2) that there exists a $\delta > 0$ such that for all $h \in (-\delta, \delta)$,

$$\sup_{t\in\mathbb{R}_+} \sup_{x\in\mathbb{R}} |\mathfrak{a}(t,x) - \mathfrak{a}(t,x+h)| \le \sup_{t\in\mathbb{R}_+} \|\mathfrak{a}(t,\cdot) - \mathfrak{a}(t,\cdot+h)\|_V < \varepsilon/6.$$

Hence there is an $h \in (0,\infty)$ such that, for all $t \in \mathbb{R}_+$ and all $x, y \in \mathbb{R}$ with $|x-y| < h$,

$$|\mathfrak{a}(t,x) - \mathfrak{a}(t,y)| < \varepsilon/6. \tag{5.8}$$

By Lemma 5.2, for every $s \in \{0,\infty\}$ and $\xi \in M_s(SO(\mathbb{R}_+))$, there is a sequence $\{t_j\}_{j\in\mathbb{N}}$ and a function $\mathfrak{a}(\xi,\cdot) \in V(\mathbb{R}) \subset C(\overline{\mathbb{R}})$ such that $t_j \to s$ as $j \to \infty$ and

$$\lim_{j\to\infty} \sup_{x\in\overline{\mathbb{R}}} |\mathfrak{a}(t_j,x) - \mathfrak{a}(\xi,x)| \le \lim_{j\to\infty} \|\mathfrak{a}(t_j,\cdot) - \mathfrak{a}(\xi,\cdot)\|_V = 0. \tag{5.9}$$

From the above inequality it follows that there is a $J \in \mathbb{N}$ such that for all $j \ge J$,

$$|\mathfrak{a}(t_j,x) - \mathfrak{a}(\xi,x)| < \varepsilon/6, \quad |\mathfrak{a}(t_j,y) - \mathfrak{a}(\xi,y)| < \varepsilon/6.$$

Combining these inequalities with (5.8), we deduce for all $x, y \in \mathbb{R}$ satisfying $|x-y| < h$, all $j \ge J$, all $s \in \{0,\infty\}$, and all $\xi \in M_s(SO(\mathbb{R}_+))$ that

$$|\mathfrak{a}(\xi,x) - \mathfrak{a}(\xi,y)| \le |\mathfrak{a}(t_j,x) - \mathfrak{a}(\xi,x)| + |\mathfrak{a}(t_j,y) - \mathfrak{a}(\xi,y)| + |\mathfrak{a}(t_j,x) - \mathfrak{a}(t_j,y)| < \varepsilon/2.$$

Therefore, for all $x, y \in \mathbb{R}$ satisfying $|x-y| < h$ we have

$$\sup_{\xi\in\Delta} |\mathfrak{a}(\xi,x) - \mathfrak{a}(\xi,y)| \le \varepsilon/2. \tag{5.10}$$

Fix $\xi \in \Delta$. Since the function $\mathfrak{a}(\cdot, x)$ belongs to the algebra $SO(\mathbb{R}_+)$, there exists an open neighborhood $U_x(\xi) \subset \Delta$ of ξ such that

$$|\mathfrak{a}(\eta, x) - \mathfrak{a}(\xi, x)| < \varepsilon/2 \quad \text{for all} \quad \eta \in U_x(\xi). \tag{5.11}$$

Consequently, we infer from (5.10) and (5.11) that

$$|\mathfrak{a}(\eta, y) - \mathfrak{a}(\xi, x)| \le |\mathfrak{a}(\eta, y) - \mathfrak{a}(\eta, x)| + |\mathfrak{a}(\eta, x) - \mathfrak{a}(\xi, x)| < \varepsilon$$

for all $(\eta, y) \in U_x(\xi) \times (x - h, x + h)$, which means that the function $(\xi, x) \mapsto \mathfrak{a}(\xi, x)$ is continuous on $\Delta \times \mathbb{R}$.

