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An Inverse Problem for Differential Operators on Hedgehog-type Graphs with General Matching Conditions

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Abstract

An inverse spectral problem is studied for Sturm-Liouville differential operators on hedgehog-type graphs with generalized matching conditions in the interior vertices and with Dirichlet boundary conditions in the boundary vertices. A uniqueness theorem of recovering potentials from given spectral characteristics is provided, and a constructive solution for the inverse problem is obtained.

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1 Introduction

We study an inverse spectral problem for second-order differential operators on the so-called hedgehog-type graphs with general matching conditions in the interior vertices and

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with Dirichlet boundary conditions in the boundary vertices. Direct and inverse problems for differential operators on graphs (spatial networks) often appear in natural sciences and engineering (see [1], [2], [4], [7] and the references therein). We note that inverse spectral problems of recovering *coefficients* of differential operators *on trees* (i.e on graphs without cycles) were solved in [1], [7]. Inverse problems for Sturm-Liouville operators on graphs with a cycle were studied in [8]–[10] and other papers but only in the case of the so-called *standard matching conditions.* In particular, in this case the uniqueness result was obtained in [9] for hedgehog-type graphs. In the present paper we consider Sturm-Liouville operators on hedgehog-type graphs with generalized matching conditions (see Section 2 for definitions). This class of matching conditions appears in applications and produces new qualitative difficulties in investigating nonlinear inverse coefficient problems. For studying this class of inverse problems we develop the ideas of the method of spectral mappings [3], [6]. We prove a uniqueness theorem for this class of nonlinear inverse problems, and provide a constructive procedure for their solutions. In order to construct the solution, we solve, in particular, an important auxiliary inverse problem for a quasi-periodic boundary value problem on the cycle with discontinuity conditions in interior points.

2 Statement of the inverse problem

Consider a compact graph G in $\mathbb{R}^m$ with the set of edges $\mathcal{E} = \{e_0, \dots, e_r\}$, where e_0 is a cycle, $\mathcal{E}' = \{e_1, \dots, e_r\}$ are boundary edges. Let $\{v_1, \dots, v_{r+N}\}$ be the set of vertices, where $V = \{v_1, \dots, v_r\}$, $v_k \in e_k$, are boundary vertices, and $U = \{v_{r+1}, \dots, v_{r+N}\}$ are internal vertices, $U = \mathcal{E}' \cap e_0$. The cycle e_0 consists of N parts: $e_0 = e_{r+1} \cup \dots \cup e_{r+N}$, $e_{r+k} = [v_{r+k}, v_{r+k+1}]$, $k = \overline{1,N}$, $v_{r+N+1} := v_{r+1}$. Each boundary edge e_j, $j = \overline{1,r}$, has the initial point at v_j, and the end point in the set U. The set $\mathcal{E}'$ consists of N groups of edges: $\mathcal{E}' = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_N$, $\mathcal{E}_k \cap e_0 = v_{r+k}$. Let r_k be the number of edges in $\mathcal{E}_k$; hence $r = r_1 + \dots + r_N$. Denote $m_0 = 1$, $m_k = r_1 + \dots + r_k$, $k = \overline{1,N}$. Then $\mathcal{E}_k = \{e_j\}$, $j = \overline{m_{k-1} + 1, m_k}$. Thus, the boundary edge $e_j \in \mathcal{E}_k$ can be viewed as the segment $e_j = [v_j, v_{r+k}]$. For example, a graph G with $N = 3$ and $r = 4$ is on Figure 1.

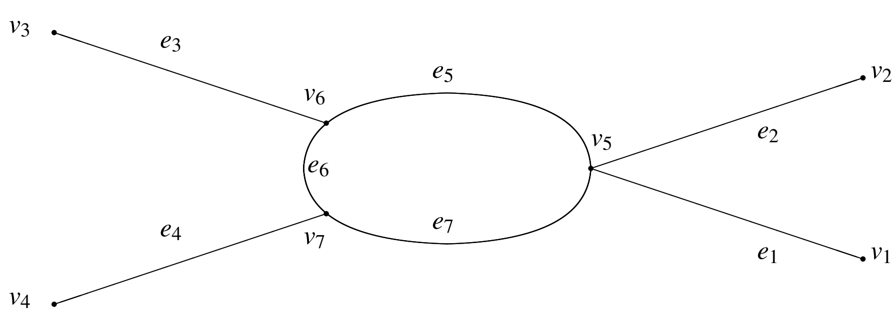

Figure 1

Let T_j be the length of the edge e_j, $j = \overline{1, r+N}$, and let $T := T_{r+1} + \dots + T_{r+N}$ be the length of the cycle e_0. Put $b_0 = 0$, $b_k = T_{r+1} + \dots + T_{r+k}$, $k = \overline{1,N}$. Then $b_N = T$. Each edge e_j, $j = \overline{1, r+N}$ is parameterized by the parameter $x_j \in [0, T_j]$, and $x_j = 0$ corresponds to the vertex v_j. The whole cycle e_0 is parameterized by the parameter $x \in [0, T]$, where $x = x_{r+j} + b_{j-1}$ for $x_{r+j} \in [0, T_{r+j}]$, $j = \overline{1,N}$.

