Communications in **M**athematical **A**nalysis
Volume 16, Number 2, pp. 1–8 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Hölder Norm of a Fractal Hilbert Transform in Douglis Analysis

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(Communicated by Palle Jorgensen)

Abstract

We establish an upper bound for the norm of a fractal Hilbert transform in the space of Hölder analytic functions in the sense of Douglis.

AMS Subject Classification: Primary 30E20, 30E25; Secondary 45E05.

Keywords: Hyperanalytic functions, fractal dimensions, Whitney decomposition, Hilbert transform, Hölder functions.

1 Introduction

The theory of hyperanalytic functions, mainly devoted by Avron Douglis in the paper [4], represents an analog of the classical theory of analytic (holomorphic) functions. This theory is focuses on the solutions, sometimes called analytic functions in the sense of Douglis, of the principal part of an elliptic system of $2r$ linear partial differential equations with $2r$

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unknowns and two independent variables. The significance and interest of this studies has long been established and we refer the reader to the basic book [6] and the survey paper [9] for a thorough treatment.

In the recent [1, 2] the authors provide conditions for the solvability of the Riemann boundary value problem for hyperanalytic functions on various fractal closed curves.

1.1 Notations and Preliminaries

The Douglis algebra is a commutative, associative algebra generated by two elements i and e, where the multiplication is governed by the rules:

$$i^2 = -1, \quad ie = ei, \quad e^r = 0.$$

Any element a of the algebra may be written as a hypercomplex number in the from $a = \sum_{k=0}^{r-1} a_k e^k$, where, by convention $e^0 = 1$ and each a_k is a complex number. a_0 is the complex part of a, meanwhile $A = \sum_{k=1}^{r-1} a_k e^k$ its nilpotent part.

The conjugation is defined by $\bar{a} = \sum_{k=0}^{r-1} \bar{a}_k e^k$ and a norm is given by $\|a\| := \sum_{k=0}^{r-1} |a_k|$.

If the complex part a_0 of a hypercomplex a is not null, then the multiplicative inverse a^{-1} of a is given by

$$a^{-1} = a_0^{-1} \sum_{k=0}^{r-1} (-1)^k \left(\frac{A}{a_0} \right)^k.$$

For $a_0 = 0$ then a is called nilpotent and it does not have multiplicative inverse.

The hypercomplex Cauchy-Riemann operator $\partial_{\bar{z}}^J$ is given by

$$\partial_{\bar{z}}^J := \partial_{\bar{z}} + J(z) \partial_z, \quad z = x + iy,$$

where $J(z)$ is a known nilpotent hypercomplex function and

$$\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y), \quad \partial_z := \frac{1}{2}(\partial_x - i\partial_y).$$

Its fundamental solution being given by

$$e_z(\zeta) := \frac{1}{2\pi} \frac{\partial_\zeta W(\zeta)}{W(\zeta) - W(z)}, \quad \zeta \neq z,$$

where $W(z) = z + \sum_{k=1}^{r-1} W_k(z) e^k$ denotes the generating solution (see [6, Section 2, pag.11]), whose nilpotent part posses bounded and continuous derivate in the complex plane. The nature of the singularity of $e_z(\zeta)$ is the same as that which the complex Cauchy kernel $\frac{1}{\zeta - z}$ has at $\zeta = z$.

We consider functions f defined in $\mathbb{C}$ and taking values in this algebra. Such a function, called hypercomplex function, may be written as $f = \sum_{k=0}^{r-1} f_k e^k$, where f_k are complex valued functions and each time we assign a property such as continuity, differentiability, etc to f is meant that all components f_k share this property. We say that the function f is hyperanalytic in the open region Ω of $\mathbb{C}$ iff f is continuously differentiable in Ω and satisfies in Ω the equation $\partial_{\bar{z}}^J f = 0$.

Let $\mathbf{E}$ be an arbitrary bounded subset of $\mathbb{R}^2 \cong \mathbb{C}$, whose diameter will be denoted by $|\mathbf{E}|$.

We stand $\mathcal{H}_\nu(\mathbf{E})$ to be the set of all Hölder continuous hypercomplex functions f of exponent ν, $0 < \nu \leq 1$ for which

$$|f|_{\nu,\mathbf{E}} := \sup_{x,y\in\mathbf{E},\ x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\nu} < \infty.$$

and then one can define

$$\|f\|_{\nu,\mathbf{E}} := |f|_{\nu,\mathbf{E}} + \max_{x\in\mathbf{E}} \|f(x)\|.$$

Notation c will be used for constants which may vary from one occurrence to the next.

1.2 d-summable sets in $\mathbb{C}$

The geometric notion of d-summable bounded set $\mathbf{E}$ in $\mathbb{R}^2$ is due to Harrison and Norton in their 1992-paper [7] by the requirement that the improper integral

$$\int_0^1 N_{\mathbf{E}}(x)x^{d-1}dx$$

converges. $N_{\mathbf{E}}(\epsilon)$ stands for the minimal number of ϵ-balls needed to cover $\mathbf{E}$. Note that the notion of d-summable set remains unchanged if $N_{\mathbf{E}}(\epsilon)$ is replaced by the number of k-squares, with $2^{-k} \leq \epsilon < 2^{-k+1}$ intersecting $\mathbf{E}$.