It remains to show that actually the function $(\xi, x) \mapsto \mathfrak{a}(\xi, x)$ is continuous on $\Delta \times \overline{\mathbb{R}}$. By (4.1), for every $\varepsilon > 0$ there is an $M > 0$ such that

$$\sup_{t \in \mathbb{R}_+} |\mathfrak{a}(t, y) - \mathfrak{a}(t, +\infty)| \le \sup_{t \in \mathbb{R}_+} \int_M^\infty |\partial_x \mathfrak{a}(t, x)|\, dx < \varepsilon/6 \quad \text{for all} \quad y > M. \tag{5.12}$$

By Lemma 5.2, for every $s \in \{0, \infty\}$ and every $\xi \in M_s(SO(\mathbb{R}_+))$ there exist a sequence $\{t_j\}_{j \in \mathbb{N}}$ and a function $\mathfrak{a}(\xi, \cdot) \in V(\mathbb{R}) \subset C(\overline{\mathbb{R}})$ such that $t_j \to s$ as $j \to \infty$ and (5.9) is fulfilled. From (5.9) it follows that there is a $J \in \mathbb{N}$ such that for all $j \ge J$, all $s \in \{0, \infty\}$, and all $\xi \in M_s(SO(\mathbb{R}_+))$,

$$|\mathfrak{a}(\xi, y) - \mathfrak{a}(\xi, +\infty)| \le |\mathfrak{a}(t_j, y) - \mathfrak{a}(\xi, y)| + |\mathfrak{a}(t_j, +\infty) - \mathfrak{a}(\xi, +\infty)| + |\mathfrak{a}(t_j, y) - \mathfrak{a}(t_j, +\infty)| < \varepsilon/2.$$

Therefore, for all $y > M$ we have

$$\sup_{\xi \in \Delta} |\mathfrak{a}(\xi, y) - \mathfrak{a}(\xi, +\infty)| \le \varepsilon/2. \tag{5.13}$$

Fix $\xi \in \Delta$. Since the function $\mathfrak{a}(\cdot, +\infty)$ belongs to $SO(\mathbb{R}_+)$, there is an open neighborhood $U_{+\infty}(\xi) \subset \Delta$ of ξ such that

$$|\mathfrak{a}(\eta, +\infty) - \mathfrak{a}(\xi, +\infty)| < \varepsilon/2 \quad \text{for all} \quad \eta \in U_{+\infty}(\xi). \tag{5.14}$$

Then similarly to (5.11) we deduce from (5.13) and (5.14) that

$$|\mathfrak{a}(\eta, y) - \mathfrak{a}(\xi, +\infty)| \le |\mathfrak{a}(\eta, y) - \mathfrak{a}(\eta, +\infty)| + |\mathfrak{a}(\eta, +\infty) - \mathfrak{a}(\xi, +\infty)| < \varepsilon \tag{5.15}$$

for all $(\eta, y) \in U_{+\infty}(\xi) \times (M, +\infty]$.

Analogously, for every $\xi \in \Delta$ there exist an open neighborhood $U_{-\infty}(\xi) \subset \Delta$ of ξ and a number $M < 0$ such that

$$|\mathfrak{a}(\eta, y) - \mathfrak{a}(\xi, -\infty)| < \varepsilon \tag{5.16}$$

for all $(\eta, y) \in U_{-\infty}(\xi) \times [-\infty, M)$.

Finally, we conclude from (5.15)–(5.16) and the continuity of $(\xi, x) \mapsto \mathfrak{a}(\xi, x)$ on the set $\Delta \times \mathbb{R}$ that this function is continuous on the compact Hausdorff space $\Delta \times \overline{\mathbb{R}}$. □

5.3 Key Construction

In this subsection we show that if $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ does not degenerate on the "boundary" (1.4), then there exists $\mathfrak{b} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that $\mathfrak{b} = 1/\mathfrak{a}$ on the "boundary" (1.4).

Lemma 5.4. *If $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \text{ for all } t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi, x) \neq 0 \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{5.17}$$

then

$$A_\pm := \sup_{t \in \mathbb{R}_+} \frac{1}{|a(t, \pm\infty)|} < \infty \tag{5.18}$$

and there exists an $r > 1$ such that

$$A(r) := \sup_{(t,x) \in T_r \times \overline{\mathbb{R}}} \left| \frac{1}{\mathfrak{a}(t, x)} \right| < \infty \tag{5.19}$$

where $T_r := (0, r^{-1}] \cup [r, \infty)$.

