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Commutators of Convolution Type Operators with Piecewise Quasicontinuous Data

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Abstract

Applying the theory of Calderón-Zygmund operators, we study the compactness of the commutators $[aI, W^0(b)]$ of multiplication operators aI and convolution operators $W^0(b)$ on weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ with $p \in (1, \infty)$ and Muckenhoupt weights w for some classes of piecewise quasicontinuous functions $a \in PQC$ and $b \in PQC_{p,w}$ on the real line $\mathbb{R}$. Then we study two C^*-algebras Z_1 and Z_2 generated by the operators $aW^0(b)$, where a, b are piecewise quasicontinuous functions admitting slowly oscillating discontinuities at ∞ or, respectively, quasicontinuous functions on $\mathbb{R}$ admitting piecewise slowly oscillating discontinuities at ∞. We describe the maximal ideal spaces and the Gelfand transforms for the commutative quotient C^*-algebras $Z_i^\pi = Z_i/\mathcal{K}$ $(i = 1, 2)$ where $\mathcal{K}$ is the ideal of compact operators on the space $L^2(\mathbb{R})$, and establish the Fredholm criteria for the operators $A \in Z_i$.

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1 Introduction

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X, let $\mathcal{K}(X)$ be the closed two-sided ideal of all compact operators in $\mathcal{B}(X)$, and let $\mathcal{B}^\pi(X) = \mathcal{B}(X)/\mathcal{K}(X)$ be the Calkin algebra of the cosets $A^\pi := A + \mathcal{K}(X)$, where $A \in \mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is said to be *Fredholm*, if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional (see, e.g., [9]). Equivalently, $A \in \mathcal{B}(X)$ is Fredholm if and only if the coset A^π is invertible in the algebra $\mathcal{B}^\pi(X)$.

A measurable function $w : \mathbb{R} \to [0,\infty]$ is called a weight if the preimage $w^{-1}(\{0,\infty\})$ of the set $\{0,\infty\}$ has measure zero. For $1 < p < \infty$, a weight w belongs to the *Muckenhoupt class* $A_p(\mathbb{R})$ if

$$c_{p,w} := \sup_I \left(\frac{1}{|I|}\int_I w^p(x)dx\right)^{1/p}\left(\frac{1}{|I|}\int_I w^{-q}(x)dx\right)^{1/q} < \infty,$$

where $1/p + 1/q = 1$, and supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length $|I|$.

In what follows we assume that $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, and consider the weighted Lebesgue space $L^p(\mathbb{R}, w)$ equipped with the norm

$$\|f\|_{L^p(\mathbb{R},w)} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x)dx\right)^{1/p}.$$

As is known (see, e.g., [11] and [5]), the Cauchy singular integral operator $S_{\mathbb{R}}$ given by

$$(S_{\mathbb{R}} f)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\mathbb{R}\setminus(x-\varepsilon,x+\varepsilon)}\frac{f(t)}{t-x}\,dt, \quad x \in \mathbb{R}, \tag{1.1}$$

is bounded on every space $L^p(\mathbb{R}, w)$ with $1 < p < \infty$ and $w \in A_p(\mathbb{R})$.

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the *Fourier transform*,

$$(\mathcal{F} f)(x) := \int_{\mathbb{R}} f(t)e^{itx}dt, \quad x \in \mathbb{R}.$$

A function $a \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^p(\mathbb{R}, w)$ if the convolution operator $W^0(a) := \mathcal{F}^{-1}a\mathcal{F}$ maps the dense subset $L^2(\mathbb{R}) \cap L^p(\mathbb{R}, w)$ of $L^p(\mathbb{R}, w)$ into itself and extends to a bounded linear operator on $L^p(\mathbb{R}, w)$. Let $M_{p,w}$ stand for the Banach algebra of all Fourier multipliers on $L^p(\mathbb{R}, w)$ equipped with the norm $\|a\|_{M_{p,w}} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R},w))}$.

Letting $\mathcal{B}_{p,w} := \mathcal{B}(L^p(\mathbb{R}, w))$ and $\mathcal{K}_{p,w} := \mathcal{K}(L^p(\mathbb{R}, w))$ for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$, we consider the Banach subalgebra

$$\mathfrak{A}_{p,w} := \operatorname{alg}(aI, W^0(b) :\ a \in PQC,\ b \in PQC_{p,w}) \subset \mathcal{B}_{p,w} \tag{1.2}$$

generated by all multiplication operators aI ($a \in PQC$) and all convolution operators $W^0(b) = \mathcal{F}^{-1}b\mathcal{F}$ ($b \in PQC_{p,w}$), where the algebras $PQC \subset L^\infty(\mathbb{R})$ and $PQC_{p,w} \subset M_{p,w}$ of piecewise quasicontinuous functions are defined in Section 2. The Banach algebra $\mathfrak{A}_{p,w}$ in the case of slowly oscillating and piecewise slowly oscillating functions a, b was studied in [16]–[18].

In the present paper, applying the theory of Calderón-Zygmund operators (see, e.g., [25], [12]), we study the compactness of the commutators

$$[aI, W^0(b)] = aW^0(b) - W^0(b)aI \ \in \mathfrak{A}_{p,w} \tag{1.3}$$

of multiplication operators aI and convolution operators $W^0(b)$ on weighted Lebesgue spaces $L^p(\mathbb{R},w)$ with $p \in (1,\infty)$ and Muckenhoupt weights w for some classes of piecewise quasicontinuous functions $a \in PQC$ and $b \in PQC_{p,w}$. Obtained results extend those in [10, Lemmas 7.1–7.4], which are related to piecewise continuous functions a,b, and those in [1, Theorem 4.2, Corollary 4.3] and [17, Theorem 4.6], which are related to piecewise slowly oscillating functions a,b, to wider classes of piecewise quasicontinuous functions a,b on weighted Lebesgue spaces $L^p(\mathbb{R},w)$. Then we study two C^*-subalgebras Z_1 and Z_2 of the C^*-algebra $\mathfrak{A}_{2,1}$ given by (1.2), which are generated by the operators $aW^0(b)$, where a,b are piecewise quasicontinuous functions admitting slowly oscillating discontinuities at ∞ or, respectively, quasicontinuous functions on $\mathbb{R}$ admitting piecewise slowly oscillating discontinuities at ∞. We describe the maximal ideal spaces and the Gelfand transforms for the commutative quotient C^*-algebras $Z_i^\pi = Z_i/\mathcal{K}$ $(i = 1,2)$ where $\mathcal{K}$ is the ideal of compact operators on the space $L^2(\mathbb{R})$, and establish the Fredholm criteria for the operators $A \in Z_i$.

The paper is organized as follows. In Section 2, following [23] and [24] (also see [9]), we introduce the algebras of quasicontinuous and piecewise quasicontinuous functions, and their subalgebras of slowly oscillating and piecewise slowly oscillating functions. In Section 3 we describe the maximal ideal spaces of these commutative algebras. In Section 4 we study the compactness of commutators (1.3) with piecewise quasicontinuous data functions a,b. Finally, in Section 5, using the results of Section 4, we describe the maximal ideal spaces and the Gelfand transforms for the commutative C^*-algebras Z_i^π $(i = 1,2)$ and study the Fredholmness of operators $A \in Z_i$.

2 Algebras of piecewise quasicontinuous functions

2.1 BMO and VMO

Let Γ be the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ or the real line $\mathbb{R}$. Given a locally integrable function $f \in L^1_{loc}(\Gamma)$ and a finite interval I on Γ, let $|I|$ denote the length of I and let

$$I(f) := |I|^{-1} \int_I f(t)dt$$

denote the average of f over I. For $a > 0$, consider the quantities

$$\begin{gathered} M_a(f) := \sup_{|I|\le a} |I|^{-1} \int_I |f(t) - I(f)|dt, \\ M_0(f) := \lim_{a\to 0} M_a(f), \quad \|f\|_* := \lim_{a\to\infty} M_a(f). \end{gathered} \tag{2.1}$$

The function $f \in L^1_{loc}(\Gamma)$ is said to have bounded mean oscillation, $f \in BMO(\Gamma)$, if $\|f\|_* < \infty$. The space $BMO(\Gamma)$ is a Banach space under the norm $\|\cdot\|_*$, provided that two functions differing by a constant are identified. A function $f \in BMO(\Gamma)$ is said to have vanishing mean oscillation, $f \in VMO(\Gamma)$, if $M_0(f) = 0$. As is well known (see, e.g., [23]), $VMO(\Gamma)$ is a closed subspace of $BMO(\Gamma)$.

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Consider the homeomorphism $\gamma : \mathbb{T} \to \dot{\mathbb{R}}$, $\gamma(t) = i(1+t)/(1-t)$. By [11, Chapter VI, Corollary 1.3], $f \in BMO(\mathbb{R})$ if and only if $f \circ \gamma \in BMO(\mathbb{T})$, and the norms of these functions are equivalent. On the other hand,

$$VMO := \{f \circ \gamma^{-1} : f \in VMO(\mathbb{T})\} \tag{2.2}$$

is a proper closed subspace of $VMO(\mathbb{R})$. Since $VMO(\mathbb{T})$ is the closure of $C(\mathbb{T})$ in $BMO(\mathbb{T})$ (see, e.g., [11, p. 253]), (2.2) implies the following property of VMO.