An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{1,r+N}}$, where the function $y_j(x_j)$, $x_j \in [0, T_j]$, is defined on the edge e_j. The function $y(x)$, $x \in [0, T]$ on the cycle e_0 is defined by $y(x) = y_{r+j}(x_{r+j})$, $j = \overline{1,N}$.

Let $Q = \{q_j\}_{j=\overline{1,r+N}}$ be an integrable real-valued function on G; Q is called the potential. The function $q(x)$, $x \in [0, T]$ is defined by $q(x) = q_{r+j}(x_{r+j})$, $j = \overline{1,N}$. Consider the following differential equation on G:

$$-y_j''(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in [0, T_j], \quad j = \overline{1, r+N}, \tag{2.1}$$

where λ is the spectral parameter, the functions y_j, y_j', $j = \overline{1, r+N}$, are absolutely continuous on $[0, T_j]$ and satisfy the following matching conditions at each internal vertex $v_{\mu+1}$, $\mu = \overline{r+1, r+N}$:

$$\left.\begin{aligned} y_{\mu+1}(0) &= \alpha_j y_j(T_j) \quad \text{for all} \quad e_j \in \mathcal{E}'_{\mu-r+1}, \\ y'_{\mu+1}(0) - h_{\mu+1}y_{\mu+1}(0) &= \sum_{e_j \in \mathcal{E}'_{\mu-r+1}} \beta_j y_j'(T_j), \end{aligned}\right\} \tag{2.2}$$

and $y_{r+N+1} := y_{r+1}$, $h_{r+N+1} := h_{r+1}$, $\mathcal{E}_{N+1} := \mathcal{E}_1$, $\mathcal{E}'_{\mu-r+1} := \mathcal{E}_{\mu-r+1} \cup e_\mu$. Here α_j, β_j and h_j are real numbers, and $\alpha_j\beta_j \neq 0$. For definiteness, let $\alpha_j\beta_j > 0$. Conditions (2.2) are a generalization of the so-called standard matching conditions (see [9]), where $\alpha_j = \beta_j = 1$, $h_j = 0$.

Let us consider the boundary value problem B_0 on G for equation (2.1) with the matching conditions (2.2) and with the Dirichlet boundary conditions at the boundary vertices $v_1, \ldots, v_r$:

$$y_j(0) = 0, \quad j = \overline{1,r}.$$

Denote by $\Lambda_0 = \{\lambda_{n0}\}_{n\geq 0}$ the eigenvalues of the boundary value problem B_0. We also consider boundary value problems $B_{\nu_1,\ldots,\nu_p}$, $p = \overline{1,r}$, $1 \leq \nu_1 < \ldots \nu_p \leq r$, for equation (2.1) with matching conditions (2.2) and with the boundary conditions

$$y_k'(0) = 0, \;\; k = \nu_1, \ldots, \nu_p, \quad y_j(0) = 0, \;\; j = \overline{1,r}, \;\; j \neq \nu_1, \ldots, \nu_p.$$

Let $\Lambda_{\nu_1,\ldots,\nu_p} := \{\lambda_{n,\nu_1,\ldots,\nu_p}\}_{n\geq 0}$ be the spectrum of $B_{\nu_1,\ldots,\nu_p}$.

An important role for solving inverse problems on graphs is played by an auxiliary quasi-periodic boundary value problem on the cycle with discontinuity conditions at interior points. The parameters of this auxiliary problem depend on the parameters of B_0. More precisely, let us introduce real numbers γ_j, η_j, $(j = \overline{1, N-1})$, h, α, β by the formulae

$$\left.\begin{aligned} &\gamma_j = \sqrt{\frac{\alpha_{r+j}}{\beta_{r+j}}}, \quad \eta_j = \gamma_j h_{r+j+1}, \;\; j = \overline{1, N-1}, \quad h = h_{r+1}, \\ &\alpha = \alpha_{r+N} \prod_{j=1}^{N-1} \gamma_j \prod_{j=1}^{N-1} \beta_{r+j}, \quad \beta = \prod_{j=1}^{N-1} \gamma_j \prod_{j=1}^{N} \beta_{r+j}. \end{aligned}\right\} \tag{2.3}$$