Proof. By Lemma 5.3, the function $(\xi, x) \mapsto \mathfrak{a}(\xi, x)$ is continuous on the compact Hausdorff space $\Delta \times \overline{\mathbb{R}}$. Therefore, we infer from (5.17) that

$$C := \min\{|\mathfrak{a}(\xi, x)| : (\xi, x) \in \Delta \times \overline{\mathbb{R}}\} > 0. \tag{5.20}$$

For every point $(\xi, x) \in \Delta \times \overline{\mathbb{R}}$ we consider its open neighborhood $U_{\mathfrak{a},\xi,x} \subset M(SO(\mathbb{R}_+)) \times \overline{\mathbb{R}}$ such that

$$|\mathfrak{a}(\eta, y) - \mathfrak{a}(\xi, x)| < C/2 \quad \text{for every} \quad (\eta, y) \in U_{\mathfrak{a},\xi,x}. \tag{5.21}$$

We claim that there exists a number $r > 1$ such that

$$T_r \times \overline{\mathbb{R}} \subset \bigcup_{(\xi,x) \in \Delta \times \overline{\mathbb{R}}} U_{\mathfrak{a},\xi,x}. \tag{5.22}$$

Assume the contrary. Then for every $n \in \mathbb{N} \setminus \{1\}$ there exists a point $(\tau_n, x_n) \in T_n \times \overline{\mathbb{R}}$ such that

$$(\tau_n, x_n) \notin \left(\bigcup_{(\xi,x) \in M_0(SO(\mathbb{R}_+)) \times \overline{\mathbb{R}}} U_{\mathfrak{a},\xi,x} \right) \cup \left(\bigcup_{(\xi,x) \in M_\infty(SO(\mathbb{R}_+)) \times \overline{\mathbb{R}}} U_{\mathfrak{a},\xi,x} \right). \tag{5.23}$$

Since $\tau_n \in T_n = (0, 1/n] \cup [n, \infty)$ for all $n \geq 2$, we can extract a subsequence $\{\tau_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{\tau_n\}_{n \in \mathbb{N} \setminus \{1\}}$ such that

$$\lim_{k \to \infty} \tau_{n_k} = s \quad \text{for some} \quad s \in \{0, \infty\}. \tag{5.24}$$

Further, we can extract a subsequence $\left\{x_{n_{k_i}}\right\}_{i \in \mathbb{N}}$ of the corresponding sequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that the limit

$$x_0 := \lim_{i \to \infty} x_{n_{k_i}} \in \overline{\mathbb{R}} \tag{5.25}$$

exists. Then, by Lemma 5.2, there exists a subsequence $\{t_j\}_{j\in\mathbb{N}} = \left\{\tau_{n_{k_{i_j}}}\right\}_{j\in\mathbb{N}}$ of the sequence $\left\{\tau_{n_{k_i}}\right\}_{i\in\mathbb{N}}$ and a point $\xi_0 \in M_s(SO(\mathbb{R}_+))$ such that

$$\lim_{j\to\infty} \|\mathfrak{a}(t_j,\cdot) - \mathfrak{a}(\xi_0,\cdot)\|_V = 0. \tag{5.26}$$

Put $\{y_j\}_{j\in\mathbb{N}} = \left\{x_{n_{k_{i_j}}}\right\}_{j\in\mathbb{N}}$. Taking into account (5.23)–(5.26), we have shown that if (5.22) is violated for all $r > 1$, then there exist $s \in \{0,\infty\}$, $\xi_0 \in M_s(SO(\mathbb{R}_+))$, and a sequence $\{(t_j, y_j)\}_{j\in\mathbb{N}}$ such that (5.26) is fulfilled,

$$\{(t_j,y_j) : j\in\mathbb{N}\} \cap \left(\bigcup_{(\xi,x)\in M_s(SO(\mathbb{R}_+))\times\overline{\mathbb{R}}} U_{\mathfrak{a},\xi,x} \right) = \emptyset, \tag{5.27}$$

and

$$\lim_{j\to\infty} y_j = x_0 \in \overline{\mathbb{R}}, \quad \lim_{j\to\infty} t_j = s. \tag{5.28}$$