Proposition 2.1. *VMO is the closure in $BMO(\mathbb{R})$ of the set $C(\dot{\mathbb{R}})$.*

2.2 The C^*-algebras $SO^\diamond$ and QC

Let $\Gamma \in \{\dot{\mathbb{R}}, \mathbb{T}\}$. For a bounded measurable function $f : \Gamma \to \mathbb{C}$ and a set $I \subset \Gamma$, let

$$\operatorname{osc}(f, I) = \operatorname{ess\,sup}\{|f(t) - f(s)| : t, s \in I\}.$$

Following [2, Section 4], we say that a function $f \in L^\infty(\Gamma)$ is *slowly oscillating at a point* $\eta \in \Gamma$ if for every $r \in (0,1)$ or, equivalently, for some $r \in (0,1)$,

$$\lim_{\varepsilon\to 0} \operatorname{osc}(f, \Gamma_{r\varepsilon,\varepsilon}(\eta)) = 0 \ \text{ for } \eta \neq \infty \ \text{ and } \ \lim_{\varepsilon\to\infty} \operatorname{osc}(f, \Gamma_{r\varepsilon,\varepsilon}(\eta)) = 0 \ \text{ for } \eta = \infty,$$

where

$$\Gamma_{r\varepsilon,\varepsilon}(\eta) := \begin{cases} \{z \in \Gamma : \ r\varepsilon \le |z - \eta| \le \varepsilon\} & \text{if } \ \eta \neq \infty, \\ \{z \in \Gamma : \ r\varepsilon \le |z| \le \varepsilon\} & \text{if } \ \eta = \infty. \end{cases}$$

For each $\eta \in \Gamma$, let $SO_\eta(\Gamma)$ denote the C^*-subalgebra of $L^\infty(\Gamma)$ defined by

$$SO_\eta(\Gamma) := \Big\{f \in C_b(\Gamma \setminus \{\eta\}) : \ f \text{ slowly oscillates at } \eta\Big\},$$

where $C_b(\Gamma \setminus \{\eta\}) := C(\Gamma \setminus \{\eta\}) \cap L^\infty(\Gamma)$. Hence, setting $SO_\lambda := SO_\lambda(\dot{\mathbb{R}})$ for all $\lambda \in \dot{\mathbb{R}}$, we conclude that

$$SO_\infty = \Big\{f \in C_b(\dot{\mathbb{R}} \setminus \{\infty\}) : \lim_{x\to+\infty} \operatorname{osc}(f, [-x, -x/2] \cup [x/2, x]) = 0\Big\},$$
$$SO_\lambda = \Big\{f \in C_b(\dot{\mathbb{R}} \setminus \{\lambda\}) : \lim_{x\to 0} \operatorname{osc}(f, \lambda + ([-x, -x/2] \cup [x/2, x])) = 0\Big\}$$

for $\lambda \in \mathbb{R}$. Let $SO^\diamond$ be the minimal C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains all the C^*-algebras SO_λ with $\lambda \in \dot{\mathbb{R}}$. In particular, $SO^\diamond$ contains $C(\dot{\mathbb{R}})$.

Lemma 2.2. [17, Lemma 2.1] *Let $\lambda \in \dot{\mathbb{R}}$, $a \in SO_\lambda$, and let $\gamma : \mathbb{T} \to \dot{\mathbb{R}}$ be the homeomorphism given by $\gamma(t) = i(1+t)/(1-t)$. Then $a \circ \gamma \in SO_\eta(\mathbb{T})$ where $\eta := \gamma^{-1}(\lambda)$.*

Corollary 2.3. [17, Corollary 2.2] *For every $\lambda \in \mathbb{R}$, the mapping $a \mapsto a \circ \beta_\lambda$ defined by the homeomorphism*

$$\beta_\lambda : \ \dot{\mathbb{R}} \to \dot{\mathbb{R}}, \ \ x \mapsto \frac{\lambda x - 1}{x + \lambda}$$

is an isometric isomorphism of the C^*-algebra SO_λ onto the C^*-algebra SO_∞.

Let H^∞ be the closed subalgebra of $L^\infty(\mathbb{R})$ that consists of all functions being non-tangential limits on $\mathbb{R}$ of bounded analytic functions on the upper half-plane. According to [23] and [24], the C^*-algebra QC of quasicontinuous functions on $\dot{\mathbb{R}}$ is defined by

$$QC := (H^\infty + C(\dot{\mathbb{R}})) \cap (\overline{H^\infty} + C(\dot{\mathbb{R}})) = VMO \cap L^\infty(\mathbb{R}). \tag{2.3}$$

Theorem 2.4. [17, Theorem 4.2] *The C^*-algebra $SO^\diamond$ is contained in the C^*-algebra QC of quasicontinuous functions on $\dot{\mathbb{R}}$.*

2.3 Fourier multipliers

Let $C^n(\mathbb{R})$ be the set of all n times continuously differentiable functions $a:\mathbb{R}\to\mathbb{C}$, and let $V(\mathbb{R})$ be the Banach algebra of all functions $a:\mathbb{R}\to\mathbb{C}$ with finite total variation

$$V(a) := \sup\Big\{\sum\nolimits_{i=1}^{n} |a(t_i)-a(t_{i-1})| : -\infty < t_0 < t_1 < \ldots < t_n < +\infty,\ n\in\mathbb{N}\Big\}$$

where the supremum is taken over all finite partitions of the real line $\mathbb{R}$ and the norm in $V(\mathbb{R})$ is given by $\|a\|_V = \|a\|_{L^\infty(\mathbb{R})} + V(a)$. As is known (see, e.g., [13, Chapter 9]), every function $a\in V(\mathbb{R})$ has finite one-sided limits at every point $t\in\dot{\mathbb{R}}$.

Let PC be the C^*-algebra of all functions on $\mathbb{R}$ having finite one-sided limits at every point $t\in\dot{\mathbb{R}}$. If $a\in PC$ has finite total variation, then $a\in M_{p,w}$ for all $p\in(1,\infty)$ and all $w\in A_p(\mathbb{R})$ according to Stechkin's inequality

$$\|a\|_{M_{p,w}} \le \|S_{\mathbb{R}}\|_{\mathcal{B}(L^p(\mathbb{R},w))}\big(\|a\|_{L^\infty(\mathbb{R})} + V(a)\big) \tag{2.4}$$

(see, e.g., [10, Theorem 2.11] and [8]), where the Cauchy singular integral operator $S_{\mathbb{R}}$ is given by (1.1).

The following result obtained in [19, Corollary 2.10] supply us with another class of Fourier multipliers in $M_{p,w}$.

Theorem 2.5. *If $a\in C^3(\mathbb{R}\setminus\{0\})$ and $\|D^k a\|_{L^\infty(\mathbb{R})} < \infty$ for all $k=0,1,2,3$, where $(Da)(x) = xa'(x)$ for $x\in\mathbb{R}$, then the convolution operator $W^0(a)$ is bounded on every weighted Lebesgue space $L^p(\mathbb{R},w)$ with $1<p<\infty$ and $w\in A_p(\mathbb{R})$, and*

$$\|a\|_{M_{p,w}} \le c_{p,w}\max\{\|D^k a\|_{L^\infty(\mathbb{R})} :\ k=0,1,2,3\} < \infty,$$

where the constant $c_{p,w}\in(0,\infty)$ depends only on p and w.

2.4 Banach algebras $C_{p,w}(\dot{\mathbb{R}})$, $C_{p,w}(\overline{\mathbb{R}})$ and $PC_{p,w}$

Let PC stand for the C^*-algebra of piecewise continuous functions $f:\mathbb{R}\to\mathbb{C}$. We denote by $C_{p,w}(\dot{\mathbb{R}})$ (resp., $C_{p,w}(\overline{\mathbb{R}})$, $PC_{p,w}$) the closure in $M_{p,w}$ of the set of all functions $a\in C(\dot{\mathbb{R}})$ (resp., $a\in C(\overline{\mathbb{R}})$, $a\in PC$) of finite total variation (see [10]). Obviously, by (2.4), $C_{p,w}(\dot{\mathbb{R}})$, $C_{p,w}(\overline{\mathbb{R}})$ and $PC_{p,w}$ are Banach subalgebras of $M_{p,w}$, and

$$C_{p,w}(\dot{\mathbb{R}})\subset C(\dot{\mathbb{R}}),\ \ C_{p,w}(\overline{\mathbb{R}})\subset C(\overline{\mathbb{R}}),\ \ PC_{p,w}\subset PC.$$

2.5 Banach algebras $SO^\diamond_{p,w}$ and $QC_{p,w}$

For $\lambda\in\dot{\mathbb{R}}$, we consider the commutative Banach algebras

$$SO^3_\lambda := \Big\{a\in SO_\lambda\cap C^3(\mathbb{R}\setminus\{\lambda\}) : \lim_{x\to\lambda}(D^k_\lambda a)(x) = 0,\ k=1,2,3\Big\}$$

equipped with the norm

$$\|a\|_{SO^3_\lambda} := \max\{\|D^k_\lambda a\|_{L^\infty(\mathbb{R})} :\ k=0,1,2,3\},$$

where $(D_\lambda a)(x) = (x-\lambda)a'(x)$ for $\lambda \in \mathbb{R}$ and $(D_\lambda a)(x) = xa'(x)$ if $\lambda = \infty$. By Theorem 2.5, $SO^3_\lambda \subset M_{p,w}$ for all $p \in (1,\infty)$ and all $w \in A_p(\mathbb{R})$. Let $SO_{\lambda,p,w}$ denote the closure of SO^3_λ in $M_{p,w}$, and let $SO^\diamond_{p,w}$ be the Banach subalgebra of $M_{p,w}$ generated by all the algebras $SO_{\lambda,p,w}$ $(\lambda \in \dot{\mathbb{R}})$. Because $M_{p,w} \subset M_2 = L^\infty(\mathbb{R})$, we conclude that $SO^\diamond_{p,w} \subset SO^\diamond$.

To define an $M_{p,w}$-analogue of the C^*-algebra QC, we need the following weighted analogue of the Krasnoselskii theorem [20, Theorem 3.10] on interpolation of compactness (see, e.g., [15, Theorem 5.2]), which follows from the Stein-Weiss interpolation theorem (see, e.g., [4, Corollary 5.5.4]).

