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Embedded Eigenvalues of a Hamiltonian in Bosonic Fock Space

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Abstract

We consider a model operator $\mathbf{H}_{\mu\lambda}$, $\mu, \lambda \geq 0$ associated with the energy operator of a lattice system describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. The existence of infinitely many negative eigenvalues of $\mathbf{H}_{0\lambda}$ is proved for the case where the associated Friedrichs model have a zero energy resonance and an asymptotics of the form $\mathcal{U}_0|\log|z||$ for the number of eigenvalues of $\mathbf{H}_{0\lambda}$ lying below $z<0$, is obtained. We find the conditions for the infiniteness of the number of eigenvalues located inside (in the gap, in the below of the bottom) of the essential spectrum of $\mathbf{H}_{\mu\lambda}$.

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1 Introduction

The main goal of the present paper is to give a thorough mathematical treatment of the spectral properties for a model operator $\mathbf{H}_{\mu\lambda}$, $\mu, \lambda \geq 0$ with emphasis on the infiniteness of the number of eigenvalues embedded in its essential spectrum. This operator is associated with a lattice system describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles.

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In the spectral theory of the continuous and discrete three-particle Schrödinger operators, there is the remarkable phenomenon known as *Efimov's effect*: if in a system of three particles, interacting by means of short-range pair potentials none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a resonance with zero energy, then this three-particle system has an infinite number of three-particle bound states with negative energy, accumulating at zero. Since its discovery by Efimov in [8] many papers have been devoted to this subject. See, for example [2, 5, 7, 25, 30, 31, 33]. The first mathematical proof of the existence of this effect was given by Yafaev [33], and the asymptotics of the number of eigenvalues near the threshold of the essential spectrum was established by Sobolev [30]. The presence of the Efimov effect for the discrete Schrödinger operators was proved in [3, 11, 12, 13] and an asymptotics analogous to [30] was obtained in [3] for the number of eigenvalues.

Perturbation problems for operators with embedded eigenvalues are generally challenging since the embedded eigenvalues cannot be separated from the rest of the spectrum. Embedded eigenvalues occur in many applications arising in physics and many works have been devoted to the study of embedded eigenvalues of the Schrödinger operators. See, for example [1, 20, 24, 28].

The number of eigenvalues in the gap of the essential spectrum and the formula for the number of eigenvalues in an arbitrary interval outside of the essential spectrum of the three-particle discrete Schrödinger operator were studied in [18], [19].

In all above mentioned papers devoted to the embedded eigenvalues, systems where the number of quasi-particles is fixed have been considered. In the theory of solid-state physics [17], quantum field theory [9], statistical physics [15, 16], fluid mechanics [6], magneto-hydrodynamics [14] and quantum mechanics [32] some important problems arise where the number of quasi-particles is finite, but not fixed. In [29] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number. Recall that the study of systems describing n particles in interaction, without conservation of the number of particles can be reduced to the investigation of the spectral properties of self-adjoint operators acting in the cut subspace of the Fock space, consisting of $r \le n$ particles [9, 16, 17, 29].

In the present paper we consider a model operator $\mathbf{H}_{\mu\lambda}$, $\mu, \lambda \ge 0$ associated with a lattice system describing two bosons and one particle another nature in interaction, without conservation of the number of particles. This operator acts in the Hilbert space $\mathcal{H}$, which is the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space and it is a lattice analogue of the spin-boson Hamiltonian [16]. Under some smoothness assumptions on the parameters of the two families of Friedrichs models $h_\mu(p)$, $\mathbf{h}_\lambda(p)$, $p \in \mathbb{T}^3 \equiv (-\pi;\pi]^3$, we obtain the following results:

(i) We describe the location and structure of the essential spectrum of $\mathbf{H}_{\mu\lambda}$ via the spectrum of $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$;

(ii) We prove the presence of an infinite number of negative eigenvalues of $\mathbf{H}_{0\lambda_0}$, and accumulating at zero, for some $\lambda_0 > 0$ (Efimov's effect). Here $\lambda = \lambda_0$ corresponds to the existence of the zero energy resonance for the Friedrichs model $\mathbf{h}_\lambda(p_1)$, $p_1 = (0,0,0) \in \mathbb{T}^3$. We also show that the number $N_\lambda(z)$ of eigenvalues of $\mathbf{H}_{0\lambda}$ on the left of z, $z < 0$ has the asymptotics $N_{\lambda_0}(z) \sim \mathcal{U}_0 |\log|z||$ as $z \to -0$, where $0 < \mathcal{U}_0 < \infty$;

(iii) We find conditions which guarantee for the infiniteness of the number of eigenvalues located inside, in the gap, and in the below of the bottom of the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$, respectively.

It is remarkable that in the assertion (ii) for the Friedrichs model $\mathbf{h}_\lambda(p_1)$ the presence of a zero energy resonance (consequently the existence of the Efimov effect for $\mathbf{H}_{0\lambda}$) is due to the annihilation and creation operators.

We discuss the case where the lattice kinetic energy $\varepsilon(\cdot)$ of a particle has a special form with non-degenerate minimum at the several points of the three-dimensional torus. In [4] the operator matrix $\mathbf{H}_{0\lambda}$ with kinetic energy which has a unique non-degenerate minimum is considered and proved the same assertion as (ii). Hence the assertion (ii) looks surprising, because it does't depends on the number of points at which the function $\varepsilon(\cdot)$ has non-degenerate minimum, that is, the asymptotics of the discrete spectrum is stable with respect to the number of that points.

Note that the operator matrices like $\mathbf{H}_{\mu\lambda}$ has been considered before in [21, 26, 27, 34] where only its essential spectrum was investigated. The lattice model operators in fermionic Fock space with the kinetic part depending on parameter γ, was considered in [22, 23] and the authors find a critical value γ^* for the parameter γ that allows or forbids the Efimov effect.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the model operator $\mathbf{H}_{\mu\lambda}$ is described as a bounded self-adjoint operator in $\mathcal{H}$ and the main results of the present paper are formulated. In Section 3 the set of negative eigenvalues of $h_\mu(p)$ is described and an expansion for the Fredholm determinant associated with the operator $\mathbf{h}_\lambda(p)$ is obtained. In Section 4, the structure of the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$ is studied. In Section 5, first we give a realization of the Birman-Schwinger principle for $\mathbf{H}_{0\lambda}$ and then we obtain an asymptotic formula for the number of negative eigenvalues of $\mathbf{H}_{0\lambda_0}$. Section 5 is devoted to the proof of the infiniteness of the number of eigenvalues of $\mathbf{H}_{\mu\lambda_0}$ lying inside (in the gap, in the below of the bottom) of its essential spectrum.

Throughout the present paper we adopt the following conventions: Denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi,\pi]^3$ with appropriately identified sides equipped with its Haar measure. The spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator will be denoted by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$, respectively. In what follows we deal with the operators in various spaces of vector-valued functions. They will be denoted by bold letters and will be written in the matrix form. For each $\delta > 0$, the notation $U_\delta(p_0) := \{p \in \mathbb{T}^3 : |p - p_0| < \delta\}$ stands for a δ-neighborhood of the point $p_0 \in \mathbb{T}^3$.

2 The model operator and main results

Let $\mathbb{C}$ be the field of complex numbers, $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^{\rm s}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_0 := \mathbb{C}$, $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 := L_2^{\rm s}((\mathbb{T}^3)^2)$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. The Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are zero-, one- and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\rm s}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively.

Let H_{ij} be annihilation (creation) operators [9] defined in the Fock space for $i<j$ $(i>j)$. We note that in physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

In this paper we consider the case, where the number of annihilations and creations of the particles of the considering system is equal to 1. It means that $H_{ij}\equiv 0$ for all $|i-j|>1$. So, an model operator (Hamiltonian) $\mathbf{H}_{\mu\lambda}$, $\mu,\lambda\geq 0$ associated to a system describing three particles in interaction, without conservation of the number of particles, acts in the Hilbert space $\mathcal{H}$ as a 3×3 block matrix operator

$$\mathbf{H}_{\mu\lambda}:=\begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{01}^* & H_{11} & \lambda H_{12} \\ 0 & \lambda H_{12}^* & H_{22}^0-\mu V \end{pmatrix}.$$

Let its components are defined by the rule

$$H_{00}f_0=w_0f_0,\quad H_{01}f_1=\int_{\mathbb{T}^3} v_1(s)f_1(s)ds,\quad (H_{11}f_1)(p)=w_1(p)f_1(p),$$

$$(H_{12}f_2)(p)=\int_{\mathbb{T}^3} f_2(p,s)ds,\quad (H_{22}^0f_2)(p,q)=w_2(p,q)f_2(p,q),$$

$$(Vf_2)(p,q)=v_2(q)\int_{\mathbb{T}^3} v_2(s)f_2(p,s)ds+v_2(p)\int_{\mathbb{T}^3} v_2(s)f_2(s,q)ds,$$

where H_{ij}^* $(i<j)$ denotes the adjoint operator to H_{ij} and $f_i\in\mathcal{H}_i$, $i=0,1,2$.