Clearly, $\alpha\beta > 0$, $\gamma_j > 0$, $j = \overline{1, N-1}$. Using these parameters, we consider the boundary value problem B on the cycle e_0:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, T), \tag{2.4}$$

$$y(0) = \alpha y(T), \quad y'(0) - hy(0) = \beta y'(T), \tag{2.5}$$

$$y(b_j + 0) = \gamma_j y(b_j - 0),\ y'(b_j + 0) = \gamma_j^{-1} y'(b_j - 0) + \eta_j y(b_j - 0),\ j = \overline{1, N-1}. \tag{2.6}$$

Let $S(x,\lambda)$ and $C(x,\lambda)$ be solutions of equation (2.4) satisfying conditions (2.6) and $S(0,\lambda) = C'(0,\lambda) = 0$, $S'(0,\lambda) = C(0,\lambda) = 1$. Put $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ of B coincide with the zeros of the characteristic function

$$a(\lambda) = \alpha\varphi(T,\lambda) + \beta S'(T,\lambda) - (1 + \alpha\beta). \tag{2.7}$$

Let $d(\lambda) := S(T,\lambda)$, $Q(\lambda) = \alpha\varphi(T,\lambda) - \beta S'(T,\lambda)$. All zeros $\{z_n\}_{n\geq 1}$ of the entire function $d(\lambda)$ are simple, i.e. $\dot{d}(z_n) \neq 0$, where $\dot{d}(\lambda) := \frac{d}{d\lambda} d(\lambda)$. Put $M_n = -d_1(z_n)\dot{d}(z_n)$, where $d_1(\lambda) := C(T,\lambda)$. The sequence $\{M_n\}_{n\geq 1}$ is called the Weyl sequence. Put $\omega_n = \operatorname{sign} Q(z_n)$, $\Omega = \{\omega_n\}_{n\geq 1}$.

We choose and fix one edge $e_{\xi_i} \in \mathcal{E}_i$ from each block $\mathcal{E}_i$, $i = \overline{1,N}$, i.e. $m_{i-1} + 1 \leq \xi_i \leq m_i$. We denote by $\xi := \{k : k = \xi_1, \ldots, \xi_N\}$ the set of indices ξ_i, $i = \overline{1,N}$. Let α_j and β_j, $j = \overline{1, r+N}$, be known a priori. The inverse problem is formulated as follows.

Inverse problem 1. Given $2^N + r - N$ spectra Λ_j, $j = \overline{0,r}$, $\Lambda_{\nu_1,\ldots,\nu_p}$, $p = \overline{2,N}$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$ and Ω, construct the potential Q on G and $H := [h_j]_{j=\overline{r+1,r+N}}$.

This inverse problem is a generalization of the classical inverse problems for Sturm-Liouville operators on intervals and on graphs.

Example 2.1. Let $N = 1$. Then we specify $r + 1$ spectra Λ_j, $j = \overline{0,r}$, and Ω. This inverse problem was solved in [8].

Example 2.2. Let $N = r$, i.e. $r_i = 1$, $i = \overline{1,N}$. Then we specify 2^N spectra Λ_0, $\Lambda_{\nu_1,\ldots,\nu_p}$, $p = \overline{1,N}$, $1 \leq \nu_1 < \ldots \nu_p \leq N$, and Ω.

Example 2.3. Let $N = 3$, $r = 4$ (see Figure 1).
Case 1. Take $\xi_1 = 2, \xi_2 = 3, \xi_3 = 4$. Then we specify Ω and the following spectra:
$\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_{23}, \Lambda_{24}, \Lambda_{34}, \Lambda_{234}$.
Case 2. Take $\xi_1 = 1, \xi_2 = 3, \xi_3 = 4$. Then we specify Ω and the following spectra:
$\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_{13}, \Lambda_{14}, \Lambda_{34}, \Lambda_{134}$.

Let us formulate the uniqueness theorem for the solution of Inverse problem 1. For this purpose together with q we consider a potential $\tilde{q}$. Everywhere below, if a symbol α denotes an object related to q, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

Theorem 2.4. *If* $\Lambda_j = \tilde{\Lambda}_j$, $j = \overline{0,r}$, $\Lambda_{\nu_1,\ldots,\nu_p} = \tilde{\Lambda}_{\nu_1,\ldots,\nu_p}$, $p = \overline{2,N}$, $1 \leq \nu_1 < \ldots < \nu_p \leq r$, $\nu_j \in \xi$, $\Omega = \tilde{\Omega}$, *then* $Q = \tilde{Q}$ *and* $H = \tilde{H}$.