Theorem 2.6. *Suppose $1 < p_i < \infty$, w_i are weights in $L^{p_i}_{loc}(\mathbb{R})$, and $T \in \mathcal{B}(L^{p_i}(\mathbb{R}, w_i))$ for $i = 1,2$. If the operator T is compact on the space $L^{p_1}(\mathbb{R}, w_1)$, then T is compact on every space $L^p(\mathbb{R}, w)$ where*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad w = w_1^{1-\theta} w_2^\theta, \quad 0 < \theta < 1. \tag{2.5}$$

Let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$. By the stability of Muckenhoupt weights (see, e.g., [5, Section 2.8]), there exists an $\varepsilon_0 \in (0, p-1)$ such that $w^{1+\varepsilon} \in A_{p_0}(\mathbb{R})$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and all $p_0 \in (p-\varepsilon_0, p+\varepsilon_0)$. Then, in particular, $w^{1+\varepsilon} \in L^{p_0}_{loc}(\mathbb{R})$ (see, e.g., [5, Lemma 4.6, Theorem 4.15]). According to the proof of [15, Corollary 5.3], let $\mathcal{E}$ denote the set of all $\varepsilon > 0$ such that $w_\varepsilon \in A_{p_\varepsilon}(\mathbb{R})$, where

$$p_\varepsilon := p/[1 + (1-p/2)\varepsilon], \quad w_\varepsilon := w^{1+\varepsilon}. \tag{2.6}$$

Taking then $p_1 = 2$, $w_1 = 1$, $p_2 = p_\varepsilon$, $w_2 = w_\varepsilon$ and $\theta = (1+\varepsilon)^{-1}$, we infer from Theorem 2.6 that (2.5) holds for all $\varepsilon \in \mathcal{E}$, which implies due to [4, Corollary 5.5.4] that

$$M_{p_\varepsilon, w_\varepsilon} \subset M_{p,w} \text{ for all } p \in (1,\infty), \ w \in A_p(\mathbb{R}) \text{ and } \varepsilon \in \mathcal{E}. \tag{2.7}$$

Thus, Theorem 2.6 gives the following.

Corollary 2.7. *If $p \in (1,\infty)$, $w \in A_p(\mathbb{R})$ and an operator T is compact on the space $L^2(\mathbb{R})$ and is bounded on the weighted Lebesgue space $L^{p_\varepsilon}(\mathbb{R}, w_\varepsilon)$ for some $\varepsilon \in \mathcal{E}$, where p_ε and w_ε are given by* (2.6)*, then the operator T is compact on the space $L^p(\mathbb{R}, w)$.*

By analogy with [14], we define the set $\mathcal{R}_{p,w} := \bigcup_{\varepsilon\in\mathcal{E}} M_{p_\varepsilon, w_\varepsilon}$. Along with QC given by (2.3), we introduce its $M_{p,w}$-analogue $QC_{p,w}$ as the closure in $M_{p,w}$ of the set $QC \cap \mathcal{R}_{p,w}$. Obviously, in view of (2.7) and the inclusion $SO^3_\lambda \subset M_{p,w}$ for all $p \in (1,\infty)$ and all $w \in A_p(\mathbb{R})$, we obtain

$$QC_{p,w} \subset QC \cap M_{p,w} \subset QC \quad \text{and} \quad SO^\diamond_{p,w} \subset QC_{p,w}.$$

2.6 Banach algebras $PSO^\diamond_{p,w}$ and $PQC_{p,w}$

Let $PSO^\diamond = \mathrm{alg}(PC, SO^\diamond)$ be the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by the C^*-algebras PC and $SO^\diamond$, and let $PSO^\diamond_{p,w} = \mathrm{alg}(PC_{p,w}, SO^\diamond_{p,w})$ be the Banach subalgebra of $M_{p,w}$ generated by the Banach algebras $PC_{p,w}$ and $SO^\diamond_{p,w}$.

Let $PQC = \mathrm{alg}(PC, QC)$ be the C^*-algebra of piecewise quasicontinuous functions generated in $L^\infty(\mathbb{R})$ by the C^*-algebras PC and QC, and let $PQC_{p,w} = \mathrm{alg}(PC_{p,w}, QC_{p,w})$ denote the Banach subalgebra of $M_{p,w}$ generated by the Banach algebras $PC_{p,w}$ and $QC_{p,w}$.

Clearly,

$$PSO^\diamond_{p,w} \subset PSO, \quad PQC_{p,w} \subset PQC, \quad PSO^\diamond_{p,w} \subset PQC_{p,w}.$$

3 The maximal ideal spaces of functional algebras

3.1 The maximal ideal space of the Banach algebra $SO^{\diamond}_{p,w}$

In what follows, let $M(\mathcal{A})$ denote the maximal ideal space of a commutative Banach algebra $\mathcal{A}$. If C is a Banach subalgebra of $\mathcal{A}$ and $\lambda \in M(C)$, then the set $M_\lambda(\mathcal{A}) := \{\xi \in M(\mathcal{A}) : \xi|_C = \lambda\}$ is called the fiber of $M(\mathcal{A})$ over λ. Hence for every Banach algebra $\mathcal{A} \subset L^\infty(\mathbb{R})$ with $M(C(\dot{\mathbb{R}}) \cap \mathcal{A}) = \dot{\mathbb{R}}$ and every $\lambda \in \dot{\mathbb{R}}$, the fiber $M_\lambda(\mathcal{A})$ denotes the set of all characters (multiplicative linear functionals) of $\mathcal{A}$ that annihilate the set $\{f \in C(\dot{\mathbb{R}}) \cap \mathcal{A} : f(\lambda) = 0\}$. As usual, for all $a \in \mathcal{A}$ and all $\xi \in M(\mathcal{A})$, we put $a(\xi) := \xi(a)$.

Identifying the points $\lambda \in \dot{\mathbb{R}}$ with the evaluation functionals δ_λ on $\dot{\mathbb{R}}$, $\delta_\lambda(f) = f(\lambda)$ for $f \in C(\dot{\mathbb{R}})$, we infer that the maximal ideal space $M(SO^\diamond)$ of $SO^\diamond$ is of the form

$$M(SO^\diamond) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(SO^\diamond), \tag{3.1}$$

where $M_\lambda(SO^\diamond) := \{\xi \in M(SO^\diamond) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\lambda\}$ are fibers of $M(SO^\diamond)$ over $\lambda \in \dot{\mathbb{R}}$. Applying Corollary 2.3 and [3, Proposition 5], we infer that for every $\lambda \in \dot{\mathbb{R}}$,

$$M_\lambda(SO^\diamond) = M_\lambda(SO_\lambda) = M_\infty(SO_\infty) = (\mathrm{clos}_{SO^*_\infty}\mathbb{R}) \setminus \mathbb{R}, \tag{3.2}$$

where $\mathrm{clos}_{SO^*_\infty}\mathbb{R}$ is the weak-star closure of $\mathbb{R}$ in SO^*_∞, the dual space of SO_∞.

The fiber $M_\infty(SO_\infty)$ is related to the partial limits of a function $a \in SO_\infty$ at infinity as follows (see [6, Corollary 4.3] and [1, Corollary 3.3]).

Proposition 3.1. *If $\{a_k\}_{k=1}^\infty$ is a countable subset of SO_∞ and $\xi \in M_\infty(SO_\infty)$, then there exists a sequence $\{g_n\} \subset \mathbb{R}_+$ such that $g_n \to \infty$ as $n \to \infty$, and for every $t \in \mathbb{R} \setminus \{0\}$ and every $k \in \mathbb{N}$, $\lim_{n\to\infty} a_k(g_n t) = \xi(a_k)$.*

Lemma 3.2. [17, Lemma 3.5] *If $1 < p < \infty$, $w \in A_p(\mathbb{R})$ and $\lambda \in \dot{\mathbb{R}}$, then the maximal ideal spaces of $SO_{\lambda,p,w}$ and SO_λ coincide as sets, that is, $M(SO_{\lambda,p,w}) = M(SO_\lambda)$.*

Fix $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$. Analogously to (3.1) we obtain

$$M(SO^\diamond_{p,w}) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(SO^\diamond_{p,w}). \tag{3.3}$$

Lemma 3.2 and relations (3.2) imply that

$$M_\lambda(SO^\diamond_{p,w}) = M_\lambda(SO_{\lambda,p,w}) = M_\lambda(SO_\lambda) = M_\infty(SO_\infty) \tag{3.4}$$

for every $\lambda \in \dot{\mathbb{R}}$. Applying (3.3), (3.4) and (3.1) we arrive at the following result.

Theorem 3.3. [17, Theorem 3.6] *If $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, then the maximal ideal spaces of $SO^\diamond_{p,w}$ and $SO^\diamond$ coincide as sets, $M(SO^\diamond_{p,w}) = M(SO^\diamond)$.*

3.2 The maximal ideal space of the C^*-algebra QC

Identifying the points $\lambda \in \dot{\mathbb{R}}$ with the evaluation functionals δ_λ on $\dot{\mathbb{R}}$, we conclude by analogy with (3.1) that the maximal ideal space $M(QC)$ of the C^*-algebra QC of quasicontinuous functions $a : \dot{\mathbb{R}} \to \mathbb{C}$ is of the form

$$M(QC) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(QC),$$

where $M_\lambda(QC) := \{\xi \in M(QC) : \xi|_{C(\dot{\mathbb{R}})} = \delta_\lambda\}$ are fibers of $M(QC)$ over $\lambda \in \dot{\mathbb{R}}$.

Let $H^\infty(\mathbb{T})$ be the C^*-subalgebra of $L^\infty(\mathbb{T})$ that consists of all functions being non-tangential limits on $\mathbb{T}$ of bounded analytic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In what follows we identify the fibers $M_\lambda(QC)$ $(\lambda \in \dot{\mathbb{R}})$ of the C^*-algebra QC with the fibers $M_t(QC(\mathbb{T}))$ for $t = (\lambda - i)/(\lambda + i) \in \mathbb{T}$ of the C^*-algebra $QC(\mathbb{T})$ of quasicontinuous functions on $\mathbb{T}$,

$$QC(\mathbb{T}) := (H^\infty(\mathbb{T}) + C(\mathbb{T})) \cap (\overline{H^\infty(\mathbb{T})} + C(\mathbb{T})) = VMO(\mathbb{T}) \cap L^\infty(\mathbb{T}). \tag{3.5}$$

Let $\mathcal{G}$ be the set of all averaging functionals of the form

$$f_I(a) = \frac{1}{|I|} \int_I a(t)|dt| \quad (a \in QC(\mathbb{T})), \tag{3.6}$$

where I runs the set $\mathcal{L}$ of all arcs of $\mathbb{T}$ and $|I|$ means the length of I. Let us identify arcs $I \subset \mathbb{T}$ with functionals f_I given by (3.6). According to [24], $M(QC(\mathbb{T}))$ consists of all functionals in the weak-star closure of $\mathcal{G}$ in the dual space $(QC(\mathbb{T}))^*$ of (3.5) that do not belong to $\mathcal{G}$.