Here μ,λ are non-negative real numbers, w_0 is a fixed real number, $v_i(\cdot)$, $i=1,2$ are real-valued continuous functions on $\mathbb{T}^3$, the functions $w_1(\cdot)$ and $w_2(\cdot,\cdot)$ are defined by the equalities

$$w_1(p):=\varepsilon(p)+1,\quad w_2(p,q):=l_1\varepsilon(p)+l_2\varepsilon(p+q)+l_1\varepsilon(q),$$

respectively, with $l_1,l_2>0$ and

$$\varepsilon(q):=\sum_{i=1}^{3}(1-\cos(3q^{(i)})),\quad q=(q^{(1)},q^{(2)},q^{(3)})\in\mathbb{T}^3.$$

Under these assumptions the operator $\mathbf{H}_{\mu\lambda}$ is bounded and self-adjoint.

To study the spectral properties of the operator $\mathbf{H}_{\mu\lambda}$ we introduce a family of bounded self-adjoint operators (Friedrichs models) $\widehat{\mathbf{h}}_{\mu\lambda}(p)$, $p\in\mathbb{T}^3$, which acts in $\mathcal{H}_0\oplus\mathcal{H}_1$ as

$$\widehat{\mathbf{h}}_{\mu\lambda}(p):=\begin{pmatrix} h_{00}(p) & \lambda h_{01} \\ \lambda h_{01}^* & h_{11}^0(p)-\mu v \end{pmatrix},$$

where

$$h_{00}(p)f_0=w_1(p)f_0,\quad h_{01}f_1=\frac{1}{\sqrt{2}}\int_{\mathbb{T}^3} f_1(s)ds,$$

$$(h_{11}^0(p)f_1)(q)=w_2(p,q)f_1(q),\quad (vf_1)(q)=v_2(q)\int_{\mathbb{T}^3} v_2(s)f_1(s)ds.$$

In [27] it was shown that the operator $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ has at most three eigenvalues.

The following theorem describes [27] the location of the essential spectrum of the operator $\mathbf{H}_{\mu\lambda}$ by the spectrum of the family $\widehat{\mathbf{h}}_{\mu\lambda}(p)$.

Theorem 2.1. *For the essential spectrum of* $\mathbf{H}_{\mu\lambda}$ *the equality*

$$\sigma_{\rm ess}(\mathbf{H}_{\mu\lambda}) = \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) \cup [0;M], \quad M := \frac{9}{2}(2l_1+l_2) \tag{2.1}$$

holds. Moreover, the set $\sigma_{\rm ess}(\mathbf{H}_{\mu\lambda})$ *is a union of at most four intervals.*

Definition 2.2. The sets $\bigcup\limits_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\widehat{\mathbf{h}}_{\mu\lambda}(p))$ and $[0;M]$ are called two-particle and three-particle branches of the essential spectrum of $\mathbf{H}_{\mu\lambda}$.

The following assumption we be needed throughout the paper: the function $v_1(\cdot)$ is periodical on each variable with period $2\pi/3$ and function $v_2(\cdot)$ satisfies the condition

$$\int_{\mathbb{T}^3} v_2(s)g(p,s)ds = 0 \tag{2.2}$$

for any function $g\in L_2^{\rm s}((\mathbb{T}^3)^2)$, which is periodical on each variable with period $2\pi/3$.

Note that the functions $v_2(p) = \sum\limits_{i=1}^{3} c_i\cos(3p^{(i)}/2)$ and $v_2(p) = \sum\limits_{i=1}^{3} c_i\cos(3p^{(i)}/2)\cos(3p^{(i)})$, where c_i, $i=1,2,3$ are arbitrary real numbers, satisfies the condition (2.2). Indeed, for $v_2(p) = \sum\limits_{i=1}^{3} c_i\cos(3p^{(i)}/2)$, we have

$$\int_{\mathbb{T}^3} v_2(s)g(p,s)ds = \int_{\mathbb{T}^3} v_2(s+2\bar\pi/3)g(p,s+2\bar\pi/3)ds = -\int_{\mathbb{T}^3} v_2(s)g(p,s)ds, \quad \bar\pi := (\pi,\pi,\pi),$$

which yields the equality (2.2).

Under the assumption (2.2) the discrete spectrum of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ coincides (see Lemma 3.1 below) with the union of discrete spectra of the operators

$$h_\mu(p) := h_{11}^0(p) - \mu v \quad \text{and} \quad \mathbf{h}_\lambda(p) :\equiv \widehat{\mathbf{h}}_{0\lambda}(p).$$

It follows from the definition of the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$ that their structure is simpler than that of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ and the equality (2.1) can be written as

$$\sigma_{\rm ess}(\mathbf{H}_{\mu\lambda}) = \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}_\lambda(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(h_\mu(p)) \cup [0;M]. \tag{2.3}$$

We introduce the points of the form $p=(p^{(1)},p^{(2)},p^{(3)})$ with $p^{(k)}\in\{0,\pm 2\pi/3\}$, $k=1,2,3$. Direct calculation shows that the number of these points is equal to 27 and for convenience we numerate that points as $p_1,\dots,p_{27}$.

Denote $\overline{l,m} := l,\dots,m$. It is easy to check that the function $w_2(\cdot,\cdot)$ has non-degenerate zero minimum at the points $(p_i,p_j)\in(\mathbb{T}^3)^2$, $i,j=\overline{1,27}$. A similar computation shows that the function $w_2(\cdot,\cdot)$ has non-degenerate maximum at the points $(q_i,q_j)\in(\mathbb{T}^3)^2$, $i,j=\overline{1,27}$, where $q_j=(q_j^{(1)},q_j^{(2)},q_j^{(3)})$ with $q_j^{(k)}\in\{\pi,\pm\pi/3\}$, $k=1,2,3$. So, the equalities $w_2(p_i,p_j)=0$ and $w_2(q_i,q_j)=M$ hold for all $i,j=\overline{1,27}$.

We remark that the definitions of $w_1(\cdot)$ and $w_2(\cdot,\cdot)$ imply the identity $\mathbf{h}_\lambda(p_1)\equiv\mathbf{h}_\lambda(p_i)$ for $i=\overline{2,27}$.

Let us denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 2.3. The operator $\mathbf{h}_\lambda(p_1)$ is said to have a zero energy resonance, if the number 1 is an eigenvalue of the integral operator given by

$$(G_\lambda\psi)(q) = \frac{\lambda^2}{2(l_1+l_2)}\int_{\mathbb{T}^3}\frac{\psi(s)}{\varepsilon(s)}ds, \quad \psi\in C(\mathbb{T}^3)$$

and at least one (up to a normalization constant) of the associated eigenfunctions ψ satisfies the condition $\psi(p_1)\neq 0$. If the number 1 is not an eigenvalue of the operator G_λ, then we say that $z=0$ is a regular type point for the operator $\mathbf{h}_\lambda(p_1)$.

Remark 2.4. We notice that in the Definition 2.3 the requirement of the presence of the eigenvalue 1 of G_λ corresponds to the existence of a solution of $\mathbf{h}_\lambda(p_1)f=0$ and the condition $\psi(p_1)\neq 0$ implies that the solution f of this equation does not belong to $\mathcal{H}_0\oplus\mathcal{H}_1$. More exactly, if $\mathbf{h}_\lambda(p_1)$ has a zero energy resonance, then the vector $f=(f_0,f_1)$, where

$$f_0=\psi(q)\equiv \text{const}, \quad f_1(q) = -\frac{\lambda f_0}{\sqrt{2}(l_1+l_2)\varepsilon(q)}, \tag{2.4}$$

obeys the equation $\mathbf{h}_\lambda(p_1)f=0$ and $f_1\in L_1(\mathbb{T}^3)\setminus L_2(\mathbb{T}^3)$ (see Lemma 3.4).

Set

$$\lambda_0 := \sqrt{2(l_1+l_2)}\Big(\int_{\mathbb{T}^3}\frac{ds}{\varepsilon(s)}\Big)^{-1/2}.$$

Remark 2.5. The number 1 is an eigenvalue of G_λ if and only if $\lambda=\lambda_0$, see Lemma 3.3.

Since the function $w_2(\cdot,\cdot)$ has non-degenerate zero minimum at the points $(p_i,p_j)\in(\mathbb{T}^3)^2$, $i,j=\overline{1,27}$ and the function $v_2(\cdot)$ is a continuous on $\mathbb{T}^3$, for any $p\in\mathbb{T}^3$ the integral

$$I(p) := \int_{\mathbb{T}^3}\frac{v_2^2(s)ds}{w_2(p,s)}$$

is positive and finite. The Lebesgue dominated convergence theorem and the equality $I(p_1)=I(p_i)$, $i=\overline{2,27}$ yield $I(p_1)=\lim_{p\to p_i} I(p_i)$, $i=\overline{1,27}$, and hence the function $I(\cdot)$ is a positive and continuous on $\mathbb{T}^3$.