This theorem will be proved in Section 4. We will also provide there a constructive procedure for the solution of Inverse problem 1. In Section 3 we study properties of spectral characteristics and prove some auxiliary assertions.

3 Properties of spectral characteristics

Let $S_j(x_j,\lambda)$, $C_j(x_j,\lambda)$, $j=\overline{1,r+N}$, $x_j\in[0,T_j]$, be the solutions of equation (2.1) on the edge e_j with the initial conditions

$$S_j(0,\lambda)=C_j'(0,\lambda)=0,\quad S_j'(0,\lambda)=C_j(0,\lambda)=1. \tag{3.1}$$

For each fixed $x_j\in[0,T_j]$, the functions $S_j^{(\nu)}(x_j,\lambda)$, $C_j^{(\nu)}(x_j,\lambda)$, $j=\overline{1,r+N}$, $\nu=0,1$, are entire in λ of order $1/2$.

Theorem 3.1. *The following relations hold for $k=\overline{1,N-1}$, $\nu=0,1$:*

$$S^{(\nu)}(b_{k+1}-0,\lambda)=\gamma_k S(b_k-0,\lambda)C^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda)+\gamma_k^{-1}S'(b_k-0,\lambda)S^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda)$$
$$+\eta_k S(b_k-0,\lambda)S^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda), \tag{3.2}$$
$$C^{(\nu)}(b_{k+1}-0,\lambda)=\gamma_k C(b_k-0,\lambda)C^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda)+\gamma_k^{-1}C'(b_k-0,\lambda)S^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda)$$
$$+\eta_k C(b_k-0,\lambda)S^{(\nu)}_{r+k+1}(T_{r+k+1},\lambda). \tag{3.3}$$

Indeed, fix $k=\overline{1,N-1}$. Let $x\in[b_k,b_{k+1}]$, i.e. $x=x_{r+k+1}+b_k$, $x_{r+k+1}\in[0,T_{r+k+1}]$. Using the fundamental system of solutions $S_{r+k+1}(x_{r+k+1},\lambda)$, $C_{r+k+1}(x_{r+k+1},\lambda)$ on e_{r+k+1}, one has

$$S^{(\nu)}(x,\lambda)=A(\lambda)C^{(\nu)}_{r+k+1}(x_{r+k+1},\lambda)+B(\lambda)S^{(\nu)}_{r+k+1}(x_{r+k+1},\lambda),\quad \nu=0,1.$$

Taking initial conditions (3.1) for $j=r+k+1$ into account, we find the coefficients $A(\lambda)$ and $B(\lambda)$, and arrive at (3.2). Relation (3.3) is proved similarly.

Let $\lambda=\rho^2$, $\tau:=Im\rho\ge 0$, $\Pi:=\{\rho:\ \tau\ge 0\}$, $\Pi_\delta:=\{\rho:\ \arg\rho\in[\delta,\pi-\delta]\}$, $\delta\in(0,\pi/2)$. The following theorem describes the asymptotic behavior of $S(x,\lambda)$ and $C(x,\lambda)$ on each interval $x\in(b_j,b_{j+1})$ (see [11]).

Theorem 3.2. *Fix $j=\overline{1,N-1}$. For $x\in(b_j,b_{j+1})$, $\nu=0,1$, $m=1,2$, $|\rho|\to\infty$,*

$$S^{(\nu)}(x,\lambda)=\Big(\prod_{k=1}^{j}\xi_k^+\Big)\frac{d^\nu}{dx^\nu}\Big(\frac{\sin\rho x}{\rho}+\sum_{k=1}^{j}\sum_{1\le\mu_1<\ldots<\mu_k\le j}\Big(\prod_{i=1}^{k}\frac{\xi^-_{\mu_i}}{\xi^+_{\mu_i}}\Big)\frac{\sin(\rho\alpha_{\mu_1,\ldots,\mu_k}(x))}{\rho}\Big)$$
$$+O(\rho^{\nu+m-3}e^{\tau x}),$$
$$C^{(\nu)}(x,\lambda)=\Big(\prod_{k=1}^{j}\xi_k^+\Big)\frac{d^\nu}{dx^\nu}\Big(\cos\rho x+\sum_{k=1}^{j}\sum_{1\le\mu_1<\ldots<\mu_k\le j}\Big(\prod_{i=1}^{k}\frac{\xi^-_{\mu_i}}{\xi^+_{\mu_i}}\Big)\cos(\rho\alpha_{\mu_1,\ldots,\mu_k}(x))\Big)$$
$$+O(\rho^{\nu+m-3}e^{\tau x}),$$

where

$$\xi_j^\pm:=\frac{\gamma_j+\gamma_j^{-1}}{2},\quad \alpha_{\mu_1,\ldots,\mu_k}(x):=2\sum_{i=1}^{k}(-1)^{i-1}b_{\mu_i}+(-1)^k x.$$