Given $t \in \mathbb{T}$, let $M_t^\pm(QC(\mathbb{T}))$ be the set of all $\xi \in M_t(QC(\mathbb{T}))$ such that $\xi(a) = 0$ if $a \in QC(\mathbb{T})$ and $\limsup_{\tau \to t^\pm} |a(\tau)| = 0$, respectively, where $\tau \to t^+$ (resp., $\tau \to t^-$) means that $\tau \in \mathbb{T}$ tends to t from the right (resp., from the left).

For $t \in \mathbb{T}$ and $c > 0$, let $\mathcal{G}_{t,c}$ denote the set of arcs $I \in \mathcal{L}$ such that the distance between t and the center of I (measured along $\mathbb{T}$) does not exceed $c|I|$. In particular, $\mathcal{G}_{t,0}$ is the set of arcs with center t. Let $M_t^0(QC(\mathbb{T}))$ be the set of functionals in the fiber $M_t(QC(\mathbb{T}))$ that lie in the weak-star closure of $\mathcal{G}_{t,0}$. By [24], $M_t^0(QC(\mathbb{T}))$ coincides with the set of functionals in $M_t(QC(\mathbb{T}))$ that lie in the weak-star closure of $\mathcal{G}_{t,c}$ for any $c > 0$.

Lemma 3.4. [24, Lemma 8] *For every* $t \in \mathbb{T}$, $M_t^+(QC(\mathbb{T})) \cap M_t^-(QC(\mathbb{T})) = M_t^0(QC(\mathbb{T}))$ *and* $M_t^+(QC(\mathbb{T})) \cup M_t^-(QC(\mathbb{T})) = M_t(QC(\mathbb{T}))$.

3.3 The maximal ideal spaces of the C^*-algebras $PSO^\diamond$ and PQC

For $\Gamma \in \{\dot{\mathbb{R}}, \mathbb{T}\}$, let $PC(\Gamma)$ be the C^*-algebra of piecewise continuous functions $f : \Gamma \to \mathbb{C}$. The maximal ideal space $M(PC(\Gamma))$ of $PC(\Gamma)$ can be identified with the set $\Gamma \times \{0, 1\}$, and its fibers over points $t \in \Gamma$ are the doubletons $M_t(PC(\Gamma)) = \{(t, 0), (t, 1)\}$, where

$$f(t, 0) = f(t - 0) \quad \text{and} \quad f(t, 1) = f(t + 0) \quad \text{for all} \;\; f \in PC(\Gamma), \tag{3.7}$$

and $f(\infty, 0) = f(+\infty)$, $f(\infty, 1) = f(-\infty)$.

By [2, Section 4] and [16, Section 3], the maximal ideal space of the C^*-algebra $PSO^\diamond \subset L^\infty(\mathbb{R})$ is of the form

$$M(PSO^\diamond) = \bigcup_{\lambda \in \dot{\mathbb{R}}} M_\lambda(PSO^\diamond), \quad M_\lambda(PSO^\diamond) = M_\lambda(SO^\diamond) \times \{0, 1\} = \bigcup_{\xi \in M_\lambda(SO^\diamond)} \{(\xi, 0), (\xi, 1)\},$$

where, for every $\lambda \in \dot{\mathbb{R}}$ and every $(\xi,\mu) \in M_\lambda(SO^\diamond) \times \{0,1\}$, we have

$$(\xi,\mu)|_{SO^\diamond} = \xi, \quad (\xi,\mu)|_{C(\dot{\mathbb{R}})} = \lambda, \quad (\xi,\mu)|_{PC} = (\lambda,\mu).$$

For all $\xi \in M(SO^\diamond)$, we put $\xi^- := (\xi,0)$ and $\xi^+ := (\xi,1)$.

Let $PQC(\mathbb{T})$ denote the C^*-subalgebra of $L^\infty(\mathbb{T})$ generated by the C^*-algebras $PC(\mathbb{T})$ and $QC(\mathbb{T})$. By [24] (also see [9, Section 3.3]), there is a natural mapping

$$w : M(PQC(\mathbb{T})) \to M(QC(\mathbb{T})) \times \{0,1\}$$

which is given as follows: for $y \in M(PQC(\mathbb{T}))$, let $\xi = y|_{QC(\mathbb{T})}$, $t = y|_{C(\mathbb{T})}$, and $v = y|_{PC(\mathbb{T})}$; if $v = (t,0)$ (resp., $v = (t,1)$), then $w(y) = (\xi,0)$ (resp., $w(y) = (\xi,1)$). Hence, $M(PQC(\mathbb{T}))$ is a subset of the set $M(QC(\mathbb{T})) \times \{0,1\}$. By analogy with (3.7), we obtain

$$M(PQC(\mathbb{T})) = \bigcup_{t\in\mathbb{T}} M_t(PQC(\mathbb{T})) = \bigcup_{t\in\mathbb{T}} \bigcup_{\xi\in M_t(QC(\mathbb{T}))} M_\xi(PQC(\mathbb{T})).$$

The fibers $M_\xi(PQC(\mathbb{T}))$ for $\xi \in M(QC(\mathbb{T}))$ are described as follows.

Theorem 3.5. [24, Section 5] *Let $\xi \in M_t(QC(\mathbb{T}))$ for $t \in \mathbb{T}$. Then*

$$M_\xi(PQC(\mathbb{T})) = \begin{cases} \{(\xi,0)\} & \text{if } \ \xi \in M_t^-(QC(\mathbb{T})) \setminus M_t^0(QC(\mathbb{T})), \\ \{(\xi,1)\} & \text{if } \ \xi \in M_t^+(QC(\mathbb{T})) \setminus M_t^0(QC(\mathbb{T})), \\ \{(\xi,0),(\xi,1)\} & \text{if } \ \xi \in M_t^0(QC(\mathbb{T})). \end{cases}$$

4 Compactness of commutators of convolution type operators

Given $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, we consider the Banach algebra $\mathcal{B}_{p,w}$ and its ideal of compact operators $\mathcal{K}_{p,w}$. In case $w \equiv 1$ we abbreviate $\mathcal{B}_{p,1}$ and $\mathcal{K}_{p,1}$ to $\mathcal{B}_p$ and $\mathcal{K}_p$, respectively. The notation $C_p(\dot{\mathbb{R}})$, $C_p(\overline{\mathbb{R}})$, PC_p and $SO_{\infty,p}$ is understood analogously.

For two algebras $\mathcal{A}$ and $\mathcal{B}$ contained in a Banach algebra C, we denote by $\mathrm{alg}(\mathcal{A},\mathcal{B})$ the Banach subalgebra of C generated by the algebras $\mathcal{A}$ and $\mathcal{B}$.

First we recall three known results on the compactness of commutators.

Lemma 4.1. [10, Lemmas 7.1–7.4] *Let $1 < p < \infty$.*

(a) *If $a \in PC$, $b \in PC_p$, and $a(\pm\infty) = b(\pm\infty) = 0$, then $aW^0(b), W^0(b)aI \in \mathcal{K}_p$.*

(b) *If $a \in C(\dot{\mathbb{R}})$ and $b \in PC_p$, or $a \in PC$ and $b \in C_p(\dot{\mathbb{R}})$, then $[aI, W^0(b)] \in \mathcal{K}_p$.*

(c) *If $a \in C(\overline{\mathbb{R}})$ and $b \in C_p(\overline{\mathbb{R}})$, then $[aI, W^0(b)] \in \mathcal{K}_p$.*

Theorem 4.2. [1, Theorem 4.2, Corollary 4.3] *If $1 < p < \infty$ and either $a \in \mathrm{alg}(SO_\infty, PC)$ and $b \in SO_{\infty,p}$, or $a \in SO_\infty$ and $b \in \mathrm{alg}(SO_{\infty,p}, PC_p)$, or $a \in \mathrm{alg}(SO_\infty, C(\overline{\mathbb{R}}))$ and $b \in \mathrm{alg}(SO_{\infty,p}, C_p(\overline{\mathbb{R}}))$, then $[aI, W^0(b)] \in \mathcal{K}_p$.*

Theorem 4.3. [17, Theorem 4.6] *Let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$. If $a \in PSO^\diamond$ and $b \in SO^\diamond_{p,w}$, or $a \in SO^\diamond$ and $b \in PSO^\diamond_{p,w}$, or $a \in \mathrm{alg}(SO_\infty, C(\overline{\mathbb{R}}))$ and $b \in \mathrm{alg}(SO_{\infty,p,w}, C_{p,w}(\overline{\mathbb{R}}))$, then $[aI, W^0(b)] \in \mathcal{K}_{p,w}$.*

We say that two functions $a, b \in L^\infty(\mathbb{R})$ are equivalent at ∞ ($a \overset{\infty}{\sim} b$) if

$$\lim_{N\to\infty} \|a-b\|_{L^\infty(\mathbb{R}\setminus[-N,N])} = 0. \tag{4.1}$$

Applying the theory of Calderón-Zygmund operators, we establish the following compactness result for weighted Lebesgue spaces.

Theorem 4.4. *If $p \in (1,\infty)$, $w \in A_p(\mathbb{R})$ and one of the following conditions holds:*

(i) $a \in PQC$ *and* $b \in SO^\diamond_{p,w}$,

(ii) $a \in SO^\diamond$ *and* $b \in PQC_{p,w}$,

(iii) $a \in PQC$, $b \in PQC_{p,w}$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ *and* $c \in SO^\diamond$, $d \in SO^\diamond_{p,w}$,

(iv) $a \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$ *and* $b \in \operatorname{alg}(SO^\diamond_{p,w}, C_{p,w}(\overline{\mathbb{R}}))$,

(v) $a \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$ *and* $b \in \operatorname{alg}(QC_{p,w}, C_{p,w}(\overline{\mathbb{R}}))$,

(vi) $a \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$, $b \in \operatorname{alg}(QC_{p,w}, C_{p,w}(\overline{\mathbb{R}}))$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ *and* $c \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$, $d \in \operatorname{alg}(SO^\diamond_{p,w}, C_{p,w}(\overline{\mathbb{R}}))$,

then the commutator $[aI, W^0(b)]$ *is compact on the space* $L^p(\mathbb{R}, w)$.