We introduce the following notations:

$$\mu_1 := \min_{p\in\mathbb{T}^3} I^{-1}(p), \quad \mu_2 := \max_{p\in\mathbb{T}^3} I^{-1}(p);$$

$$a_\mu := \min\Big\{\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(h_\mu(p))\cap(-\infty;0]\Big\}, \quad b_\mu := \max\Big\{\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(h_\mu(p))\cap(-\infty;0]\Big\},$$

for $\mu>\mu_1$.

Note that the operator $\mathbf{h}_{\lambda_0}(p)$ is a non-negative (see Lemma 3.5) and hence

$$\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(\mathbf{h}_{\lambda_0}(p))\cap(-\infty;0)=\emptyset. \tag{2.5}$$

Therefore, the study of the structure of the set $\sigma_{\rm ess}(\mathbf{H}_{\mu\lambda_0})$ is reduced to the study of the structure of the set $\bigcup_{p\in\mathbb{T}^3}\sigma_{\rm disc}(h_\mu(p))\cup[0;M]$.

The following theorem describes the structure of the part of the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$ located in $(-\infty;M]$.

Theorem 2.6. *The following assertions hold.*
(i) *If* $\mu \in [0;\mu_1]$, *then* $(-\infty;M] \cap \sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = [0;M]$;
(ii) *If* $\mu \in (\mu_1;\mu_2]$, *then* $(-\infty;M] \cap \sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = [a_\mu;M]$ *and* $a_\mu < 0$;
(iii) *If* $\mu \in (\mu_2;+\infty)$, *then* $(-\infty;M] \cap \sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = [a_\mu;b_\mu] \cup [0;M]$ *and* $a_\mu < b_\mu < 0$.

As in the introduction, let us denote by $\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$ the bottom of the essential spectrum $\sigma_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$ of $\mathbf{H}_{0\lambda}$ and by $N_\lambda(z)$ the number of eigenvalues of $\mathbf{H}_{0\lambda}$ on the left of z, $z < \tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$. The assertion (i) of Theorem 2.6 implies that $\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda_0}) = 0$.

We will denote by $\mathbb{N}$ the set of all positive integers.

The main results of the present paper as follows.

Theorem 2.7. *The operator* $\mathbf{H}_{0\lambda_0}$ *has infinitely many negative eigenvalues* $(E_n)_{n\in\mathbb{N}}$ *such that* $\lim\limits_{n\to\infty} E_n = 0$, *and the function* $N_{\lambda_0}(\cdot)$ *obeys the relation*

$$\lim_{z\to -0} \frac{N_{\lambda_0}(z)}{|\log|z||} = \mathcal{U}_0, \quad 0 < \mathcal{U}_0 < \infty. \tag{2.6}$$

Remark 2.8. The constant $\mathcal{U}_0$ does not depend on the function $v_1(\cdot)$ and is given as a positive function depending only on the ratio l_1/l_2.

Remark 2.9. Clearly, by equality (2.6) the infinite cardinality of the negative discrete spectrum of $\mathbf{H}_{0\lambda_0}$ follows automatically from the positivity of $\mathcal{U}_0$.

Remark 2.10. In fact, a result similar to Theorem 2.7 was proved in [4] for $\mathbf{H}_{0\lambda_0}$ provided that all third order partial derivatives of $w_2(\cdot,\cdot)$ are Hölder continuous functions on $(\mathbb{T}^3)^2$ with a unique non-degenerate minimum. Therefore, Theorem 2.7 can be considered as a generalization of Theorem 2.14 in [4], since in our case the function $w_2(\cdot,\cdot)$ has non-degenerate minimum at 729 different points of $(\mathbb{T}^3)^2$ and it is surprising that the asymptotics (2.6) doesn't depends on the number of these points.

For $n \in \mathbb{N}$ denote by $f^{(n)}$ the eigenvector corresponding to the eigenvalue E_n of $\mathbf{H}_{0\lambda}$ defined in Theorem 2.7.

In the following theorem, we precisely describe the dependence of the location of the eigenvalues $(E_n)_{n\in\mathbb{N}}$ as the eigenvalues of $\mathbf{H}_{\mu\lambda_0}$ on the parameter $\mu \ge 0$.

Theorem 2.11. *For any* $\mu \ge 0$ *the numbers* $(E_n)_{n\in\mathbb{N}}$ *are eigenvalues of* $\mathbf{H}_{\mu\lambda_0}$ *with the eigenvector* $f^{(n)}$, $n \in \mathbb{N}$. *Moreover,*
(i) *if* $\mu \in [0;\mu_1]$, *then the set* $\{E_n : n \in \mathbb{N}\}$ *is located on the below of the bottom of the essential spectrum of* $\mathbf{H}_{\mu\lambda_0}$;
(ii) *if* $\mu \in (\mu_1;\mu_2]$, *then the countable subset of* $\{E_n : n \in \mathbb{N}\}$ *located in the essential spectrum of* $\mathbf{H}_{\mu\lambda_0}$;
(iii) *if* $\mu \in (\mu_2;+\infty)$, *then the countable subset of* $\{E_n : n \in \mathbb{N}\}$ *located in the gap of the essential spectrum of* $\mathbf{H}_{\mu\lambda_0}$.

Since $\lim\limits_{\mu\to\mu_2+0} b_\mu = 0$, it follows from the parts (iii) of Theorems 2.6 and 2.11 that for any given finite number $k \in \mathbb{N}$ there exists $\mu'(k) \in (\mu_2;+\infty)$ such that for $\mu = \mu'(k)$ the set $\{E_n : n \in \mathbb{N}\} \cap [a_\mu;b_\mu]$ consists of k elements.

3 Some spectral properties of the family of Friedrichs models $\widehat{\mathbf{h}}_{\mu\lambda}(p)$

In this section we study some spectral properties of the family of Friedrichs models $\widehat{\mathbf{h}}_{\mu\lambda}(p)$, which plays an important role in the study of spectral properties of $\mathbf{H}_{\mu\lambda}$.

3.1 Spectrum of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$

Let the operator $\mathbf{h}^0(p)$ acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathbf{h}^0(p) := \begin{pmatrix} 0 & 0 \\ 0 & h_{11}^0(p) \end{pmatrix}.$$

The perturbation $\widehat{\mathbf{h}}_{\mu\lambda}(p) - \mathbf{h}^0(p)$ of the operator $\mathbf{h}^0(p)$ is a self-adjoint operator of rank at most 3, and thus, according to the Weyl theorem, the essential spectrum of the operator $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ coincides with the essential spectrum of $\mathbf{h}^0(p)$. It is evident that $\sigma_{\text{ess}}(\mathbf{h}^0(p)) = [m(p); M(p)]$, where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) := \min_{q \in \mathbb{T}^3} w_2(p,q), \quad M(p) := \max_{q \in \mathbb{T}^3} w_2(p,q).$$

This yields $\sigma_{\text{ess}}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) = [m(p); M(p)]$.

For any fixed $\mu, \lambda > 0$ and $p \in \mathbb{T}^3$, we define the analytic functions in $\mathbb{C} \setminus [m(p); M(p)]$ by

$$\Delta_\mu(p\,; z) := 1 - \mu \int_{\mathbb{T}^3} \frac{v_2^2(s) ds}{w_2(p,s) - z}, \quad \boldsymbol{\Delta}_\lambda(p\,; z) := w_1(p) - z - \frac{\lambda^2}{2} \int_{\mathbb{T}^3} \frac{ds}{w_2(p,s) - z};$$

these functions are the Fredholm determinants associated with the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$, respectively.

The following lemma describes the relation between the eigenvalues of the operators $\widehat{\mathbf{h}}_{\mu\lambda}(p)$, $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$.

Lemma 3.1. *The number $z \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ if and only if the number z is an eigenvalue of at least one of the operators $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$.*

Proof. Suppose $(f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is an eigenvector of the operator $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ associated with the eigenvalue $z \in \mathbb{C} \setminus [m(p); M(p)]$. Then f_0 and f_1 satisfy the following system of equations

$$(w_1(p) - z) f_0 + \frac{\lambda}{\sqrt{2}} \int_{\mathbb{T}^3} f_1(s) ds = 0;$$

$$\frac{\lambda}{\sqrt{2}} f_0 + (w_2(p,q) - z) f_1(q) - \mu v_2(q) \int_{\mathbb{T}^3} v_2(s) f_1(s) ds = 0. \tag{3.1}$$

Since for any $z \in \mathbb{C} \setminus [m(p); M(p)]$ and $q \in \mathbb{T}^3$ the relation $w_2(p,q) - z \neq 0$ holds for all $p \in \mathbb{T}^3$, from the second equation of the system (3.1) for f_1 we have

$$f_1(q) = \frac{\mu C_{f_1} v_2(q)}{w_2(p,q) - z} - \frac{\lambda}{\sqrt{2}} \frac{f_0}{w_2(p,q) - z}, \tag{3.2}$$

where

$$C_{f_1} = \int_{\mathbb{T}^3} v_2(s) f_1(s) ds. \tag{3.3}$$

Substituting the expression (3.2) for f_1 into the first equation of the system (3.1) and the equality (3.3), and then using the condition (2.2), we conclude that the system of equations (3.1) has a nontrivial solution if and only if the system of equations

$$\begin{aligned} &\mathbf{\Delta}_\lambda(p\,;z) f_0 = 0; \\ &\Delta_\mu(p\,;z) C_{f_1} = 0 \end{aligned}$$

has a nontrivial solution, i.e., if the condition $\Delta_\mu(p\,;z)\mathbf{\Delta}_\lambda(p\,;z) = 0$ is satisfied.