Using Theorem 2.4, we obtain for $|\rho| \to \infty$, $\rho \in \Pi_\delta$:

$$a(\lambda) = \frac{(\alpha+\beta)\xi}{2} e^{-i\rho T} \ [4], \quad d(\lambda) = -\frac{\xi}{2i\rho} e^{-i\rho T} \ [4], \quad \xi := \prod_{j=1}^{N-1} \xi_j^+. \tag{3.4}$$

Moreover,

$$a(\lambda) = O(e^{\tau T}), \quad d(\lambda) = O(\rho^{-1} e^{\tau T}), \quad |\rho| \to \infty, \quad \rho \in \Pi. \tag{3.5}$$

Fix $k = \overline{1,r}$. Let $\Phi_k = \{\Phi_{kj}\}_{j=\overline{1,r+N}}$ be the solution of equation (2.1) satisfying (2.2) and the boundary conditions

$$\Phi_{kj}(0,\lambda) = \delta_{jk}, \quad j = \overline{1,r}, \tag{3.6}$$

where δ_{jk} is the Kronecker symbol. Put $M_k(\lambda) := \Phi'_{kk}(0,\lambda)$, $k = \overline{1,r}$. The function $M_k(\lambda)$ is called the *Weyl function* with respect to the boundary vertex v_k. Put $M^1_{kj}(\lambda) := \Phi_{kj}(0,\lambda)$, $M^0_{kj}(\lambda) := \Phi'_{kj}(0,\lambda)$. Then

$$\Phi_{kj}(x_j,\lambda) = M^1_{kj}(\lambda) C_j(x_j,\lambda) + M^0_{kj}(\lambda) S_j(x_j,\lambda), \quad x_j \in [0,T_j], \ j = \overline{1,r+N}, \ k = \overline{1,r}. \tag{3.7}$$

In particular, $M^1_{kk}(\lambda) = 1$, $M^0_{kk}(\lambda) = M_k(\lambda)$. Substituting (3.7) into (2.2) and (3.6), we obtain a linear algebraic system D_k with respect to $M^\nu_{kj}(\lambda)$, $\nu = 0,1$, $j = \overline{1,r+N}$. The determinant $\Delta_0(\lambda)$ of D_k does not depend on k and has the form

$$\Delta_0(\lambda) = \sigma(\lambda)\Big(a_0(\lambda) + \sum_{k=1}^{N} \sum_{1\le\mu_1<\ldots<\mu_k\le N} a_{\mu_1\ldots\mu_k}(\lambda) \prod_{i=1}^{k}\Big(\sum_{e_j\in\mathcal{E}_{\mu_i}} \Omega_j(\lambda)\Big)\Big), \tag{3.8}$$

where

$$\sigma(\lambda) = \prod_{j=1}^{r} (\alpha_j S_j(T_j,\lambda)), \quad \Omega_j(\lambda) = \frac{\beta_j S'_j(T_j,\lambda)}{\alpha_j S_j(T_j,\lambda)}, \tag{3.9}$$

$$a_0(\lambda) = a(\lambda), \quad a_1(\lambda) = \alpha d(\lambda). \tag{3.10}$$

We note that the coefficients $a_0(\lambda)$ and $a_{\mu_1\ldots\mu_k}(\lambda)$ in (3.10) depend only on $S^{(\nu)}_j(T_j,\lambda)$ and $C^{(\nu)}_j(T_j,\lambda)$ for $j = \overline{r+1,r+N}$, and (3.10) follows from Theorem 3.1. We do not need concrete formulae for other coefficients $a_{\mu_1\ldots\mu_k}(\lambda)$. The function $\Delta_0(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem B_0. The function $\Delta_0(\lambda)$ is called the characteristic function for the boundary value problems B_0. Let $\Delta_{\nu_1,\ldots,\nu_p}(\lambda)$, $p = \overline{1,r}$, $1 \le \nu_1 < \ldots < \nu_p \le r$, be the function obtained from $\Delta_0(\lambda)$ by the replacement of $S^{(\nu)}_j(T_j,\lambda)$ with $C^{(\nu)}_j(T_j,\lambda)$ for $j = \nu_1,\ldots,\nu_p$, $\nu = 0,1$. More precisely,