Proof. Since every function $b \in QC_{p,w}$ can be approximated in $M_{p,w}$ by functions $b_n \in QC \cap M_{p_\varepsilon,w_\varepsilon}$ for some $\varepsilon \in \mathcal{E}$, where p_ε and w_ε are given by (2.6), and since all functions b in the algebras $SO_{p,w}$, $C_{p,w}(\overline{\mathbb{R}})$ and $PC_{p,w}$ can be also approximated in $M_{p,w}$ by functions b_n in $SO \cap M_{p_\varepsilon,w_\varepsilon}$, $C(\overline{\mathbb{R}}) \cap M_{p_\varepsilon,w_\varepsilon}$ and $PC \cap M_{p_\varepsilon,w_\varepsilon}$, respectively, we conclude from Corollary 2.7 that the commutators $[aI, W^0(b_n)]$ will be compact on the space $L^p(\mathbb{R}, w)$ for all functions a and b in conditions (i)–(vi) of the theorem if these commutators will be compact on the space $L^2(\mathbb{R})$. Consequently, in that case, in view of the equality

$$\lim_{n\to\infty} \big\|[aI, W^0(b_n)] - [aI, W^0(b)]\big\|_{\mathcal{B}(L^p(\mathbb{R},w))} = 0,$$

the commutator $[aI, W^0(b)]$ will be compact on the space $L^p(\mathbb{R}, w)$ as well.

Thus, according to Corollary 2.7, it is sufficient to prove the compactness of the commutator $[aI, W^0(b)]$ under conditions (i)–(vi) on functions a and b only on the space $L^2(\mathbb{R})$, which implies its compactness on all the spaces $L^p(\mathbb{R}, w)$. Then conditions (i)–(vi) can be rewritten in the form

(i') $a \in PQC$ and $b \in SO^\diamond$,

(ii') $a \in SO^\diamond$ and $b \in PQC$,

(iii') $a, b \in PQC$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ and $c, d \in SO^\diamond$,

(iv') $a \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$ and $b \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$,

(v') $a \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$ and $b \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$, (vi') $a, b \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ and $c, d \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$.

Under the transform $A \mapsto \mathcal{F}A\mathcal{F}^{-1}$, the cases (ii') and (v') are reduced to the cases (i') and (iv'), respectively. Indeed, $\mathcal{F}a\mathcal{F}^{-1} = W^0(\widetilde{b})$ and $\mathcal{F}W^0(b)\mathcal{F}^{-1} = \widetilde{a}I$ where $\widetilde{b}(x) = a(-x)$ and $\widetilde{a} = b$. Thus, it only remains to prove the assertion in the cases (i'), (iii'), (iv') and (vi').

Case (i'). Since PQC is the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by the C^*-algebras PC and QC, it is sufficient to prove part (i') for the pair $a \in QC$, $b \in SO^\diamond$ only, because for the pair $a \in PC$, $b \in SO^\diamond$ the compactness of the commutator $[aI, W^0(b)]$ follows from Theorem 4.3. Since $SO^\diamond$ is the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by all the C^*-algebras SO_λ ($\lambda \in \dot{\mathbb{R}}$), and since SO_λ is the closure of SO^3_λ in $L^\infty(\mathbb{R})$, it remains to prove part (i') for the pair $a \in QC$, $b \in SO^3_\lambda$.

If $\lambda \in \{0, \infty\}$, then we proceed similarly to the proof of [17, Theorem 4.6]. It follows from [19, Lemma 2.2] that the distribution $K = \mathcal{F}^{-1}b$ for $b \in SO^3_\lambda$ agrees with a function $K(\cdot)$ differentiable on $\mathbb{R} \setminus \{0\}$ and such that

$$\big|K(x)\big| \le A_0|x|^{-1}, \quad \big|K'(x)\big| \le A_1|x|^{-2} \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\}, \tag{4.2}$$

where the constants A_α ($\alpha = 0, 1$) are estimated by

$$A_\alpha \le C_\alpha \max\{\|D^k b\|_{L^\infty(\mathbb{R})} : \ k = 0, 1, 2, 3\},$$

$(Db)(x) = xb'(x)$ for $x \in \mathbb{R}$ and the constants $C_\alpha \in (0, \infty)$ depend only on α. Hence $K(\cdot)$ is a classical Calderón-Zygmund kernel, and the convolution operator $W^0(b)$ can be considered as the Calderón-Zygmund operator given by

$$(Tf)(x) = \text{v.p.} \int_{\mathbb{R}} K(x-y)f(y)dy \quad \text{for } \ x \in \mathbb{R}, \tag{4.3}$$

where T is bounded on every weighted Lebesgue space $L^p(\mathbb{R}, w)$ with $1 < p < \infty$ and $w \in A_p(\mathbb{R})$ (see, e.g., Theorem 2.5). In particular, the second condition in (4.2) implies that there is a constant $A_2 \in (0, \infty)$ such that

$$|K(x-y) - K(x)| \le A_2|y|^\delta |x|^{-1-\delta} \quad \text{for } \ |x| \ge 2|y| > 0, \tag{4.4}$$

where $\delta \in (0, 1)$. Moreover, because the convolution operator $W^0(b)$ is bounded on the space $L^2(\mathbb{R})$, we conclude from [25, p. 291, Proposition 2] that

$$\sup_{0<r<R<\infty} \left| \int_{r<|x|<R} K(x)dx \right| < \infty. \tag{4.5}$$

Since conditions (4.2), (4.4) and (4.5) for the operator $T = W^0(b)$ represented in the form (4.3) are fulfilled, we infer from [12, Theorem 7.5.6] that there exists a constant $C \in (0, \infty)$ such that

$$\big\|[aI, W^0(b)]\big\|_{\mathcal{B}_2} \le C\|a\|_* \tag{4.6}$$

for every $a \in BMO(\mathbb{R})$, where $\mathcal{B}_2 = \mathcal{B}(L^2(\mathbb{R}))$ and $\|\cdot\|_*$ is given by (2.1). On the other hand, by Theorem 2.4, every function $a \in QC$ belongs to the Banach space VMO. Hence, in view of Proposition 2.1, for every $a \in QC$ there exists a sequence $\{a_n\} \in C(\dot{\mathbb{R}})$ such that $\lim_{n\to\infty} \|a - a_n\|_* = 0$, and therefore, by (4.6),

$$\lim_{n\to\infty} \big\|[aI, W^0(b)] - [a_nI, W^0(b)]\big\|_{\mathcal{B}_2} = \lim_{n\to\infty} \big\|[(a-a_n)I, W^0(b)]\big\|_{\mathcal{B}_2} = 0. \tag{4.7}$$

But $[a_nI, W^0(b)] \in \mathcal{K}_2$ for all $a_n \in C(\dot{\mathbb{R}})$ and all $b \in SO_\lambda$ $(\lambda \in \dot{\mathbb{R}})$ in virtue of Theorem 4.3. Thus, we deduce from (4.7) that the commutator $[aI, W^0(b)]$ is compact on the space $L^2(\mathbb{R})$ for every $a \in QC$ and every $b \in SO_\lambda$ with $\lambda \in \{0, \infty\}$. Note that the compactness of the commutator $[aI, W^0(b)]$ for such a, b also follows from [26, Theorem 2] because $QC \subset VMO$ and $W^0(b)$ is a classical Calderón-Zygmund operator.

Let $e_\mu(x) := e^{i\mu x}$ for all $\mu, x \in \mathbb{R}$. The case $a \in QC$ and $b \in SO_\lambda$ $(\lambda \in \mathbb{R} \setminus \{0\})$ is reduced to the previous one for $\lambda = 0$ according to the equality

$$e_\lambda[aI, W^0(b)]e_{-\lambda}I = [aI, W^0(b_0)],$$

where $b_0(x) = b(x+\lambda)$ for $x \in \mathbb{R}$ and hence $b_0 \in SO_0$, which completes the proof of part (i').

Case (iii'). Since $a, b \in PQC$ and $a \overset{\infty}{\sim} c \overset{\infty}{\sim} \widetilde{c}$, $b \overset{\infty}{\sim} d \overset{\infty}{\sim} \widetilde{d}$, where $c, d \in SO^\diamond$ and $\widetilde{c}, \widetilde{d} \in SO_\infty$, we conclude that

$$a = \widetilde{c} + (a - \widetilde{c}), \quad b = \widetilde{d} + (b - \widetilde{d}), \quad a - \widetilde{c},\, b - \widetilde{d} \in QC, \tag{4.8}$$

and, according to (4.1),

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|\ge N} |a(x) - \widetilde{c}(x)| = 0, \quad \lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|\ge N} |b(x) - \widetilde{d}(x)| = 0. \tag{4.9}$$

By (4.8), the commutator $[aI, W^0(b)]$ is represented in the form

$$[aI, W^0(b)] = [\widetilde{c}I, W^0(\widetilde{d})] + [\widetilde{c}I, W^0(b-\widetilde{d})] + [(a-\widetilde{c})I, W^0(\widetilde{d})] + [(a-\widetilde{c})I, W^0(b-\widetilde{d})]. \tag{4.10}$$

By Theorem 4.2, the commutator $[\widetilde{c}I, W^0(\widetilde{d})]$ with $\widetilde{c}, \widetilde{d} \in SO_\infty$ is compact on the space $L^2(\mathbb{R})$. By part (i'), the commutator $[(a-\widetilde{c})I, W^0(\widetilde{d})]$ is also compact on $L^2(\mathbb{R})$ because $a - \widetilde{c} \in QC$ and $\widetilde{d} \in SO_\infty$. This implies due to part (ii'), which is equivalent to part (i'), that the commutator $[\widetilde{c}I, W^0(b-\widetilde{d})]$ with $\widetilde{c} \in SO_\infty$ and $b - \widetilde{d} \in QC$ is also compact on $L^2(\mathbb{R})$.