If in above analysis we set $\mu = 0$, then $\widehat{\mathbf{h}}_{\mu\lambda}(p) = \mathbf{h}_\lambda(p)$; in this case the number $z \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $\mathbf{h}_\lambda(p)$ if and only if $\mathbf{\Delta}_\lambda(p\,;z) = 0$.

It can be similarly shown that the number $z \in \mathbb{C} \setminus [m(p); M(p)]$ is an eigenvalue of $h_\mu(p)$ if and only if $\Delta_\mu(p\,;z) = 0$. The lemma is proved. □

From Lemma 3.1 it follows that

$$\sigma_{\rm disc}(\widehat{\mathbf{h}}_{\mu\lambda}(p)) = \sigma_{\rm disc}(h_\mu(p)) \cup \sigma_{\rm disc}(\mathbf{h}_\lambda(p)), \tag{3.4}$$

where

$$\sigma_{\rm disc}(h_\mu(p)) = \{z \in \mathbb{C} \setminus [m(p); M(p)] : \Delta_\mu(p\,;z) = 0\}; \tag{3.5}$$

$$\sigma_{\rm disc}(\mathbf{h}_\lambda(p)) = \{z \in \mathbb{C} \setminus [m(p); M(p)] : \mathbf{\Delta}_\lambda(p\,;z) = 0\}. \tag{3.6}$$

So by Lemma 3.1, that is, by the equality (3.4), the study of spectrum of $\widehat{\mathbf{h}}_{\mu\lambda}(p)$ is reduced to the study of spectra of $h_\mu(p)$ and $\mathbf{h}_\lambda(p)$.

The following lemma describes the set of negative eigenvalues of $h_\mu(p)$.

Lemma 3.2. (i) *Let $\mu \in [0;\mu_1]$. Then for any $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ has no negative eigenvalues;*
(ii) *Let $\mu \in (\mu_1;\mu_2]$. Then there exists a non empty open set $D_\mu \subset \mathbb{T}^3$ such that $D_\mu \neq \mathbb{T}^3$ and for any $p \in D_\mu$ the operator $h_\mu(p)$ has a unique negative eigenvalue and for any $p \in \mathbb{T}^3 \setminus D_\mu$ the operator $h_\mu(p)$ has no negative eigenvalues;*
(iii) *Let $\mu > \mu_2$. Then for any $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ has a unique negative eigenvalue.*

Proof. First we prove part (ii). Let $\mu \in (\mu_1;\mu_2]$. Since $\Delta_\mu(\cdot\,;0)$ is a continuous on $\mathbb{T}^3$, the definition of the numbers μ_i, $i = 1,2$ imply $\min\limits_{p\in\mathbb{T}^3} \Delta_\mu(p\,;0) < 0$ and $\max\limits_{p\in\mathbb{T}^3} \Delta_\mu(p\,;0) \geq 0$ for any $\mu \in (\mu_1;\mu_2]$. Then there exist the points $\theta_1, \theta_2 \in \mathbb{T}^3$ such that

$$\min_{p\in\mathbb{T}^3} \Delta_\mu(p\,;0) = \Delta_\mu(\theta_1\,;0) \quad \text{and} \quad \max_{p\in\mathbb{T}^3} \Delta_\mu(p\,;0) = \Delta_\mu(\theta_2\,;0).$$

Therefore, $\Delta_\mu(\theta_1\,;0) < 0$ and $\Delta_\mu(\theta_1\,;0) \geq 0$ for any $\mu \in (\mu_1;\mu_2]$.

Let us introduce the notation: $D_\mu := \{p \in \mathbb{T}^3 : \Delta_\mu(p\,;0) < 0\}$. The continuity of the function $\Delta_\mu(\cdot\,;0)$ and the assertion $\Delta_\mu(\theta_1\,;0) < 0$ implies that D_μ is a non empty open set, and $D_\mu \neq \mathbb{T}^3$ because $\theta_1 \notin D_\mu$.

For any $\mu > 0$ and $p \in \mathbb{T}^3$ the function $\Delta_\mu(p\,;\cdot)$ is continuous and decreasing on $(-\infty;0]$ and $\lim\limits_{z\to-\infty} \Delta_\mu(p\,;z) = 1$. Then for any $p \in D_\mu$ there exists a unique point $e_\mu(p) \in (-\infty;0)$ such that $\Delta_\mu(p\,;e_\mu(p)) = 0$. By the equality (3.5) for any $p \in D_\mu$ the point $e_\mu(p)$ is the unique negative eigenvalue of the operator $h_\mu(p)$.

For any $p \in \mathbb{T}^3 \setminus D_\mu$ and $z < 0$ we have $\Delta_\mu(p\,;z) > \Delta_\mu(p\,;0) \geq 0$. Hence again by the equality (3.5) for each $p \in \mathbb{T}^3 \setminus D_\mu$ the operator $h_\mu(p)$ has no negative eigenvalues.

It is immediate that if $\mu \in [0;\mu_1]$ (resp. $\mu \in (\mu_2;+\infty)$), then $D_\mu = \emptyset$ (resp. $D_\mu = \mathbb{T}^3$), and the above analysis leads again to the case (i) (resp. (iii)). The straightforward details are omitted. The lemma is completely proved. □

3.2 Threshold analysis of the family of Friedrichs models $\mathbf{h}_\lambda(p)$

First we remark that the definitions of $w_1(\cdot)$ and $w_2(\cdot,\cdot)$ imply $\mathbf{\Delta}_\lambda(p_1\,;0) = \mathbf{\Delta}_\lambda(p_i\,;0)$, $i = \overline{2,27}$.

Lemma 3.3. *The following statements are equivalent:*
(i) *the operator* $\mathbf{h}_\lambda(p_1)$ *has a zero energy resonance;*
(ii) $\mathbf{\Delta}_\lambda(p_1\,;0) = 0$;
(iii) $\lambda = \lambda_0$.

Proof. Let the operator $\mathbf{h}_\lambda(p_1)$ have a zero energy resonance for some $\lambda > 0$. Then by Definition 2.3 the equation $G_\lambda\psi = \psi$ has a simple solution $\psi \in C(\mathbb{T}^3)$ which satisfies the condition $\psi(p_1) \neq 0$. This solution is equal to the function $\psi(q) \equiv 1$ (up to a constant factor). Therefore we see that

$$1 = \frac{\lambda^2}{2(l_1+l_2)} \int_{\mathbb{T}^3} \frac{ds}{\varepsilon(s)}$$

and hence

$$\mathbf{\Delta}_\lambda(p_1\,;0) = 1 - \frac{\lambda^2}{2(l_1+l_2)} \int_{\mathbb{T}^3} \frac{ds}{\varepsilon(s)}$$

and so $\lambda = \lambda_0$.

Let for some $\lambda > 0$ the equality $\mathbf{\Delta}_\lambda(p_1\,;0) = 0$ holds and consequently $\lambda = \lambda_0$. Then it is clear that only the function $\psi(q) \equiv 1$ (up to a constant factor) is a solution of the equation $G_\lambda\psi = \psi$, that is, by Definition 2.3 the operator $\mathbf{h}_\lambda(p_1)$ has a zero energy resonance. □

Henceforth, we shall denote by C_1, C_2, C_3 different positive numbers and set

$$\mathbb{T}_\delta := \mathbb{T}^3 \setminus \bigcup_{j=1}^{27} U_\delta(p_j).$$

Lemma 3.4. *The vector* $f = (f_0, f_1)$, *where* f_0 *and* f_1 *are given by* (2.4), *obeys the equation* $\mathbf{h}_{\lambda_0}(p_1)f = 0$ *and* $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$.