$$\Delta_{\nu_1,\ldots,\nu_p}(\lambda) = \sigma_{\nu_1,\ldots,\nu_p}(\lambda)\Big(a_0(\lambda) + \sum_{k=1}^{N} \sum_{1\le\mu_1<\ldots<\mu_k\le N} a_{\mu_1\ldots\mu_k}(\lambda)$$
$$\times \prod_{i=1}^{k}\Big(\sum_{e_j\in\mathcal{E}_{\mu_i},\ j\ne\nu_1,\ldots,\nu_p} \Omega_j(\lambda) + \sum_{e_j\in\mathcal{E}_{\mu_i},\ j=\nu_1,\ldots,\nu_p} \Omega^0_j(\lambda)\Big)\Big), \tag{3.11}$$

where

$$\sigma_{\nu_1,\dots,\nu_p}(\lambda) = \prod_{j=1,\, j\neq\nu_1,\dots,\nu_p}^{r} (\alpha_j S_j(T_j,\lambda)) \prod_{j=\nu_1,\dots,\nu_p} (\alpha_j C_j(T_j,\lambda)),\ \Omega_j^0(\lambda) = \frac{\beta_j C_j'(T_j,\lambda)}{\alpha_j C_j(T_j,\lambda)}. \quad (3.12)$$

The function $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem $B_{\nu_1,\dots,\nu_p}$. The function $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$ is called the characteristic function for the boundary value problem $B_{\nu_1,\dots,\nu_p}$.

Solving the algebraic system D_k, we get by Cramer's rule: $M_{kj}^s(\lambda) = \Delta_{kj}^s(\lambda)/\Delta_0(\lambda)$, $s = 0,1$, $j = \overline{1,r+N}$, where the determinant $\Delta_{kj}^s(\lambda)$ is obtained from $\Delta_0(\lambda)$ by the replacement of the column which corresponds to $M_{kj}^s(\lambda)$ with the column of free terms. In particular,

$$M_k(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}, \quad k = \overline{1,r}. \quad (3.13)$$

Uniformly on $x_j \in [0,T_j]$, we have (see [3]):

$$S_j^{(\nu)}(x_j,\lambda) = \frac{1}{2i\rho}\Big((i\rho)^\nu \exp(i\rho x_j)[1] - (-i\rho)^\nu \exp(-i\rho x_j)[1]\Big),\ \ \rho\in\Pi,\ \ |\rho|\to\infty, \quad (3.14)$$

$$C_j^{(\nu)}(x_j,\lambda) = \frac{1}{2}\Big((i\rho)^\nu \exp(i\rho x_j)[1] + (-i\rho)^\nu \exp(-i\rho x_j)[1]\Big),\ \ \rho\in\Pi,\ \ |\rho|\to\infty. \quad (3.15)$$

Moreover, for each fixed $x_k \in [0,T_k)$:

$$\Phi_{kk}^{(\nu)}(x_k,\lambda) = (i\rho)^\nu \exp(i\rho x_k)[1], \quad \rho\in\Pi_\delta,\ \ |\rho|\to\infty.$$

In particular, $M_k(\lambda) = (i\rho)$ [4], $\rho\in\Pi_\delta$, $|\rho|\to\infty$.

Using (3.8), (3.14), (3.15), (3.4) and (3.5), by the well-known method (see, for example, [6]), one can obtain the following properties of the characteristic function $\Delta_0(\lambda)$ and the eigenvalues Λ_0 of the boundary value problem B_0.

1) For $\rho\in\Pi$, $|\rho|\to\infty$,

$$\Delta_0(\lambda) = O\Big(|\rho|^{-r}\exp\Big(\tau\sum_{j=1}^{r+N} T_j\Big)\Big).$$

2) There exist $h>0$, $C_h>0$ such that

$$|\Delta_0(\lambda)| \ge C_h|\rho|^{-r}\exp\Big(\tau\sum_{j=1}^{r+N} T_j\Big)$$

for $\tau\ge h$. Hence, the eigenvalues $\lambda_{n0} = \rho_{n0}^2$ lie in the domain $0\le\tau<h$.

3) For $n\to\infty$,

$$\rho_{n0} = \rho_{n0}^0 + O\Big(\frac{1}{\rho_{n0}^0}\Big),$$

where $\lambda_{n0}^0 = (\rho_{n0}^0)^2$ are the eigenvalues of the boundary value problem B_0 with $Q=0$ and $H=0$.