Finally, in view of (4.10), it remains to prove the compactness on $L^2(\mathbb{R})$ of the commutator $[(a-\widetilde{c})I, W^0(b-\widetilde{d})]$ with functions $a - \widetilde{c}, b - \widetilde{d} \in QC$ that vanish at ∞. We infer from (4.9) that

$$\big\|(a-\widetilde{c})(1-\widetilde{\chi}_n)\big\|_{L^\infty(\mathbb{R})} = 0, \quad \big\|(b-\widetilde{d})(1-\widetilde{\chi}_n)\big\|_{L^\infty(\mathbb{R})} = 0, \tag{4.11}$$

where the functions $\widetilde{\chi}_n \in C(\dot{\mathbb{R}})$ for $n \in \mathbb{N}$ are given by

$$\widetilde{\chi}_n(x) = \begin{cases} 1 & \text{if } |x| \le n, \\ n+1-|x| & \text{if } n < |x| < n+1, \\ 0 & \text{if } |x| \ge n+1. \end{cases}$$

Then from (4.11) it follows that

$$[(a-\widetilde{c})I, W^0(b-\widetilde{d})] = \lim_{n\to\infty} [(a-\widetilde{c})\widetilde{\chi}_nI, W^0(\widetilde{\chi}_n(b-\widetilde{d}))], \tag{4.12}$$

where the limit is taken in the operator norm. Since

$$[(a-\widetilde{c})\widetilde{\chi}_n I, W^0(\widetilde{\chi}_n(b-\widetilde{d}))] = (a-\widetilde{c})(\widetilde{\chi}_n W^0(\widetilde{\chi}_n))W^0(b-\widetilde{d}) - W^0(b-\widetilde{d})(W^0(\widetilde{\chi}_n)\widetilde{\chi}_n I)(a-\widetilde{c})I,$$

and since the operators $\widetilde{\chi}_n W^0(\widetilde{\chi}_n)$ and $W^0(\widetilde{\chi}_n)\widetilde{\chi}_n I$ are compact on the space $L^2(\mathbb{R})$ due to Lemma 4.1(a), we obtain the compactness of all commutators

$$[(a-\widetilde{c})\widetilde{\chi}_n I, W^0(\widetilde{\chi}_n(b-\widetilde{d}))] \quad (n \in \mathbb{N}).$$

Then from (4.12) it follows that the commutator $[(a-\widetilde{c})I, W^0(b-\widetilde{d})]$ is also compact on the space $L^2(\mathbb{R})$, which completes the proof of part (iii').

Case (iv'). The compactness of the commutator $[aI, W^0(b)]$ on the space $L^2(\mathbb{R})$ for $a \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$ and $b \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$ follows from the same property for the pairs: $a \in QC$ and $b \in SO^\diamond$, $a \in QC$ and $b \in C(\overline{\mathbb{R}})$, $a \in C(\overline{\mathbb{R}})$ and $b \in SO^\diamond$, and $a, b \in C(\overline{\mathbb{R}})$. For $a \in QC$ and $b \in SO^\diamond$, this was proved in part (i'), for $a \in C(\overline{\mathbb{R}})$ and $b \in SO^\diamond$ this follows from Theorem 4.3, for $a, b \in C(\overline{\mathbb{R}})$ this is given by Lemma 4.1(c).

Thus, it remains to prove the compactness of the commutator $[aI, W^0(b)]$ for $a \in QC$ and $b \in C(\overline{\mathbb{R}})$. Given $b \in C(\overline{\mathbb{R}})$, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ of piecewise constant functions with finite sets of discontinuities that uniformly converges to b in $L^\infty(\mathbb{R})$. Then

$$[aI, W^0(b)] = \lim_{n\to\infty} [aI, W^0(b_n)],$$

and therefore the compactness of the commutator $[aI, W^0(b)]$ on $L^2(\mathbb{R})$ will follow from the compactness of the commutators $[aI, W^0(b_n)]$. Since every function b_n is of the form

$$b_n(x) = \sum_{k=1}^{m} c_k \operatorname{sgn}(x - t_k) \quad (x \in \mathbb{R}),$$

where c_k are complex constants and $-\infty < t_1 < t_2 < \ldots < t_m < +\infty$, we conclude from the equality $W^0(\operatorname{sgn}((\cdot) - t_k)) = -e_{-t_k} S_{\mathbb{R}} e_{t_k} I$ that

$$[aI, W^0(b_n)] = -\sum_{k=1}^{m} c_k e_{-t_k} [aI, S_{\mathbb{R}}] e_{t_k} I. \tag{4.13}$$

Because $a \in QC = (H^\infty + C(\dot{\mathbb{R}})) \cap (\overline{H^\infty} + C(\dot{\mathbb{R}}))$ in view of Theorem 2.4, it immediately follows from the Hartman compactness result (see, e.g., [7, Theorem 2.18]) that $[aI, S_{\mathbb{R}}] \in \mathcal{K}_2$ (also see [21, Section 2]). Consequently, we conclude from (4.13) that the commutators $[aI, W^0(b_n)]$ are compact on the space $L^2(\mathbb{R})$, which completes the proof of part (iv').

Case (vi'). By analogy with part (iii'), if $a, b \in \operatorname{alg}(QC, C(\overline{\mathbb{R}}))$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ and $c, d \in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$, then there are functions $\widetilde{c}, \widetilde{d} \in \operatorname{alg}(SO_\infty, C(\overline{\mathbb{R}}))$ such that $a \overset{\infty}{\sim} \widetilde{c}$, $b \overset{\infty}{\sim} \widetilde{d}$. Then we infer from (4.8) and (4.10) that the commutator $[aI, W^0(b)]$ will be compact on $L^2(\mathbb{R})$ if the following commutators will be compact:

1) $[\widetilde{c}I, W^0(\widetilde{d})]$ with $\widetilde{c}, \widetilde{d} \in \operatorname{alg}(SO_\infty, C(\overline{\mathbb{R}}))$,

2) $[\widetilde{c}I, W^0(b - \widetilde{d})]$ with $\widetilde{c} \in \operatorname{alg}(SO_\infty, C(\overline{\mathbb{R}}))$ and $b - \widetilde{d} \in QC$,

3) $[(a-\widetilde{c})I, W^0(\widetilde{d})]$ with $a-\widetilde{c}\in QC$ and $\widetilde{d}\in \operatorname{alg}(SO_\infty, C(\overline{\mathbb{R}}))$,

4) $[(a-\widetilde{c})I, W^0(b-\widetilde{d})]$ with $a-\widetilde{c}, b-\widetilde{d}\in QC$ that satisfy (4.9).

Case 1) is covered by Theorem 4.2, case 2) was considered in part (iv'), case 3) is reduced to case 2) under the transform $A\mapsto \mathcal{F}A\mathcal{F}^{-1}$, and case 4) was treated in part (iii'). Consequently, the commutator $[aI, W^0(b)]$ is compact on $L^2(\mathbb{R})$ under conditions (vi') as well, which completes the proof of the theorem. □

Open problem. Let $p\in(1,\infty)$ and $w\in A_p(\mathbb{R})$. Is the commutator $[aI, W^0(b)]$ compact on the space $L^p(\mathbb{R}, w)$ for all $a, b\in QC$?

5 Fredholm study of the commutative C^*-algebras Z_1 and Z_2

Let $p=2$ and $w=1$. Consider the C^*-subalgebras

$$Z_1 := \operatorname{alg}\big(aI, W^0(b):\ a,b\in PQC,\ a\overset{\infty}{\sim}c,\ b\overset{\infty}{\sim}d,\ c,d\in SO^\diamond\big), \tag{5.1}$$

$$Z_2 := \operatorname{alg}\big(aI, W^0(b):\ a,b\in QC,\ a\overset{\infty}{\sim}c,\ b\overset{\infty}{\sim}d,\ c,d\in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))\big) \tag{5.2}$$

of the C^*-algebra $\mathcal{B}_2=\mathcal{B}(L^2(\mathbb{R}))$ generated by the operators aI and $W^0(b)$ with corresponding data $a,b\in PQC$ or $a,b\in QC$. As is known (see, e.g., [17, Lemma 6.1]), the ideal $\mathcal{K}:=\mathcal{K}(L^2(\mathbb{R}))$ of compact operators is contained in both the C^*-algebras Z_1 and Z_2. By Theorem 4.4, the quotient C^*-algebras $Z_i^\pi := Z_i/\mathcal{K}$ $(i=1,2)$ are commutative.

Let $e_\lambda(x)=e^{i\lambda x}$ for all $\lambda, x\in\mathbb{R}$, and let $U_\lambda = W^0(e_\lambda)$ be the translation operator acting by the rule $(U_\lambda f)(x) = f(x-\lambda)$ for $x\in\mathbb{R}$.

To study the maximal ideal spaces of the commutative C^*-algebras $Z_i^\pi := Z_i/\mathcal{K}$ $(i=1,2)$ we need the following two evident results on limit operators (see, e.g., [17, Lemma 5.1]).

Lemma 5.1. *If $p=2$, and $a,b\in SO^\diamond$, then for every $\xi\in M_\infty(SO^\diamond)$ there is a sequence $\{h_n\}\subset(0,\infty)$ such that $\lim_{n\to\infty} h_n=+\infty$, $\lim_{n\to\infty} a(h_n)=a(\xi)$, $\lim_{n\to\infty} b(h_n)=b(\xi)$ and on $L^2(\mathbb{R})$,*

$$\operatorname*{s-lim}_{n\to\infty}(e_{h_n}(aI)e_{h_n}^{-1}I) = aI,\quad \operatorname*{s-lim}_{n\to\infty}(e_{h_n}W^0(b)e_{h_n}^{-1}I) = b(\xi)I, \tag{5.3}$$

$$\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}(aI)U_{h_n}) = a(\xi)I,\quad \operatorname*{s-lim}_{n\to\infty}(U_{h_n}(aI)U_{-h_n}) = a(\xi)I, \tag{5.4}$$

$$\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}W^0(b)U_{h_n}) = W^0(b),\quad \operatorname*{s-lim}_{n\to\infty}(U_{h_n}W^0(b)U_{-h_n}) = W^0(b). \tag{5.5}$$

Lemma 5.2. *If $p=2$, and $a,b\in \operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}}))$, then for every $\xi^\pm\in M_\infty(\operatorname{alg}(SO^\diamond, C(\overline{\mathbb{R}})))$ there is a sequence $\{h_n\}\subset(0,\infty)$ such that $\lim_{n\to\infty} h_n=+\infty$, $\lim_{n\to\infty} a(\mp h_n)=a(\xi^\pm)$, $\lim_{n\to\infty} b(\mp h_n)=b(\xi^\pm)$ and, on the space $L^2(\mathbb{R})$,*

$$\operatorname*{s-lim}_{n\to\infty}(e_{h_n}(aI)e_{h_n}^{-1}I) = aI,\quad \operatorname*{s-lim}_{n\to\infty}(e_{\mp h_n}W^0(b)e_{\mp h_n}^{-1}I) = b(\xi^\pm)I,$$

$$\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}(aI)U_{h_n}) = a(\xi^-)I,\quad \operatorname*{s-lim}_{n\to\infty}(U_{h_n}(aI)U_{-h_n}) = a(\xi^+)I,$$

$$\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}W^0(b)U_{h_n}) = W^0(b),\quad \operatorname*{s-lim}_{n\to\infty}(U_{h_n}W^0(b)U_{-h_n}) = W^0(b).$$

We identify the fibers $M_\lambda(QC)$ and $M_\tau(QC(\mathbb{T}))$, where $\tau = (\lambda - i)/(\lambda + i)$, by the rule $\xi \in M_\lambda(QC) \mapsto \zeta \in M_\tau(QC(\mathbb{T}))$, which implies the identification of the fibers $M_\xi(PQC)$ and $M_\zeta(PQC(\mathbb{T}))$. Thus, the fibers $M_\xi(PQC)$ for $\xi \in M(QC)$ are actually described by Theorem 3.5.