Proof. Since the fact that the vector f defined as in Lemma 3.4 satisfies $\mathbf{h}_{\lambda_0}(p_1)f = 0$ is obvious, we show that $f_1 \in L_1(\mathbb{T}^3)\setminus L_2(\mathbb{T}^3)$. The definition of the function $\varepsilon(\cdot)$ implies that there exist $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that

$$C_1|q-p_j|^2 \le \varepsilon(q) \le C_2|q-p_j|^2, \quad q \in U_\delta(p_j), \quad j = \overline{1,27}; \tag{3.7}$$

$$\varepsilon(q) \ge C_3, \quad q \in \mathbb{T}_\delta. \tag{3.8}$$

Using the estimates (3.7) and (3.8) we have

$$\int_{\mathbb{T}^3} |f_1(s)|^2 ds \ge \frac{\lambda_0^2|f_0|^2}{2(l_1+l_2)^2}\int_{U_\delta(p_1)}\frac{ds}{\varepsilon^2(s)} \ge C_2\int_{U_\delta(p_1)}\frac{ds}{|s-p_1|^4} = \infty;$$
$$\int_{\mathbb{T}^3} |f_1(s)| ds = \frac{\lambda_0|f_0|}{\sqrt{2}(l_1+l_2)}\Big(\sum_{j=1}^{27}\int_{U_\delta(p_j)}\frac{ds}{\varepsilon(s)} + \int_{\mathbb{T}_\delta}\frac{ds}{\varepsilon(s)}\Big) \le C_1\sum_{j=1}^{27}\int_{U_\delta(p_j)}\frac{ds}{|s-p_j|^2} + C_3 < \infty.$$

Therefore, $f_1 \in L_1(\mathbb{T}^3)\setminus L_2(\mathbb{T}^3)$. □

Lemma 3.5. *For any $p \in \mathbb{T}^3$ the operator $\mathbf{h}_{\lambda_0}(p)$ has no negative eigenvalues, that is, the operator $\mathbf{h}_{\lambda_0}(p)$ is nonnegative.*

Proof. Let the function $\Lambda(\cdot)$ be defined in $\mathbb{T}^3$ as

$$\Lambda(p) := \int_{\mathbb{T}^3}\frac{ds}{w_2(p,s)}.$$

Using the properties of the function $w_2(\cdot,\cdot)$, one can easily verify that $\Lambda(p_1) = \Lambda(p_i)$ for $i = \overline{2,27}$. First we prove that the inequality $\Lambda(p) < \Lambda(p_1)$ holds for any $p \in \mathbb{T}^3\setminus\{p_1,\dots,p_{27}\}$.

Since the function $w_2(\cdot,\cdot)$ is even the function $\Lambda(\cdot)$ is also even. Then we get

$$\Lambda(p) - \Lambda(p_1) = \frac{1}{4}\int_{\mathbb{T}^3}\frac{2w_2(p_1,s)-(w_2(p,s)+w_2(-p,s))}{w_2(p,s)w_2(-p,s)w_2(p_1,s)}[w_2(p,s)+w_2(-p,s)]ds$$
$$-\frac{1}{4}\int_{\mathbb{T}^3}\frac{[w_2(p,s)-w_2(-p,s)]^2}{w_2(p,s)w_2(-p,s)w_2(p_1,s)}ds.$$

Applying the latter equality and the equality

$$w_2(p_1,s) - \frac{w_2(p,q)+w_2(-p,q)}{2} = \sum_{k=1}^{3}(\cos(3p^{(k)})-1)(1+\cos(3q^{(k)}))$$

we obtain the inequality $\Lambda(p) - \Lambda(p_1) < 0$ for all $p \in \mathbb{T}^3\setminus\{p_1,\dots,p_{27}\}$. It means that the function $\Lambda(\cdot)$ has maximum at the points $p_i \in \mathbb{T}^3$, $i = \overline{1,27}$. Since the function $w_1(\cdot)$ has minimum at the points $p_i \in \mathbb{T}^3$, $i = \overline{1,27}$ the function $\mathbf{\Delta}_\lambda(\cdot\,;0)$ also has minimum at these points. So, $\mathbf{\Delta}_\lambda(p\,;0) > \mathbf{\Delta}_\lambda(p_1\,;0)$ holds for any $p \in \mathbb{T}^3\setminus\{p_1,\dots,p_{27}\}$.

By Lemma 3.3 we have $\mathbf{\Delta}_{\lambda_0}(p_1\,;0) = 0$ and hence

$$\mathbf{\Delta}_{\lambda_0}(p\,;z) > \mathbf{\Delta}_{\lambda_0}(p\,;0) \ge \min_{p\in\mathbb{T}^3}\mathbf{\Delta}_{\lambda_0}(p\,;0) = \mathbf{\Delta}_{\lambda_0}(p_1\,;0) = 0$$

for all $p \in \mathbb{T}^3$ and $z < 0$. Now the equality (3.6) yields that the operator $\mathbf{h}_{\lambda_0}(p)$ has no negative eigenvalues. □

Now we formulate a lemma (zero energy expansion for the Fredholm determinant, leading to behaviors of the zero energy resonance), which is important in the proof of Theorem 2.7, that is, the asymptotics (2.6).

Lemma 3.6. *The following decomposition*

$$\mathbf{\Delta}_{\lambda_0}(p\,;z) = \frac{6\lambda_0^2\pi^2}{(l_1+l_2)^{3/2}}\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p_i|^2-\frac{2z}{9}}+O(|p-p_i|^2)+O(|z|)$$

holds for all $|p-p_i|\to 0$, $i=\overline{1,27}$ *and* $z\to -0$.

Remark 3.7. An analogue lemma for the two-body discrete Schrödinger operator has been proven in [3] in the case where the function $\varepsilon(\cdot)$ is of the form

$$\varepsilon(q)=\sum_{i=1}^{3}(1-\cos q^{(i)}).$$

Proof of Lemma 3.6. Let us sketch the main idea of the proof. Take a sufficiently small $\delta>0$ such that $U_\delta(p_i)\cap U_\delta(p_j)=\emptyset$ for all $i\neq j$, $i,j=\overline{1,27}$.

Using the additivity property of the integral we represent the function $\mathbf{\Delta}_{\lambda_0}(\cdot\,;\cdot)$ as

$$\mathbf{\Delta}_{\lambda_0}(p\,;z)=w_1(p)-z-\frac{\lambda_0^2}{2}\sum_{j=1}^{27}\int_{U_\delta(p_j)}\frac{ds}{w_2(p,s)-z}-\frac{\lambda_0^2}{2}\int_{\mathbb{T}_\delta}\frac{ds}{w_2(p,s)-z}. \tag{3.9}$$

Since the function $w_2(\cdot,\cdot)$ has non-degenerate minimum at the points (p_i,p_j), $i,j=\overline{1,27}$, analysis similar to that in the proof of Lemma 3.5 in [4] show that

$$\int_{U_\delta(p_j)}\frac{ds}{w_2(p,s)-z}=\int_{U_\delta(p_j)}\frac{ds}{w_2(p_i,s)}$$
$$-\frac{4\pi^2}{9(l_1+l_2)^{3/2}}\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p_i|^2-\frac{2z}{9}}+O(|p-p_i|^2)+O(|z|);$$
$$\int_{\mathbb{T}_\delta}\frac{ds}{w_2(p,s)-z}=\int_{\mathbb{T}_\delta}\frac{ds}{w_2(p_i,s)}+O(|p-p_i|^2)+O(|z|)$$

as $|p-p_i|\to 0$ for $i=\overline{1,27}$ and $z\to -0$. Substituting the last two expressions and the the expansion

$$w_1(p)=1+\frac{9}{2}|p-p_i|^2+O(|p-p_i|^4)$$

as $|p-p_i|\to 0$ for $i=\overline{1,27}$, to the equality (3.9) we obtain

$$\mathbf{\Delta}_{\lambda_0}(p\,;z)=\mathbf{\Delta}_{\lambda_0}(p_1\,;0)+\frac{6\lambda_0^2\pi^2}{(l_1+l_2)^{3/2}}\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p_i|^2-\frac{2z}{9}}+O(|p-p_i|^2)+O(|z|)$$

as $|p-p_i|\to 0$ for $i=\overline{1,27}$ and $z\to -0$. Now the equality $\mathbf{\Delta}_{\lambda_0}(p_1\,;0)=0$ completes the proof of the Lemma 3.6. □

Corollary 3.8. *For some* $C_1, C_2, C_3 > 0$ *and* $\delta > 0$ *the following inequalities hold*
(i) $C_1|p-p_i| \le \mathbf{\Delta}_{\lambda_0}(p;0) \le C_2|p-p_i|$, $p \in U_\delta(p_i)$, $i=\overline{1,27}$;
(ii) $\mathbf{\Delta}_{\lambda_0}(p;0) \ge C_3$, $p \in \mathbb{T}_\delta$.

Proof. Lemma 3.6 yields the assertion (i) for some positive numbers C_1, C_2. The positivity and continuity of the function $\mathbf{\Delta}_{\lambda_0}(\cdot\,;0)$ on the compact set $\mathbb{T}_\delta$ imply the assertion (ii). □

4 The structure of the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$

In this section we will prove Theorem 2.6.

Proof of Theorem 2.6. First we recall that by Lemma 3.5 for any $\mathbb{T}^3$ the operator $\mathbf{h}_{\lambda_0}(p)$ is nonnegative and hence the equality holds

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathbf{h}_{\lambda_0}(p)) \cap (-\infty; M] = [0; M]. \tag{4.1}$$

Therefore, by the equality (2.3) it is enough to study the structure of the set

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(h_\mu(p)) \cup [0; M].$$

We consider the following three cases.

(i) Let $\mu \in [0; \mu_1]$. Then by assertion (i) of Lemma 3.2 it follows that for any $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ has no negative eigenvalues, that is,

$$\bigcup_{p\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(h_\mu(p)) \cap (-\infty; 0) = \emptyset.$$

Then equalities (2.3) and (4.1) complete the proof of assertion (i) of Theorem 2.6.