Since $(\xi_0, x_0) \in M_s(SO(\mathbb{R}_+)) \times \overline{\mathbb{R}} \subset \Delta \times \overline{\mathbb{R}}$, from Lemma 5.3 and the first equality in (5.28) we deduce that

$$\lim_{j\to\infty} |\mathfrak{a}(\xi_0, y_j) - \mathfrak{a}(\xi_0, x_0)| = 0. \tag{5.29}$$

For every $j \in \mathbb{N}$, we have

$$\begin{aligned} |\mathfrak{a}(t_j,y_j) - \mathfrak{a}(\xi_0,x_0)| &\le |\mathfrak{a}(t_j,y_j) - \mathfrak{a}(\xi_0,y_j)| + |\mathfrak{a}(\xi_0,y_j) - \mathfrak{a}(\xi_0,x_0)| \\ &\le \sup_{y\in\overline{\mathbb{R}}} |\mathfrak{a}(t_j,y) - \mathfrak{a}(\xi_0,y)| + |\mathfrak{a}(\xi_0,y_j) - \mathfrak{a}(\xi_0,x_0)| \\ &\le \|\mathfrak{a}(t_j,\cdot) - \mathfrak{a}(\xi_0,\cdot)\|_V + |\mathfrak{a}(\xi_0,y_j) - \mathfrak{a}(\xi_0,x_0)|. \end{aligned}$$

From (5.26), (5.29), and the above inequality we deduce that

$$\lim_{j\to\infty} \mathfrak{a}(t_j,y_j) = \mathfrak{a}(\xi_0,x_0).$$

This means that for all sufficiently large j the points (t_j, y_j) belong to the neighborhood $U_{\mathfrak{a},\xi_0,x_0}$ of the point $(\xi_0, x_0) \in M_s(SO(\mathbb{R}_+)) \times \overline{\mathbb{R}}$, which is impossible in view of (5.27). Hence, we arrive at the contradiction.

Thus, condition (5.22) is fulfilled for some $r > 1$. Therefore, in view of (5.20) and (5.21), we obtain

$$\inf_{(t,x)\in T_r\times\overline{\mathbb{R}}} |\mathfrak{a}(t,x)| > C/2 > 0.$$

This inequality immediately yields (5.19). Finally, (5.19) and the first condition in (5.17) imply (5.18). □

Lemma 5.5. *Suppose $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ satisfies* (5.17) *and $r > 1$ is a number such that* (5.19) *holds (the existence of this number is guaranteed by Lemma* 5.4*). Put*

$$\ell_\pm(t) := \frac{\ln r \pm \ln t}{2\ln r}, \quad c_\pm(t) := \frac{1}{\mathfrak{a}(t,\pm\infty)} - \frac{\ell_-(t)}{\mathfrak{a}(r^{-1},\pm\infty)} - \frac{\ell_+(t)}{\mathfrak{a}(r,\pm\infty)}, \quad t\in[r^{-1},r], \tag{5.30}$$

and consider the functions $p_\pm$ *given by* (5.1). *Then the function*

$$\mathfrak{b}(t,x) := \begin{cases} \dfrac{1}{\mathfrak{a}(t,x)}, & (t,x) \in (\mathbb{R}_+ \setminus [r^{-1}, r]) \times \overline{\mathbb{R}}, \\ \dfrac{\ell_-(t)}{\mathfrak{a}(r^{-1},x)} + \dfrac{\ell_+(t)}{\mathfrak{a}(r,x)} + c_-(t)p_-(x) + c_+(t)p_+(x), & (t,x) \in [r^{-1}, r] \times \overline{\mathbb{R}}, \end{cases} \tag{5.31}$$

is continuous on $\mathbb{R}_+ \times \overline{\mathbb{R}}$ *and is equal to* $1/\mathfrak{a}$ *on the set* $((\mathbb{R}_+ \setminus (r^{-1}, r)) \times \overline{\mathbb{R}}) \cup ((r^{-1}, r) \times \{\pm\infty\})$.