Theorem 5.3. *The maximal ideal space $M(Z_1^\pi)$ of the commutative quotient C^*-algebra Z_1^π is homeomorphic to the set*

$$\Omega_1 := \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \times M_\infty(SO^\diamond)\Big) \cup \Big(M_\infty(SO^\diamond) \times \bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC)\Big) \cup \Big(M_\infty(SO^\diamond) \times M_\infty(SO^\diamond)\Big) \tag{5.6}$$

equipped with topology induced by the product topology of

$$\Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond)\Big) \times \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond)\Big),$$

where $M_\lambda(PQC) = \bigcup_{\xi\in M_\lambda(QC)} M_\xi(PQC)$. The Gelfand transform $\Gamma_1 : Z_1^\pi \to C(\Omega_1)$, $A^\pi \mapsto \mathcal{A}(\cdot,\cdot)$ is defined on the generators $A^\pi = (aW^0(b))^\pi$ of the algebra Z_1^π, where $a,b \in PQC$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ and $c,d \in SO^\diamond$, by

$$\mathcal{A}(\xi,\eta) = a(\xi)b(\eta) \quad \textit{for all} \quad (\xi,\eta) \in \Omega_1. \tag{5.7}$$

Proof. If J is a maximal ideal of the commutative C^*-algebra Z_1^π, then

$$J \cap \{aI + \mathcal{K} :\ a \in PQC,\ a \overset{\infty}{\sim} c,\ c \in SO^\diamond\} \ \text{ and } \ J \cap \{W^0(b) + \mathcal{K} :\ b \in PQC,\ b \overset{\infty}{\sim} d,\ d \in SO^\diamond\}$$

are maximal ideals of the commutative C^*-algebras

$$\{aI + \mathcal{K} :\ a \in PQC,\ a \overset{\infty}{\sim} c,\ c \in SO^\diamond\} \ \text{ and } \ \{W^0(b) + \mathcal{K} :\ b \in PQC,\ b \overset{\infty}{\sim} d,\ d \in SO^\diamond\}, \tag{5.8}$$

respectively (see [9, Lemma 1.33]). Therefore, taking into account the relations

$$\begin{aligned} M(\{aI + \mathcal{K} :\ a \in PQC,\ a \overset{\infty}{\sim} c,\ c \in SO^\diamond\}) &= \bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond), \\ M(\{W^0(b) + \mathcal{K} :\ b \in PQC,\ b \overset{\infty}{\sim} d,\ d \in SO^\diamond\}) &= \bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond), \end{aligned} \tag{5.9}$$

we conclude that for every point

$$(\xi,\eta) \in \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond)\Big) \times \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC) \cup M_\infty(SO^\diamond)\Big),$$

there exists the closed two-sided (not necessarily maximal) ideal $I_{\xi,\eta}^\pi$ of the C^*-algebra Z_1^π generated by the maximal ideals

$$\begin{gathered} \{aI + \mathcal{K} :\ a \in PQC,\ a \overset{\infty}{\sim} c,\ c \in SO^\diamond,\ \xi(a) = 0\}, \\ \{W^0(b) + \mathcal{K} :\ b \in PQC,\ b \overset{\infty}{\sim} d,\ d \in SO^\diamond,\ \eta(b) = 0\} \end{gathered} \tag{5.10}$$

of the commutative C^*-algebras (5.8), respectively. Thus, in virtue of (5.9), the maximal ideal space of Z_1^π can be identified with a subset of

$$\Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC)\cup M_\infty(SO^\diamond)\Big)\times\Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC)\cup M_\infty(SO^\diamond)\Big).$$

Fix $(\xi,\eta)\in\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC)\times\bigcup_{\tau\in\mathbb{R}} M_\tau(PQC)$. Then $\xi\in M_\lambda(PQC)$ and $\eta\in M_\tau(PQC)$ for some $\lambda,\tau\in\mathbb{R}$. Given $a,b\in PQC$, we choose functions $a_1,b_1\in C(\dot{\mathbb{R}})$ such that $a_1(\lambda)=a(\xi)$, $b_1(\tau)=b(\eta)$, and $a_1(\infty)=b_1(\infty)=0$. Then

$$aW^0(b)=T_1+T_2+T_3+T_4, \tag{5.11}$$

where

$$T_1=(a-a_1)W^0(b-b_1),\ T_2=(a-a_1)W^0(b_1),\ T_3=a_1W^0(b-b_1),\ T_4=a_1W^0(b_1).$$

The operator T_4 is compact by Lemma 4.1(a), and the cosets T_1^π,T_2^π,T_3^π belong to the ideal $I_{\xi,\eta}^\pi$. Thus, the smallest closed two-sided ideal of Z_1^π which corresponds to the point $(\xi,\eta)\in\bigcup_{\lambda\in\mathbb{R}} M_\lambda(PQC)\times\bigcup_{\tau\in\mathbb{R}} M_\tau(PQC)$ coincides with the whole C^*-algebra Z_1^π, and therefore the ideal $I_{\xi,\eta}^\pi$ is not maximal. So, the maximal ideals of the commutative C^*-algebra Z_1^π can only correspond to points $(\xi,\eta)\in\Omega_1$, where Ω_1 is given by (5.6).

It remains to show that for all $(\xi,\eta)\in\Omega_1$, the closed two-sided ideals $I_{\xi,\eta}^\pi$ generated by the maximal ideals (5.10) are maximal ideals of the commutative C^*-algebra Z_1^π.

First, let us prove that these ideals are proper. To this end we need to show that for all $(\xi,\eta)\in\Omega_1$ the ideals $I_{\xi,\eta}^\pi$ do not contain the coset $I^\pi=I+\mathcal{K}$. By [22, Proposition 2.2.5], the ideals $I_{\xi,\eta}^\pi$ consist of the cosets

$$[aI]^\pi A^\pi+[W^0(b)]^\pi B^\pi, \tag{5.12}$$

where

$$a,b\in PQC,\ \ a\overset{\infty}{\sim}\widetilde{c},\ \ b\overset{\infty}{\sim}\widetilde{d},\ \ \widetilde{c},\widetilde{d}\in SO_\infty,\ \ \xi(a)=0,\ \ \eta(b)=0,\ \ A,B\in Z_1. \tag{5.13}$$

Given $\lambda\in\dot{\mathbb{R}}$, let $(\xi,\eta)\in M_\lambda(PQC)\times M_\infty(SO^\diamond)$. Assume that $I^\pi\in I_{\xi,\eta}^\pi$. Hence, by (5.12),

$$I=aA+W^0(b)B+K, \tag{5.14}$$

where (5.13) holds and $K\in\mathcal{K}$. Since for every $\eta\in M_\infty(SO^\diamond)=M_\infty(SO_\infty)$ and every $\widetilde{d}\in SO_\infty$ there is a sequence $h_n\to+\infty$ in $\mathbb{R}$ such that $\lim\limits_{n\to\infty}\widetilde{d}(h_n)=\eta(\widetilde{d})$ (see, e.g., [3, Proposition 6]), and therefore

$$\lim_{n\to\infty} b(x+h_n)=\lim_{n\to\infty}\widetilde{d}(x+h_n)=\eta(\widetilde{d})=\eta(b)=0$$

for almost all $x\in\mathbb{R}$, we conclude from (5.3) that

$$\operatorname*{s-lim}_{n\to\infty}(e_{h_n}W^0(b)e_{-h_n}I)=0. \tag{5.15}$$

Moreover, from (5.14), the algebraic properties of limit operators (see [6, Proposition 6.1]) and [7, Lemma 10.1] it follows that we can choose the sequence $\{h_n\}$ in such a way that there exist the strong limits

$$\operatorname*{s-lim}_{n\to\infty}(e_{h_n}Ae_{-h_n}I)=\widetilde{a}I \quad (\widetilde{a}\in PQC), \quad \operatorname*{s-lim}_{n\to\infty}(e_{h_n}Ke_{-h_n}I)=0. \tag{5.16}$$

Consequently, by (5.15) and (5.16), we obtain

$$I=\operatorname*{s-lim}_{n\to\infty}(e_{h_n}(aA+W^0(b)B+K)e_{-h_n}I)=a\widetilde{a}I,$$

which is impossible because $\xi(a)=0$ and therefore $a\widetilde{a}\neq 1$.

Given $\lambda\in\mathbb{R}$, let now $(\xi,\eta)\in M_\infty(SO^\diamond)\times M_\lambda(PQC)$, and we again assume that $I^\pi\in I^\pi_{\xi,\eta}$. Then we have (5.14), where (5.13) holds and $K\in\mathcal{K}$.