(ii) Let $\mu \in (\mu_1; \mu_2]$. Then by assertion (ii) of Lemma 3.2 there exists a non empty open set $D_\mu \subset \mathbb{T}^3$ such that $D_\mu \neq \mathbb{T}^3$ and for any $p \in D_\mu$ the operator $h_\mu(p)$ has a unique negative eigenvalue $e_\mu(p)$. Since the function $v_2(\cdot)$ is a continuous on $\mathbb{T}^3$ and the function $w_2(\cdot,\cdot)$ is an analytic on $(\mathbb{T}^3)^2$, the function $e_\mu : p \in D_\mu \to e_\mu(p)$ is a continuous on D_μ.

Since for any fixed $\mu > 0$ and $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ is a bounded and $\mathbb{T}^3$ is a compact set, there exists a positive number C_μ such that $\sup\limits_{p\in\mathbb{T}^3} \|h_\mu(p)\| \le C_\mu$. Consequently, for any $p \in \mathbb{T}^3$ we have

$$\sigma(h_\mu(p)) \subset [-C_\mu; C_\mu]. \tag{4.2}$$

For any $q \in \partial D_\mu = \{p \in \mathbb{T}^3 : \Delta_\mu(p;0) = 0\}$ there exist $\{p_n\}_{n\in\mathbb{N}} \subset D_\mu$ such that $p_n \to q$ as $n \to \infty$. If we set $e_\mu^{(n)} = e_\mu(p_n)$, then by Lemma 3.2 for any $n \in \mathbb{N}$ the inequality $e_\mu^{(n)} < 0$ holds and from (4.2) we get $\{e_\mu^{(n)}\}_{n\in\mathbb{N}} \subset [-C_\mu; 0)$. Without loss of generality (otherwise we would have to take a subsequence) we assume that $e_\mu^{(n)} \to e_\mu^{(0)}$ as $n \to \infty$ for some $e_\mu^{(0)} \in [-C_\mu; 0]$.

From the continuity of the function $\Delta_\mu(\cdot\,;\cdot)$ in $\mathbb{T}^3 \times (-\infty; 0]$, and $p_n \to q$ and $e_\mu^{(n)} \to e_\mu^{(0)}$ as $n \to \infty$ it follows that

$$0 = \lim_{n\to\infty} \Delta_\mu(p_n; e_\mu^{(n)}) = \Delta_\mu(q; e_\mu^{(0)}).$$

Since for any $\mu > 0$ and $p \in \mathbb{T}^3$ the function $\Delta_\mu(p;\cdot)$ is decreasing in $(-\infty;0]$ and $q \in \partial D_\mu$ we see that $\Delta_\mu(q;e_\mu^{(0)}) = 0$ if and only if $e_\mu^{(0)} = 0$.

Now for $q \in \partial D_\mu$ we define

$$e_\mu(q) = \lim_{p' \to q,\, p' \in D_\mu} e_\mu(p') = 0.$$

Since the function $e_\mu(\cdot)$ is a continuous on the compact set $D_\mu \cup \partial D_\mu$ and $e_\mu(q) = 0$ for all $q \in \partial D_\mu$ we conclude that $\mathrm{Im}(e_\mu) = [a_\mu;0]$ and $a_\mu < 0$, where $\mathrm{Im}(e_\mu)$ denotes the range of the function $e_\mu(\cdot)$.

Hence the set

$$\bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(h_\mu(p)) \cap (-\infty;0]$$

coincides with the set $\mathrm{Im}(e_\mu) = [a_\mu;0]$. Then equalities (2.3) and (4.1) complete the proof of assertion (ii) of Theorem 2.6.

(iii) Let $\mu > \mu_2$. Then by assertion (iii) of Lemma 3.2 for all $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ has a unique negative eigenvalue $e_\mu(p)$. Since the function $e_\mu : p \in D_\mu \to e_\mu(p)$ is a continuous on the compact set $\mathbb{T}^3$ the set $\mathrm{Im}(e_\mu)$ is a connected closed subset of $(-\infty;0)$, that is, $\mathrm{Im}(e_\mu) = [a_\mu;b_\mu]$ with $a_\mu < b_\mu < 0$ and hence

$$\bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(h_\mu(p)) \cap (-\infty;0] = [a_\mu;b_\mu].$$

Then again the equalities (2.3) and (4.1) complete the proof of assertion (iii) of Theorem 2.6. □

5 Asymptotics for the number of negative eigenvalues of $\mathbf{H}_{0\lambda_0}$

In this section first we review the corresponding Birman-Schwinger principle for the operator $\mathbf{H}_{0\lambda}$ and then we prove the asymptotic relation (2.6) for the number of negative eigenvalues of $\mathbf{H}_{0\lambda_0}$.

5.1 The Birman-Schwinger principle.

For a bounded self-adjoint operator A acting in the Hilbert space $\mathcal{R}$, we define the number $n(\gamma, A)$ by the rule

$$n(\gamma,A) := \sup\{\dim F : (Au,u) > \gamma,\, u \in F \subset \mathcal{R},\, \|u\| = 1\}.$$

The number $n(\gamma,A)$ is equal to the infinity if $\gamma < \max \sigma_{\mathrm{ess}}(A)$; if $n(\gamma,A)$ is finite, then it is equal to the number of the eigenvalues of A bigger than γ.

By the definition of $N_\lambda(z)$, we have

$$N_\lambda(z) = n(-z, -\mathbf{H}_{0\lambda}),\ -z > -\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda}).$$

Since the function $\mathbf{\Delta}_\lambda(\cdot;\cdot)$ is a positive on $\mathbb{T}^3 \times (-\infty;\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda}))$, the positive square root of $\mathbf{\Delta}_\lambda(p;z)$ exists for any $p \in \mathbb{T}^3$ and $z < \tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$.

In our analysis of the discrete spectrum of $\mathbf{H}_{0\lambda}$ the crucial role is played by the self-adjoint compact 2×2 block operator matrix $\widehat{\mathbf{T}}_\lambda(z)$, $z<\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$ acting on $\mathcal{H}_0\oplus\mathcal{H}_1$ as

$$\widehat{\mathbf{T}}_\lambda(z) := \begin{pmatrix} \widehat{T}_{00}(\lambda;z) & \widehat{T}_{01}(\lambda;z) \\ \widehat{T}^*_{01}(\lambda;z) & \widehat{T}_{11}(\lambda;z) \end{pmatrix}$$

with the entries

$$\widehat{T}_{00}(\lambda;z)g_0 = (1+z-w_0)g_0, \quad \widehat{T}_{01}(\lambda;z)g_1 = -\int_{\mathbb{T}^3} \frac{v(s)g_1(s)ds}{\sqrt{\Delta_\lambda(s;z)}};$$

$$(\widehat{T}_{11}(\lambda;z)g_1)(p) = \frac{1}{2\sqrt{\Delta_\lambda(p;z)}}\int_{\mathbb{T}^3} \frac{g_1(s)ds}{\sqrt{\Delta_\lambda(s;z)}(w_2(p,s)-z)}.$$

The following lemma is a modification of the well-known Birman-Schwinger principle for the operator $\mathbf{H}_{0\lambda}$ (see [3, 4, 18, 19, 30]).

Lemma 5.1. *The operator $\widehat{\mathbf{T}}_\lambda(z)$ is compact and continuous in $z<\tau_{\mathrm{ess}}(\mathbf{H}_{0\lambda})$ and*

$$N_\lambda(z) = n(1,\widehat{\mathbf{T}}_\lambda(z)).$$

For the proof of this lemma, see Lemma 5.1 of [4].

5.2 Proof of Theorem 2.7.

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\sigma := L_2(\mathbb{S}^2)$. As we shall see, the discrete spectrum asymptotics of the operator $\widehat{\mathbf{T}}_{\lambda_0}(z)$ as $z\to -0$ is determined by the integral operator $S_{\mathbf{r}}$, $\mathbf{r} = 1/2|\log|z||$ in $L_2((0,\mathbf{r}),\sigma)$ with the kernel

$$S(y,t) := \frac{1}{4\pi^2}\frac{(l_1+l_2)^2}{\sqrt{l_1^2+2l_1l_2}}\frac{1}{(l_1+l_2)\cosh y + l_2 t},$$

where $y = x-x'$, $x,x'\in(0,\mathbf{r})$, $t=\langle\xi,\eta\rangle$, $\xi,\eta\in\mathbb{S}^2$.

The eigenvalues asymptotics for the operator $S_{\mathbf{r}}$ have been studied in detail by Sobolev [30], by employing an argument used in the calculation of the canonical distribution of Toeplitz operators.

Let us recall some results of [30] which are important in our work.

The coefficient in the asymptotics (2.6) of $N_{\lambda_0}(z)$ will be expressed by means of the self-adjoint integral operator $\widehat{S}(\theta)$, $\theta\in\mathbb{R}$, in the space σ, whose kernel is of the form

$$\widehat{S}(\theta,t) := \frac{1}{4\pi^2}\frac{(l_1+l_2)^2}{l_1^2+2l_1l_2}\frac{\sinh[\theta\arccos\frac{l_2}{l_1+l_2}t]}{\sinh(\pi\theta)},$$

and depends on the inner product $t=\langle\xi,\eta\rangle$ of the arguments $\xi,\eta\in\mathbb{S}^2$. For $\gamma>0$, define

$$U(\gamma) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} n(\gamma,\widehat{S}(\theta))d\theta.$$

This function was studied in detail in [30]; where it was used in showing existence proof of the Efimov effect. In particular, as it was shown in [30], the function $U(\cdot)$ is continuous in $\gamma > 0$, and the limit

$$\lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}) = U(\gamma) \tag{5.1}$$

exists and the number $U(1)$ is positive.