Proof. Since $\ell_\pm(r^{\mp 1}) = 0$ and $\ell_\pm(r^{\pm 1}) = 1$, we have $c_\pm(r) = c_\pm(r^{-1}) = 0$. Therefore

$$\mathfrak{b}(r^{\pm 1}, x) = 1/\mathfrak{a}(r^{\pm 1}, x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{5.32}$$

Taking into account that $p_\mp(\pm\infty) = 0$ and $p_\pm(\pm\infty) = 1$, we get from (5.30)–(5.31)

$$\mathfrak{b}(t, \pm\infty) = \frac{\ell_-(t)}{\mathfrak{a}(r^{-1}, \pm\infty)} + \frac{\ell_+(t)}{\mathfrak{a}(r, \pm\infty)} + c_\pm(t) = \frac{1}{\mathfrak{a}(t, \pm\infty)} \quad \text{for all} \quad t \in [r^{-1}, r]. \tag{5.33}$$

Thus, the assertion of the lemma follows from (5.32)–(5.33) and and the equality $\mathfrak{b}(t,x) = 1/\mathfrak{a}(t,x)$ for all $(t,x) \in (\mathbb{R}_+ \setminus [r^{-1}, r]) \times \overline{\mathbb{R}}$ (see (5.31)). □

Lemma 5.6. *Suppose* $a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *satisfies* (5.17) *and* $\mathfrak{b}$ *is the function defined by* (5.30)–(5.31) *with* $r > 1$ *such that* (5.19) *holds (the existence of this number is guaranteed by Lemma* 5.4*). Then* $\mathfrak{b} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *and*

$$\mathfrak{b}(t, \pm\infty) = 1/\mathfrak{a}(t, \pm\infty) \ \textit{for all} \ t \in \mathbb{R}_+, \quad \mathfrak{b}(\xi, x) = 1/\mathfrak{a}(\xi, x) \ \textit{for all} \ (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{5.34}$$

Proof. We divide the proof into five steps:

(a) First we prove that the function $\mathfrak{b}$ belongs to the algebra $C_b(\mathbb{R}_+, V(\mathbb{R}))$. Let

$$T_r := (0, r^{-1}] \cup [r, +\infty).$$

By Lemma 5.5,

$$\mathfrak{b}(t,x) = 1/\mathfrak{a}(t,x), \quad (t,x) \in T_r \times \overline{\mathbb{R}}. \tag{5.35}$$

Since $\mathfrak{a}(t,\cdot)$ belongs to $V(\mathbb{R})$ for all $t \in \mathbb{R}_+$, by analogy with (4.2), we infer from (5.19) that

$$\|\mathfrak{b}(t,\cdot)\|_V \le A^2(r) \sup_{t \in T_r} \|\mathfrak{a}(t,\cdot)\|_V, \quad t \in T_r. \tag{5.36}$$

From (5.18) and (5.30) it follows that

$$0 \le \ell_\pm(t) \le 1, \quad |c_\pm(t)| \le 3A_\pm, \quad t \in [r^{-1}, r]. \tag{5.37}$$

From (5.31), (5.35)–(5.37), and Lemma 5.1(a) it follows that for $t \in (r^{-1}, r)$,

$$\begin{aligned} \|\mathfrak{b}(t,\cdot)\|_V &\le \ell_-(t)\|\mathfrak{b}(r^{-1},\cdot)\|_V + \ell_+(t)\|\mathfrak{b}(r,\cdot)\|_V + |c_-(t)|\,\|p_-\|_V + |c_+(t)|\,\|p_+\|_V \\ &\le 2A^2(r) \sup_{t \in T_r} \|\mathfrak{a}(t,\cdot)\|_V + 6A_- + 6A_+. \end{aligned} \tag{5.38}$$

Combining (5.36) and (5.38), we arrive at

$$\|\mathfrak{b}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} = \sup_{t\in\mathbb{R}_+}\|\mathfrak{b}(t,\cdot)\|_V \le 2A^2(r)\sup_{t\in T_r}\|\mathfrak{a}(t,\cdot)\|_V + 6A_- + 6A_+ < +\infty. \tag{5.39}$$