Since for every $\xi\in M_\infty(SO^\diamond)=M_\infty(SO_\infty)$ and every $\widetilde{c}\in SO_\infty$ there is a sequence $\{h_n\}\subset\mathbb{R}$ such that $\lim_{n\to\infty}h_n=+\infty$, $\lim_{n\to\infty}\widetilde{c}(h_n)=\xi(\widetilde{c})$, and hence

$$\lim_{n\to\infty}a(x+h_n)=\lim_{n\to\infty}\widetilde{c}(x+h_n)=\xi(\widetilde{c})=\xi(a)=0$$

for almost all $x\in\mathbb{R}$, we conclude from (5.4) that

$$\operatorname*{s-lim}_{\nu\to\infty}(U_{-h_n}(aI)U_{h_n})=0, \tag{5.17}$$

where $U_{h_n}=W^0(e_{h_n})$ is a translation operator. On the other hand, we infer from (5.5) that

$$\operatorname*{s-lim}_{\nu\to\infty}(U_{-h_n}W^0(b)U_{h_n})=W^0(b).$$

Using then (5.14), the algebraic properties of limit operators (see [6, Proposition 6.1]) and [7, Lemma 18.9], we can choose the sequence $\{h_n\}$ in such a way that there exists the strong limits

$$\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}BU_{h_n})=W^0(\widetilde{b}) \quad (\widetilde{b}\in PQC), \quad \operatorname*{s-lim}_{n\to\infty}(U_{-h_n}KU_{h_n})=0. \tag{5.18}$$

Then from (5.17) and (5.18), we obtain

$$I=\operatorname*{s-lim}_{n\to\infty}(U_{-h_n}(aA+W^0(b)B+K)U_{h_n}I)=W^0(b)W^0(\widetilde{b})=W^0(b\widetilde{b}),$$

which is impossible because $\eta(b)=0$ and therefore $b\widetilde{b}\neq 1$.

Thus, for all $(\xi,\eta)\in\Omega_1$ the ideals $I^\pi_{\xi,\eta}$ do not contain the unit coset I^π, and hence these ideals are proper. Suppose, contrary to our claim on the maximality of the ideal $I^\pi_{\xi,\eta}$, that for a point $(\xi,\eta)\in\Omega_1$ there is a proper closed two-sided ideal $\widetilde{I}^\pi_{\xi,\eta}$ of the algebra Z^π_1 that properly contains the ideal $I^\pi_{\xi,\eta}$. Then there is a coset $A^\pi\in Z^\pi_1$ which belongs to $\widetilde{I}^\pi_{\xi,\eta}\setminus I^\pi_{\xi,\eta}$. Since in view of (5.11),

$$(aW^0(b))^\pi-(a(\xi)W^0(b(\eta)))^\pi=(aW^0(b))^\pi-(a(\xi)b(\eta)I)^\pi\in I^\pi_{\xi,\eta} \tag{5.19}$$

for all $a,b\in PQC$ such that $a\overset{\infty}{\sim}c$, $b\overset{\infty}{\sim}d$ and $c,d\in SO^\diamond$, and since $A^\pi\notin I^\pi_{\xi,\eta}$, there exists a complex number $\nu\neq 0$ such that $A^\pi-(\nu I)^\pi\in I^\pi_{\xi,\eta}$. Hence $(\nu I)^\pi\in\widetilde{I}^\pi_{\xi,\eta}$ because $A^\pi\in\widetilde{I}^\pi_{\xi,\eta}$ and

$I^\pi_{\xi,\eta} \subset \widetilde{I}^\pi_{\xi,\eta}$. But the coset $(vI)^\pi$ is invertible in the algebra Z^π_1, which implies that the ideal $\widetilde{I}^\pi_{\xi,\eta}$ coincides with the whole algebra Z^π_1. Thus the ideal $\widetilde{I}^\pi_{\xi,\eta}$ is not proper, a contradiction. Consequently, all the ideals $I^\pi_{\xi,\eta}$ for $(\xi,\eta) \in \Omega_1$ are maximal, and therefore $M(Z^\pi_1)$ can be identified with Ω_1 given by (5.6).

Furthermore, by (5.19), the value of the Gelfand transform of the coset $A^\pi = (aW^0(b))^\pi$ at a point $(\xi,\eta) \in \Omega_1$ equals $a(\xi)b(\eta)$ for each choice of functions $a, b \in PQC$ being equivalent to functions $c, d \in SO^\diamond$ at ∞. This defines the Gelfand transform for the whole algebra Z^π_1 by formula (5.7). □

Making use of the equality $M_\infty(\mathrm{alg}\,(SO^\diamond, C(\overline{\mathbb{R}}))) = M_\infty(PSO^\diamond)$ and applying Lemma 5.2 instead of Lemma 5.1, we obtain the following result by analogy with Theorem 5.3.

Theorem 5.4. *The maximal ideal space $M(Z^\pi_2)$ of the commutative quotient C^*-algebra Z^π_2 is homeomorphic to the set*

$$\Omega_2 := \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(QC) \times M_\infty(PSO^\diamond)\Big) \cup \Big(M_\infty(PSO^\diamond) \times \bigcup_{\lambda\in\mathbb{R}} M_\lambda(QC)\Big) \cup \Big(M_\infty(PSO^\diamond) \times M_\infty(PSO^\diamond)\Big)$$

equipped with topology induced by the product topology of

$$\Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(QC) \cup M_\infty(PSO^\diamond)\Big) \times \Big(\bigcup_{\lambda\in\mathbb{R}} M_\lambda(QC) \cup M_\infty(PSO^\diamond)\Big),$$

and the Gelfand transform $\Gamma_2 : Z^\pi_2 \to C(\Omega_2)$, $A^\pi \mapsto \mathcal{A}(\cdot,\cdot)$ is defined on the generators $A^\pi = (aW^0(b))^\pi$ of the algebra Z^π_2, where $a, b \in QC$, $a \overset{\infty}{\sim} c$, $b \overset{\infty}{\sim} d$ and $c, d \in \mathrm{alg}\,(SO^\diamond, C(\overline{\mathbb{R}}))$, by

$$\mathcal{A}(\xi,\eta) = a(\xi)b(\eta) \quad \textit{for all} \quad (\xi,\eta) \in \Omega_2.$$

Theorems 5.3 and 5.4 imply the following Fredholm criteria for the C^*-algebras Z_1 and Z_2 given by (5.1) and (5.2), respectively.

Corollary 5.5. *An operator $A \in Z_1$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if the Gelfand transform of the coset A^π is invertible, that is, if $\mathcal{A}(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \Omega_1$.*

Corollary 5.6. *An operator $A \in Z_2$ is Fredholm on the space $L^2(\mathbb{R})$ if and only if the Gelfand transform of the coset A^π is invertible, that is, if $\mathcal{A}(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \Omega_2$.*

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References

[1] M. A. Bastos, A. Bravo, and Yu. I. Karlovich, Convolution type operators with symbols generated by slowly oscillating and piecewise continuous matrix functions. In: *Operator Theoretical Methods and Applications to Mathematical Physics. The Erhard Meister Memorial Volume*, Oper. Theory: Adv. Appl. **147** (2004), pp 151-174.

[2] M. A. Bastos, C. A. Fernandes, and Yu. I. Karlovich, C^*-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely. *Integr. Equ. Oper. Theory* **55** (2006), pp 19-67.

[3] M. A. Bastos, Yu. I. Karlovich, and B. Silbermann, Toeplitz operators with symbols generated by slowly oscillating and semi-almost periodic matrix functions. *Proc. London Math. Soc.* **89** (2004), pp 697-737.

[4] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin 1976.

[5] A. Böttcher and Yu. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Progr. Math. **154**, Birkhäuser, Basel 1997.

[6] A. Böttcher, Yu. I. Karlovich, and V. S. Rabinovich, The method of limit operators for one-dimensional singular integrals with slowly oscillating data. *J. Operator Theory* **43** (2000), pp 171-198.

[7] A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Oper. Theory: Adv. Appl. **131**, Birkhäuser, Basel 2002.

[8] A. Böttcher and I. M. Spitkovsky, Wiener-Hopf integral operators with *PC* symbols on spaces with Muckenhoupt weight. *Revista Matemática Iberoamericana* **9** (1993), pp 257-279.

[9] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, 2nd edition, Springer, Berlin 2006.

[10] R. V. Duduchava, *Integral Equations with Fixed Singularities*, Teubner, Leipzig 1979.

[11] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York 1981.

[12] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson/Prentice Hall, Upper Saddle River, NJ, 2004.

[13] N. B. Haaser and J. A. Sullivan, *Real Analysis*, Dover Publications, New York 1991.

[14] L. Hörmander, Estimates for translation invariant operators in L^p spaces. *Acta Math.* **104** (1960), pp 93-140.

[15] Yu. I. Karlovich, Boundedness and compactness of pseudodifferential operators with non-regular symbols on weighted Lebesgue spaces. *Integr. Equ. Oper. Theory* **73** (2012), pp 217-254.

[16] Yu. I. Karlovich and I. Loreto Hernández, Algebras of convolution type operators with piecewise slowly oscillating data. I: Local and structural study. *Integr. Equ. Oper. Theory* **74** (2012), pp 377-415.

[17] Yu. I. Karlovich and I. Loreto Hernández, On convolution type operators with piecewise slowly oscillating data. In: *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics. The Vladimir Rabinovich Anniversary Volume*, Oper. Theory: Adv. Appl. **228** (2013), pp 185-207.

[18] Yu. I. Karlovich and I. Loreto Hernández, Algebras of convolution type operators with piecewise slowly oscillating data. II: Local spectra and Fredholmness. *Integr. Equ. Oper. Theory* **75** (2013), pp 49-86.

[19] Yu. I. Karlovich and J. Loreto Hernández, Wiener-Hopf operators with slowly oscillating matrix symbols on weighted Lebesgue spaces. *Integr. Equ. Oper. Theory* **64** (2009), pp 203-237.

[20] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik, and P. E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Nauka, Moscow 1966 (Russian); English transl.: Noordhoff I.P., Leyden 1976.

[21] S. C. Power, Hankel Operators on Hilbert Space. *Bull. London Math. Soc.* **12** (1980), pp 422-442.

[22] S. Roch, P. A. Santos, and B. Silbermann, *Non-commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts*, Springer, London 2011.

[23] D. Sarason, Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* **207** (1975), pp 391-405.

[24] D. Sarason, Toeplitz operators with piecewise quasicontinuous symbols. *Indiana Univ. Math. J.* **26** (1977), pp 817-838.

[25] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[26] A. Uchiyama, On the compactness of operators of Hankel type. *Tôhoku Math. Journ.* **30** (1978), pp 163-171.