For completeness, we reproduce the following lemma, which has been proven in [30].

Lemma 5.2. *Let $A(z) := A_0(z) + A_1(z)$, where $A_0(z)$ $(A_1(z))$ is compact and continuous for $z < 0$ (for $z \le 0$). Assume that the limit*

$$\lim_{z\to -0} f(z)\, n(\gamma, A_0(z)) = l(\gamma)$$

exists and $l(\cdot)$ is continuous in $(0;+\infty)$ for some function $f(\cdot)$, where $f(z) \to 0$ as $z \to -0$. Then the same limit exists for $A(z)$ and

$$\lim_{z\to -0} f(z)\, n(\gamma, A(z)) = l(\gamma).$$

Remark 5.3. Since the function $U(\cdot)$ is continuous with respect to γ, it follows from Lemma 5.2 that any perturbation of $A_0(z)$ treated in Lemma 5.2 (which is compact and continuous up to $z = 0$) does not contribute to the asymptotic relation (2.6). In the rest part of this subsection we use this fact without further comments.

Now we are going to reduce the study of the asymptotics for the operator $\widehat{\mathbf{T}}_{\lambda_0}(z)$ to that of the asymptotics $S_{\mathbf{r}}$.

Let $\mathbf{T}(\delta;|z|)$ be the operator in $\mathcal{H}_0 \oplus \mathcal{H}_1$ defined by

$$\mathbf{T}(\delta;|z|) := \begin{pmatrix} 0 & 0 \\ 0 & T_{11}(\delta;|z|) \end{pmatrix},$$

where $T_{11}(\delta;|z|)$ is the integral operator in $\mathcal{H}_1$ with the kernel

$$\frac{(l_1+l_2)^{3/2}}{54\pi^2} \sum_{i,j=1}^{27} \frac{\chi_\delta(p-p_i)\chi_\delta(q-p_j)(m|p-p_i|^2 + 2|z|/9)^{-\frac{1}{4}}(m|q-p_j|^2 + 2|z|/9)^{-\frac{1}{4}}}{(l_1+l_2)|p-p_i|^2 + 2l_2(p-p_i, q-p_j) + (l_1+l_2)|q-p_j|^2 + 2|z|/9}.$$

Here $m := (l_1^2 + 2l_1 l_2)/(l_1+l_2)$ and $\chi_\delta(\cdot)$ is the characteristic function of the domain $U_\delta(\mathbf{0})$.

The operator $\mathbf{T}(\delta;|z|)$ is called singular part of $\widehat{\mathbf{T}}_{\lambda_0}(z)$.

The main technical point to apply Lemma 5.2 is the following lemma.

Lemma 5.4. *For any $z \le 0$ and small $\delta > 0$ the difference $\widehat{\mathbf{T}}_{\lambda_0}(z) - \mathbf{T}(\delta;|z|)$ is compact and is continuous with respect to $z \le 0$.*

Proof. Since the function $w_2(\cdot,\cdot)$ has non-degenerate zero minimum at the points $(p_i, p_j) \in (\mathbb{T}^3)^2$, $i, j = \overline{1,27}$, we obtain the following expansion

$$w_2(p,q) = \frac{9}{2}\Big[(l_1+l_2)|p-p_i|^2 + 2l_2(p-p_i, q-p_j) + (l_1+l_2)|q-p_j|^2\Big] + O(|p-p_i|^4) + O(|q-p_j|^4)$$

as $|p-p_i|, |q-p_j| \to 0$, for $i,j=\overline{1,27}$. Then there exist $C_1, C_2>0$ and $\delta>0$ such that

$$C_1(|p-p_i|^2+|q-p_j|^2) \le w_2(p,q) \le C_2(|p-p_i|^2+|q-p_j|^2),$$
$$(p,q)\in U_\delta(p_i)\times U_\delta(p_j) \text{ for } i,j=\overline{1,27};$$
$$w_2(p,q)\ge C_1, \quad (p,q)\in \mathbb{T}^2_\delta.$$

Applying last inequalities and Corollary 3.8, one can estimate the kernel of the operator $\widehat{T}_{11}(\lambda_0;z)-T_{11}(\delta;|z|)$, $z\le 0$, by the square-integrable function

$$C_1\sum_{i,j=1}^{27}\Big[\frac{1}{|p-p_i|^{1/2}}+\frac{1}{|q-p_j|^{1/2}}+\frac{|p-p_i|+|q-p_j|}{|p-p_i|^{1/2}(|p-p_i|^2+|q-p_j|^2)|q-p_j|^{1/2}}$$
$$+\frac{|z|^{1/2}}{(|p-p_i|^2+|z|)^{1/4}(|p-p_i|^2+|q-p_j|^2)(|q-p_j|^2+|z|)^{1/4}}+1\Big].$$

Hence, the operator $\widehat{T}_{11}(\lambda_0;z)-T_{11}(\delta;|z|)$ belongs to the Hilbert-Schmidt class for all $z\le 0$. In combination with the continuity of the kernel of the operator with respect to $z<0$, this implies the continuity of $\widehat{T}_{11}(\lambda_0;z)-T_{11}(\delta;|z|)$ with respect to $z\le 0$.

It is easy to see that $\widehat{T}_{00}(\lambda_0;z)$, $\widehat{T}_{01}(\lambda_0;z)$ and $\widehat{T}^*_{01}(\lambda_0;z)$ are rank 1 operators and they are continuous from the left up to $z=0$. Consequently $\widehat{\mathbf{T}}_{\lambda_0}(z)-\mathbf{T}(\delta;|z|)$ is compact and continuous in $z\le 0$. □

From definition of $\mathbf{T}(\delta;|z|)$ it follows that $\sigma(\mathbf{T}(\delta;|z|))=\{0\}\cup\sigma(T_{11}(\delta;|z|))$ and hence $n(\gamma,\mathbf{T}(\delta;|z|))=n(\gamma,T_{11}(\delta;|z|))$ for all $\gamma>0$.

The following theorem is fundamental for the proof of the asymptotic relation (2.6).

Theorem 5.5. *We have the relation*

$$\lim_{|z|\to 0}\frac{n(\gamma,T_{11}(\delta;|z|))}{|\log|z||}=U(\gamma), \quad \gamma>0. \tag{5.2}$$

Proof. The subspace of functions ψ, supported by the set $\Omega_\delta:=\bigcup_{i=1}^{27}U_\delta(p_i)$ is invariant with respect to the operator $T_{11}(\delta;|z|)$.

Let $T^0_{11}(\delta;|z|)$ be the restriction of the integral operator $T_{11}(\delta;|z|)$ to the subspace $L_2(\Omega_\delta)$, that is, the integral operator in $L_2(\Omega_\delta)$ with the kernel $T^0_{11}(\delta;|z|;\cdot,\cdot)$ defined on $\Omega_\delta\times\Omega_\delta$ as

$$T^0_{11}(\delta;|z|;p,q):=\frac{(l_1+l_2)^{3/2}}{54\pi^2}\frac{(m|p-p_i|^2+2|z|/9)^{-\frac14}(m|q-p_j|^2+2|z|/9)^{-\frac14}}{(l_1+l_2)|p-p_i|^2+2l_2(p-p_i,q-p_j)+(l_1+l_2)|q-p_j|^2+2|z|/9},$$

$(p,q)\in U_\delta(p_i)\times U_\delta(p_j)$ for $i,j=\overline{1,27}$.

Since $L_2(\bigcup_{i=1}^{27}U_\delta(p_i))\cong\bigoplus_{i=1}^{27}L_2(U_\delta(p_i))$, we can express the integral operator $T^0_{11}(\delta;|z|)$ as the 27×27 block operator matrix $\mathbf{T}_0(\delta;|z|)$ acting on $\bigoplus_{i=1}^{27}L_2(U_\delta(p_i))$ as

$$\mathbf{T}_0(\delta;|z|):=\begin{pmatrix} T_0^{(1,1)}(\delta;|z|) & \dots & T_0^{(1,27)}(\delta;|z|)\\ \vdots & \ddots & \vdots \\ T_0^{(27,1)}(\delta;|z|) & \dots & T_0^{(27,27)}(\delta;|z|)\end{pmatrix},$$

where for $i, j = \overline{1,27}$ the operator $T_0^{(i,j)}(\delta;|z|) : L_2(U_\delta(p_j)) \to L_2(U_\delta(p_i))$ is the integral operator with the kernel $T_0(\delta;|z|;p,q)$, $(p,q) \in U_\delta(p_i) \times U_\delta(p_j)$.