From (5.19) and (5.35)–(5.36), by analogy with (4.4), we obtain for $t,\tau\in T_r$,

$$\begin{aligned}\|\mathfrak{b}(t,\cdot)-\mathfrak{b}(\tau,\cdot)\|_V &\le \|\mathfrak{b}(t,\cdot)\|_V\|\mathfrak{b}(\tau,\cdot)\|_V\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\|_V\\ &\le A^4(r)\left(\sup_{t\in T_r}\|\mathfrak{a}(t,\cdot)\|_V\right)^2\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\|_V.\end{aligned}$$

Since $\mathfrak{a}$ is a continuous $V(\mathbb{R})$-valued function, from the above inequality we conclude that $t\mapsto\mathfrak{b}(t,\cdot)$ is a continuous $V(\mathbb{R})$-valued function for $t\in T_r$.

Obviously, $\ell_\pm$ are continuous on $[r^{-1},r]$. Since $\mathfrak{a}$ is a continuous $V(\mathbb{R})$-valued function, taking into account (5.18), we also have for $t,\tau\in[r^{-1},r]$,

$$\left|\frac{1}{\mathfrak{a}(t,\pm\infty)}-\frac{1}{\mathfrak{a}(\tau,\pm\infty)}\right| = \frac{|\mathfrak{a}(t,\pm\infty)-\mathfrak{a}(\tau,\pm\infty)|}{|\mathfrak{a}(t,\pm\infty)|\,|\mathfrak{a}(\tau,\pm\infty)|} \le A_\pm^2\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\|_V.$$

From this inequality and the definitions of $c_\pm$ in (5.30) we see that the functions $c_\pm$ are continuous on $[r^{-1},r]$. Therefore, from the definition (5.31) we conclude that $t\mapsto\mathfrak{b}(t,\cdot)$ is a continuous $V(\mathbb{R})$-valued function on $[r^{-1},r]$. From the continuity of the $V(\mathbb{R})$-valued function $t\mapsto\mathfrak{b}(t,\cdot)$ on $\mathbb{R}_+$ and inequality (5.39) we conclude that $\mathfrak{b}\in C_b(\mathbb{R}_+,V(\mathbb{R}))$.

(b) Now we prove that $\mathfrak{b}\in SO(\mathbb{R}_+,V(\mathbb{R}))$. By analogy with (4.5), from (5.19) and (5.35) we obtain

$$\|\mathfrak{b}(t,\cdot)-\mathfrak{b}(\tau,\cdot)\|_{L^\infty(\mathbb{R})} \le A^2(r)\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\|_{L^\infty(\mathbb{R})},\quad t,\tau\in T_r.$$

Since $a\in SO(\mathbb{R}_+,V(\mathbb{R}))$, from this estimate we obtain

$$\lim_{\nu\to s}\mathrm{cm}_\nu^C(\mathfrak{b}) \le A^2(r)\lim_{\nu\to s}\mathrm{cm}_\nu^C(\mathfrak{a}) = 0,$$

which means that $\mathfrak{b}\in SO(\mathbb{R}_+,V(\mathbb{R}))$.

(c) On this step we show that $\mathfrak{b}\in\mathcal{E}(\mathbb{R}_+,V(\mathbb{R}))$. By analogy with (4.6), taking into account that the norm of $V(\mathbb{R})$ is translation-invariant, from (5.19) and (5.35)–(5.36) we get for $h\in\mathbb{R}$ and $t\in T_r$,

$$\begin{aligned}\|\mathfrak{b}(t,\cdot)-\mathfrak{b}^h(t,\cdot)\|_V &\le \|\mathfrak{b}(t,\cdot)\|_V\|\mathfrak{b}^h(t,\cdot)\|_V\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\|_V\\ &\le C(\mathfrak{a})\sup_{t\in\mathbb{R}_+}\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\|_V,\end{aligned} \tag{5.40}$$

where

$$C(\mathfrak{a}) := A^4(r)\left(\sup_{t\in T_r}\|\mathfrak{a}(t,\cdot)\|_V\right)^2.$$