The characteristic functions $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$ have similar properties. In particular,

$$\Delta_{\nu_1,\dots,\nu_p}(\lambda) = O\Big(|\rho|^{p-r}\exp\Big(\tau\sum_{j=1}^{r+N}T_j\Big)\Big) \quad \text{for } \rho\in\Pi, \ |\rho|\to\infty.$$

Using properties of the characteristic functions and Hadamard's factorization theorem, one gets that the specification of the spectrum Λ_0 uniquely determines the characteristic function $\Delta_0(\lambda)$, i.e. if $\Lambda_0 = \tilde{\Lambda}_0$, then $\Delta_0(\lambda) \equiv \tilde{\Delta}_0(\lambda)$. Analogously, if $\Lambda_{\nu_1,\dots,\nu_p} = \tilde{\Lambda}_{\nu_1,\dots,\nu_p}$, then $\Delta_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde{\Delta}_{\nu_1,\dots,\nu_p}(\lambda)$. The characteristic functions can be constructed as the corresponding infinite products (see [3] for details).

4 Solution of the inverse problem

In this section we provide a constructive procedure for the solution of Inverse problem 1, and prove its uniqueness.

Fix $k = \overline{1,r}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k). Given two spectra Λ_0 and Λ_k, construct $q_k(x_k)$, $x_k \in [0,T_k]$.

Theorem 4.1. *Fix $k = \overline{1,r}$. The specification of two spectra Λ_0 and Λ_k uniquely determines the potential q_k on the edge e_k.*

The solution of IP(k) can be obtained by the method of spectral mappings [3], [6]. Here we only explain ideas briefly; for details and proofs see [3], [6]. Take a boundary value problem $\tilde{B}_0$ with $\tilde{Q} = 0$. Take a fixed $c_1 > 0$ such that $|\mathrm{Im}\rho_{n0}|, |\mathrm{Im}\tilde{\rho}_{n0}| < c_1$. In the ρ-plane we consider the contour γ (with counterclockwise circuit) of the form $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^\pm = \{\rho : \ \pm\mathrm{Im}\rho = c_1\}$. For each fixed $x_k \in [0,T_k]$, the function $S_k(x_k,\lambda)$ is a unique solution of the linear integral equation

$$S_k(x_k,\lambda) = \tilde{S}_k(x_k,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{D}_k(x_k,\lambda,\mu)S_k(x_k,\mu)\,d\mu, \tag{4.1}$$

where $\tilde{D}_k(x,\lambda,\mu) = \displaystyle\int_0^x \tilde{S}_k(t,\lambda)\tilde{S}_k(t,\mu)\hat{M}_k(\mu)\,dt$, $\hat{M}_k(\mu) := M_k(\mu) - \tilde{M}_k(\mu)$. The Weyl function $M_k(\lambda)$ is calculated by (3.13). The potential q_k on the edge e_k can be constructed from the solution of (4.1) by the formula

$$q_k(x_k) = \frac{1}{2\pi i}\int_\gamma (S_k(x_k,\lambda)\tilde{S}_k(x_k,\lambda))'\hat{M}_k(\lambda)\,d\lambda$$

or by $q_k(x_k) = \lambda + S_k''(x_k,\lambda)/S_k(x_k,\lambda)$.

Now we study the following auxiliary inverse problem on the cycle e_0, which is called IP(0). Consider the boundary value problem B of the form (2.4)–(2.6), where the parameters of B_0 are defined by (2.3), and α,β are known.

IP(0). Given $a(\lambda)$, $d(\lambda)$ and Ω, construct $q(x)$, $x \in [0,T]$, h, γ_j and η_j, $j = \overline{1,N-1}$.

The inverse problem IP(0) was solved in [11], where the following theorem is established.

Theorem 4.2. *The specification $a(\lambda)$, $d(\lambda)$ and Ω uniquely determines $q(x)$, h, γ_j and η_j, $j = \overline{1,N-1}$. The solution of IP(0) can be found by the following algorithm.*

Algorithm 1.