Set

$$L_2^{(27)}(U_r(\mathbf{0})) := \{\phi = (\phi_1, \cdots, \phi_{27}) : \phi_i \in L_2(U_r(\mathbf{0})), i = \overline{1,27}\}.$$

It is easy to show that $\mathbf{T}_0(\delta;|z|)$ is unitarily equivalent to the 27×27 block operator matrix $\mathbf{T}_1(r)$, $r = |z|^{-\frac{1}{2}}$, acting on $L_2^{(27)}(U_r(\mathbf{0}))$ as

$$\mathbf{T}_1(r) := \begin{pmatrix} T_1(r) & \dots & T_1(r) \\ \vdots & \ddots & \vdots \\ T_1(r) & \dots & T_1(r) \end{pmatrix},$$

where $T_1(r)$ is the integral operator on $L_2(U_r(\mathbf{0}))$ with the kernel

$$\frac{(l_1+l_2)^{3/2}}{54\pi^2} \frac{(m|p|^2+2/9)^{-\frac{1}{4}}(m|q|^2+2/9)^{-\frac{1}{4}}}{(l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2+2/9}.$$

The equivalence is realized by the unitary dilation (27×27 diagonal matrix)

$$\mathbf{B}_r := \mathrm{diag}\{B_r^{(1)}, \dots, B_r^{(27)}\} : \bigoplus_{i=1}^{27} L_2(U_\delta(p_i)) \to L_2^{(27)}(U_r(\mathbf{0})),$$

Here for $i = \overline{1,27}$ the operator $B_r^{(i)} : L_2(U_\delta(p_i)) \to L_2(U_r(\mathbf{0}))$ acts as

$$(B_r^{(i)} f)(p) = (r/\delta)^{-3/2} f(\delta p/r + p_i).$$

Let $\mathbf{A}_r$ and $\mathbf{E}$ be the 27×1 and 1×27 matrices of the form

$$\mathbf{A}_r := \begin{pmatrix} T_1(r) \\ \vdots \\ T_1(r) \end{pmatrix}, \quad \mathbf{E} := (I \dots I),$$

respectively, where I is the identity operator on $L_2(U_r(\mathbf{0}))$.

It is well known that if B_1, B_2 are bounded operators and $\gamma \neq 0$ is an eigenvalue of $B_1 B_2$, then γ is an eigenvalue for $B_2 B_1$ as well of the same algebraic and geometric multiplicities (see *e.g.* [10]). Therefore, $n(\gamma, \mathbf{A}_r\mathbf{E}) = n(\gamma, \mathbf{E}\mathbf{A}_r)$, $\gamma > 0$. Direct calculation shows that $\mathbf{T}_1(r) = \mathbf{A}_r\mathbf{E}$ and $\mathbf{E}\mathbf{A}_r = 27T_1(r)$. So, for $\gamma > 0$ we have $n(\gamma, \mathbf{T}_1(r)) = n(\gamma, 27T_1(r))$.

Furthermore, replacing

$$(m|p|^2+2/9)^{\frac{1}{4}}, \quad (m|q|^2+2/9)^{\frac{1}{4}} \quad \text{and} \quad (l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2+2/9$$

by the expressions

$$(m|p|^2)^{\frac{1}{4}}(1-\chi_1(p))^{-1}, \quad (m|q|^2)^{\frac{1}{4}}(1-\chi_1(q))^{-1} \quad \text{and} \quad (l_1+l_2)|p|^2+2l_2(p,q)+(l_1+l_2)|q|^2,$$

respectively, we obtain the integral operator $T_2(r)$. The error $27T_1(r) - T_2(r)$ is a Hilbert-Schmidt operator and continuous up to $z = 0$.

Using the dilation

$$M : L_2(U_r(\mathbf{0}) \setminus U_1(\mathbf{0})) \to L_2((0,\mathbf{r}),\sigma), \quad (Mf)(x,w) = e^{3x/2} f(e^x w),$$

where $\mathbf{r} = 1/2|\log|z||$, $x \in (0,\mathbf{r})$, $w \in \mathbb{S}^2$, one sees that the operator $T_2(r)$ is unitarily equivalent to the integral operator $S_{\mathbf{r}}$.

Since the difference of the operators $S_{\mathbf{r}}$ and $T_{11}(\delta;|z|)$ is compact (up to unitary equivalence) and hence, since $\mathbf{r} = 1/2|\log|z||$, we obtain the equality

$$\lim_{|z|\to 0} \frac{n(\gamma, T_{11}(\delta;|z|))}{|\log|z||} = \lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}), \quad \gamma > 0.$$

Now Lemma 5.2 and the equality (5.1) completes the proof of Theorem 5.5. □

We are now ready for the

Proof of Theorem 2.7. Using Lemmas 5.2, 5.4 and Theorem 5.5 we have that

$$\lim_{|z|\to 0} \frac{n(1, \mathbf{T}_{\lambda_0}(z))}{|\log|z||} = U(1).$$

Taking into account the last equality and Lemma 5.1, and setting $\mathcal{U}_0 = U(1)$, we complete the proof of Theorem 2.7. □

6 Infiniteness of the number of embedded eigenvalues of $\mathbf{H}_{\mu\lambda_0}$

In this section we shall prove Theorem 2.11.

Proof of Theorem 2.11. First we recall that by the assertion (i) of Theorem 2.6 we have $\min \sigma_{\text{ess}}(\mathbf{H}_{0\lambda_0}) = 0$ and by Theorem 2.7 the operator $\mathbf{H}_{0\lambda_0}$ has infinitely many negative eigenvalues $E_1, \ldots, E_n, \ldots$, accumulating at zero. Let $f^{(1)}, \ldots, f^{(n)}, \ldots$ be the corresponding eigenvectors.

Denote by $\mathcal{L}_0$ the subspace of all eigenvectors of $\mathbf{H}_{0\lambda_0}$, corresponding to the all negative eigenvalues. We show that $V\Big|_{\mathcal{L}_0} = 0$.

Let $z < 0$ be an eigenvalue of the operator $\mathbf{H}_{0\lambda_0}$ and $f = (f_0, f_1, f_2) \in \mathcal{H}$ be the corresponding eigenvector. Then f_0, f_1 and f_2 are satisfy

$$(w_0 - z) f_0 + \int_{\mathbb{T}^3} v_1(s) f_1(s) ds = 0;$$

$$v_1(p) f_0 + (w_1(p) - z) f_1(p) + \lambda \int_{\mathbb{T}^3} f_2(p,s) ds = 0; \tag{6.1}$$

$$\frac{\lambda}{2}(f_1(p) + f_1(q)) + (w_2(p,q) - z) f_2(p,q) = 0.$$

Since $z \notin [0; M]$, from the third equation of the system (6.1) for f_2 we have

$$f_2(p,q) = -\frac{\lambda(f_1(p) + f_1(q))}{2(w_2(p,q) - z)}. \tag{6.2}$$

Substituting the expression (6.2) for f_2 into the second equation of the system (6.1), we obtain

$$f_1(p) = \frac{\lambda_0^2}{2\Delta_{\lambda_0}(p;z)} \int_{\mathbb{T}^3} \frac{f_1(s)ds}{w_2(p,s)-z} - \frac{v_1(p)f_0}{\Delta_{\lambda_0}(p;z)}.$$

Since the functions $v_1(\cdot)$, $w_1(\cdot)$ and $w_2(\cdot,\cdot)$ are periodic of each variable with period $2\pi/3$ this implies that $f_1(\cdot)$ is also a periodic function on each variable with period $2\pi/3$. Therefore, the function $f_2(\cdot,\cdot)$ defined by (6.2), is a periodic function on each six variables with period $2\pi/3$. Hence this function satisfies the condition (2.2), that is, $Vf_2 = 0$ for any $f_2 \in \mathcal{L}_0$. From here, in particular, it follows that $Vf_2^{(n)} = 0$ for any $n \in \mathbb{N}$. Therefore, for any $\mu \geq 0$ the numbers $E_1, \ldots, E_n, \ldots$ are eigenvalues of $\mathbf{H}_{\mu\lambda_0}$ with the same eigenvectors $f^{(n)}$, $n \in \mathbb{N}$.

If $\mu \in [0;\mu_1]$, then by the assertion (i) of Theorem 2.6 we have $\min \sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = 0$. In this case, the set $\{E_n : n \in \mathbb{N}\}$ is located in the below of the bottom of the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$ and $\lim\limits_{n\to\infty} E_n = 0$. Let $\mu \in (\mu_1;\mu_2]$. Then the assertion (ii) of Theorem 2.6 implies that $\sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = [a_\mu; M]$ with $a_\mu < 0$. Hence, the countable part of the set $\{E_n : n \in \mathbb{N}\}$ is located in the essential spectrum of $\mathbf{H}_{\mu\lambda_0}$. If $\mu > \mu_2$, then by the assertion (iii) of Theorem 2.6 we obtain $\sigma_{\mathrm{ess}}(\mathbf{H}_{\mu\lambda_0}) = [a_\mu; b_\mu] \cup [0; M]$ with $a_\mu < b_\mu < 0$. It means that the countable part of the set $\{E_n : n \in \mathbb{N}\}$ is located in $(b_\mu; 0)$. Theorem 2.11 is proved. □

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