On the other hand, from (5.31), (5.35), (5.37), (5.40), and Lemma 5.1(b) it follows that for $h\in\mathbb{R}$ and $t\in(r^{-1},r)$,

$$\begin{aligned}\|\mathfrak{b}(t,\cdot)-\mathfrak{b}^h(t,\cdot)\|_V \le\,&\ell_-(t)\|\mathfrak{b}(r^{-1},\cdot)-\mathfrak{b}^h(r^{-1},\cdot)\|_V+\ell_+(t)\|\mathfrak{b}(r,\cdot)-\mathfrak{b}^h(r,\cdot)\|_V\\ &+|c_-(t)|\,\|p_--p_-^h\|_V+|c_+(t)|\,\|p_+-p_+^h\|_V\\ \le\,&2C(\mathfrak{a})\sup_{t\in\mathbb{R}_+}\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\|_V+\frac{15\pi}{2}(A_-+A_+)|h|.\end{aligned} \tag{5.41}$$

Combining (5.40)–(5.41), we arrive at

$$\sup_{t\in\mathbb{R}_+}\|\mathfrak{b}(t,\cdot)-\mathfrak{b}^h(t,\cdot)\|_V \le 2C(\mathfrak{a})\sup_{t\in\mathbb{R}_+}\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\|_V+\frac{15\pi}{2}(A_-+A_+)|h|.$$

Since $\mathfrak{a}\in\mathcal{E}(\mathbb{R}_+,V(\mathbb{R}))$, the right-hand side of the above inequality tends to zero as $|h|\to 0$. Hence

$$\lim_{|h|\to 0}\sup_{t\in\mathbb{R}_+}\|\mathfrak{b}(t,\cdot)-\mathfrak{b}^h(t,\cdot)\|_V=0.$$

Thus, $b\in\mathcal{E}(\mathbb{R}_+,V(\mathbb{R}))$.

(d) Now we prove that $\mathfrak{b}\in\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$. From (5.35) we obtain

$$\partial_x\mathfrak{b}(t,x)=-\mathfrak{a}^{-2}(t,x)\partial_x\mathfrak{a}(t,x),\quad (t,x)\in T_r\times\mathbb{R}.$$

From this identity and (5.19) it follows that for all $m>0$ and $t\in T_r$,

$$\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(t,x)|\,dx\le A^2(r)\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{a}(t,x)|\,dx. \tag{5.42}$$

On the other hand, from (5.35), (5.37), (5.42), and Lemma 5.1(c) it follows that for all $t\in(r^{-1},r)$ and $m>0$,

$$\begin{aligned}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(t,x)|\,dx\le&\ell_-(t)\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(r^{-1},x)|\,dx+\ell_+(t)\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(r,x)|\,dx\\&+|c_-(t)|\int_{\mathbb{R}\setminus[-m,m]}|p'_-(x)|\,dx+|c_+(t)|\int_{\mathbb{R}\setminus[-m,m]}|p'_+(x)|\,dx\\\le&2A^2(r)\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{a}(t,x)|\,dx+6(A_-+A_+)e^{-2\pi m}.\end{aligned}\tag{5.43}$$

Combining (5.42)–(5.43), we obtain for $m>0$,

$$\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(t,x)|\,dx\le 2A^2(r)\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{a}(t,x)|\,dx+6(A_-+A_+)e^{-2\pi m}.$$

Since $\mathfrak{a}\in\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$, the right-hand side of the above inequality tends to zero as $m\to\infty$. This implies that

$$\lim_{m\to\infty}\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}|\partial_x\mathfrak{b}(t,x)|\,dx=0.$$

Thus, $\mathfrak{b}\in\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$.