1) Construct $D(\lambda) = a(\lambda) + (1+\alpha\beta)$.
2) Find the zeros $\{z_n\}_{n\ge1}$ of the entire function $d(\lambda)$.
3) Calculate $Q(z_n)$ by the formula $Q(z_n) = \omega_n \sqrt{D^2(z_n) - 4\alpha\beta}$.
4) Construct $d_1(z_n) = (D(z_n) + Q(z_n))/(2\alpha)$.
5) Calculate the Weyl sequence $\{M_n\}_{n\ge1}$ via $M_n = -d_1(z_n)/\dot{d}(z_n)$.
6) From the given data $\{z_n, M_n\}_{n\ge1}$ construct $q(x), \gamma_j, \eta_j$, $j = \overline{1,N-1}$, by solving the inverse Dirichlet problem with discontinuities inside the interval (see [5]).
7) Find $S(T,\lambda)$, $S'(T,\lambda)$ and $C(T,\lambda)$.
8) Calculate h, using (2.7).

Let us go on to the solution of Inverse problem 1. Firstly, we give the proof of Theorem 2.4. Let $\Lambda_k = \tilde{\Lambda}_k$, $k = \overline{0,r}$, $\Lambda_{\nu_1,\dots,\nu_p} = \tilde{\Lambda}_{\nu_1,\dots,\nu_p}$, $p = \overline{2,N}$, $1 \le \nu_1 < \dots < \nu_p \le r$, $\nu_j \in \xi$ and $\Omega = \tilde{\Omega}$. Then

$$\Delta_k(\lambda) \equiv \tilde{\Delta}_k(\lambda), \quad k = \overline{0,r},$$

$$\Delta_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde{\Delta}_{\nu_1,\dots,\nu_p}(\lambda), \quad p = \overline{2,N},\ 1 \le \nu_1 < \dots < \nu_p \le r,\ \nu_j \in \xi.$$

Moreover, by virtue of (2.3) one has $\gamma_j = \tilde{\gamma}_j$, $j = \overline{1,N-1}$, $\alpha = \tilde{\alpha}, \beta = \tilde{\beta}$. Using Theorem 4.1, we obtain $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0,T_k]$, and consequently,

$$C_k(x_k,\lambda) \equiv \tilde{C}_k(x_k,\lambda), \quad S_k(x_k,\lambda) \equiv \tilde{S}_k(x_k,\lambda), \quad k = \overline{1,r}. \tag{4.2}$$

In view of (3.9), (3.12) and (4.2) we get

$$\sigma(\lambda) \equiv \tilde{\sigma}(\lambda), \quad \sigma_{\nu_1,\dots,\nu_p}(\lambda) \equiv \tilde{\sigma}_{\nu_1,\dots,\nu_p}(\lambda), \quad \Omega_j(\lambda) \equiv \tilde{\Omega}_j(\lambda), \quad \Omega_j^0(\lambda) \equiv \tilde{\Omega}_j^0(\lambda), \quad j = \overline{1,r}.$$

Using (3.8) and (3.11), we obtain, in particular, $a_0(\lambda) = \tilde{a}(\lambda)$, $a_1(\lambda) = \tilde{a}_1(\lambda)$. Together with (3.10) this yields

$$a(\lambda) = \tilde{a}(\lambda), \quad d(\lambda) = \tilde{d}(\lambda).$$

It follows from Theorem 4.2 that $q_k(x_k) = \tilde{q}_k(x_k)$ a.e. on $[0,T_k]$, $k = \overline{r+1,r+N}$, $h = \tilde{h}$, $\eta_j = \tilde{\eta}_j$, $j = \overline{1,N-1}$. Taking (2.3) into account, we deduce $H = \tilde{H}$. Theorem 2.4 is proved.

The solution of Inverse problem 1 can be constructed by the following algorithm.

Algorithm 2.

Given Λ_k, $k = \overline{0,r}$, $\Lambda_{\nu_1,\dots,\nu_p}$, $p = \overline{2,N}$, $1 \le \nu_1 < \dots < \nu_p \le r$, $\nu_j \in \xi$, Ω.
1) Construct $\Delta_k(\lambda)$ and $\Delta_{\nu_1,\dots,\nu_p}(\lambda)$.
2) Calculate γ_j, $j = \overline{1,N-1}$, α and β, using (2.3).
3) For each fixed $k = \overline{1,r}$, solve the inverse problem IP(k) and find $q_k(x_k)$, $x_k \in [0,T_k]$ on the edge e_k.
4) For $k = \overline{1,r}$ construct $C_k(x_k,\lambda)$ and $S_k(x_k,\lambda)$, $x_k \in [0,T_k]$.
5) Calculate $a(\lambda)$ and $d(\lambda)$, using (3.8), (3.10) and (3.11).
6) From the given $a(\lambda), d(\lambda)$ and Ω construct $q_k(x_k)$, $[0,T_k]$, $k = \overline{r+1,r+N}$, h and η_j, $j = \overline{1,N-1}$.
7) Using (2.3), construct H.

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