(e) Finally, we prove (5.34). The first equality in (5.34) was proved in Lemma 5.5. Fix $s\in\{0,\infty\}$. Since $\mathfrak{a},\mathfrak{b}\in\widetilde{\mathcal{E}}(\mathbb{R}_+,V(\mathbb{R}))$, from Lemma 3.2 it follows that for each $\xi\in M_s(SO(\mathbb{R}_+))\subset\Delta$ there exists a sequence $\{t_j\}_{j\in\mathbb{N}}\subset\mathbb{R}_+$ and functions $\mathfrak{a}(\xi,\cdot),\mathfrak{b}(\xi,\cdot)\in V(\mathbb{R})$ such that $t_j\to s$ as $j\to\infty$ and

$$\mathfrak{a}(\xi,x)=\lim_{j\to\infty}\mathfrak{a}(t_j,x),\quad \mathfrak{b}(\xi,x)=\lim_{j\to\infty}\mathfrak{b}(t_j,x),\quad x\in\overline{\mathbb{R}}. \tag{5.44}$$

For all sufficiently large j, one has $t_j\in T_r$. Then from (5.35) we get $\mathfrak{b}(t_j,x)=1/\mathfrak{a}(t_j,x)$ for all sufficiently large j and all $x\in\overline{\mathbb{R}}$. From this equality and (5.44) we obtain the second equality in (5.34). □

5.4 Regularization of Mellin PDO's with Symbols in $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$

From [12, Theorem 4.1] we can extract the following.

Lemma 5.7. *If* $\mathfrak{c} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$*, then* $\mathrm{Op}(\mathfrak{c}) \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$ *if and only if*

$$\mathfrak{c}(t, \pm\infty) = 0 \text{ for all } t \in \mathbb{R}_+, \quad \mathfrak{c}(\xi, x) = 0 \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{5.45}$$

Now we are in a position to prove the main result of the paper.

Theorem 5.8. *Suppose* $\mathfrak{a} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

(a) *If the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$ *is Fredholm on the space* $L^p(\mathbb{R}_+, d\mu)$*, then*

$$\mathfrak{a}(t, \pm\infty) \neq 0 \text{ for all } t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi, x) \neq 0 \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{5.46}$$

(b) *If* (5.46) *holds, then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$ *is Fredholm on the space* $L^p(\mathbb{R}_+, d\mu)$ *and each its regularizer has the form* $\mathrm{Op}(\mathfrak{b}) + K$*, where* K *is a compact operator on the space* $L^p(\mathbb{R}_+, d\mu)$ *and* $\mathfrak{b} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ *is such that*

$$\mathfrak{b}(t, \pm\infty) = 1/\mathfrak{a}(t, \pm\infty) \text{ for all } t \in \mathbb{R}_+, \ \mathfrak{b}(\xi, x) = 1/\mathfrak{a}(\xi, x) \text{ for all } (\xi, x) \in \Delta \times \overline{\mathbb{R}}. \tag{5.47}$$

Proof. Part (a) follows from the necessity portion of [12, Theorem 4.3], which was obtained on the base of [10, Theorem 12.2] and (3.7)–(3.9).

The proof of part (b) is analogous to the proof of the sufficiency portion of [10, Theorem 12.2]. If (5.46) holds, then by Lemma 5.6 there exists a function $\mathfrak{b} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ such that (5.47) is fulfilled. Therefore, the function $\mathfrak{c} := \mathfrak{a}\mathfrak{b} - 1$ belongs to $\widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and (5.45) holds. By Lemma 5.7, the operator $\mathrm{Op}(\mathfrak{c}) = \mathrm{Op}(\mathfrak{a}\mathfrak{b}) - I$ is compact on $L^p(\mathbb{R}_+, d\mu)$. From this observation and Theorem 3.4 we obtain

$$\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b}) \simeq \mathrm{Op}(\mathfrak{a}\mathfrak{b}) \simeq I, \quad \mathrm{Op}(\mathfrak{b})\,\mathrm{Op}(\mathfrak{a}) \simeq \mathrm{Op}(\mathfrak{a}\mathfrak{b}) \simeq I.$$

Thus, the operator $\mathrm{Op}(\mathfrak{a})$ is Fredholm and each its regularizer is of the form $\mathrm{Op}(\mathfrak{b}) + K$, where $K \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$. □

For a symbol $\mathfrak{a} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfying (1.1)–(1.2) the corresponding result was obtained in [16, Theorem 2.6].

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