To deal with appropriated extension for hypercomplex functions f defined on a compact set $\mathbf{E}$ to the whole complex plane $\mathbb{C}$ we will consider the Whitney extension operator denoted by $\mathcal{E}_0$, see [10]. Indeed, if $f \in \mathcal{H}_\nu(\mathbf{E})$, then its Whitney extension $\mathcal{E}_0(f)$ belongs to $\mathcal{H}_\nu(\mathbb{C})$ and has partial derivatives of all orders at any point $z \in \mathbb{C}\setminus\mathbf{E}$. Moreover, there exists a constant $c > 0$ depending only on ν such that

$$\|\partial_{\bar z}^J \mathcal{E}_0(f)(z)\| \leq c|f|_{\nu,\mathbf{E}}(\mathrm{dist}(z,\mathbf{E}))^{\nu-1}, \quad z \in \mathbb{C}\setminus\mathbf{E}. \tag{1.1}$$

In what follows, we will take $\Omega \subset \mathbb{C}$ a Jordan domain, the boundary of which is denoted by γ. If $\mathcal{X}(z)$ denotes the characteristic function of the set $\Omega\cup\gamma$, we shall write $f^\omega(z) := \mathcal{X}(z)\mathcal{E}_0(f)(z)$.

For the convenience of the reader we recall the main lines in the construction of Whitney decomposition $\mathcal{W}$ of Ω by k-squares Q; for further details, we refer to [10]. Consider the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$ as well as the collection of closed unit squares defined by it, and let $\mathcal{M}_1$ be the mesh consisting of those unit squares having a non-empty intersection with Ω. We may then recursively define a chain of meshes $\mathcal{M}_k$, $k = 2, 3, \ldots$, each time bisecting the sides of the squares of the foregoing mesh. The squares in the mesh $\mathcal{M}_k$ thus have side length 2^{-k+1} and diameter $|Q| = \sqrt{2}\,2^{-k+1}$. The Whitney decomposition $\mathcal{W}$ of Ω is then obtained by defining, for $k = 2, 3, \ldots$,

$$\begin{aligned} \mathcal{W}^1 &= \{Q \in \mathcal{M}_1 \,|\, \text{all neighbouring squares of } Q \text{ belong to } \Omega\} \\ \mathcal{W}^k &= \{Q \in \mathcal{M}_k \,|\, \text{all neighbouring squares of } Q \text{ belong to } \Omega, \\ &\qquad \text{and} \nexists\, Q^* \in \mathcal{W}^{k-1} : Q \subset Q^*\Big\} \end{aligned}$$

for which it can be proven that

$$\Omega = \bigcup_{k=1}^{+\infty} \mathcal{W}^k = \bigcup_{k=1}^{+\infty} \bigcup_{Q\in\mathcal{W}^k} Q \equiv \bigcup_{Q\in\mathcal{W}} Q$$

all squares Q in $\mathcal{W}$ having disjoint interiors. It holds that

$$\operatorname{dist}(z,\Gamma) \geq \frac{1}{\sqrt{2}}|Q| = 2^{-k+1}, \quad z \in Q,\ Q \in \mathcal{W}^k. \tag{1.2}$$

The following lemma reveals the specific importance of the notion of d-summability applied to the boundary γ of a Jordan domain Ω and relating to the Whitney decomposition $\mathcal{W}$ of Ω.

Lemma 1.1. *[7, Lemma 2] If Ω is a Jordan domain of $\mathbb{C}$ and its boundary γ is d-summable, then the expression $s(d) := \sum_{Q\in\mathcal{W}} |Q|^d$, called d-sum of the Whitney decomposition $\mathcal{W}$ of Ω, is finite.*

For the case of γ be a rectifiable curve, we have the following useful lemma.

Lemma 1.2. *If γ is a rectifiable curve, then for every $\varepsilon > 0$*

$$s(1+\varepsilon) \leq c\frac{length(\gamma)}{\varepsilon}. \tag{1.3}$$

Proof. If we denote by $P_\gamma(\tau)$ the packing number of γ, i.e., the biggest number of disjoint τ-balls with center in γ, (e.g., [8]), then $N_\gamma(\tau) \leq P_\gamma\left(\frac{\tau}{2}\right)$. On the other hand, the rectifiability of γ yields that there exists a constant c such that $c\tau P_\gamma\left(\frac{\tau}{2}\right) \leq length(\gamma)$ and therefore $N_\gamma(\tau) \leq c\tau^{-1} length(\gamma)$. The proof now proceeds similarly to that of [7, Lemma 2]. □

2 The hypercomplex Cauchy type integral and Hilbert transform on d-summable curves

The following definition can be founded in [2].

Definition 2.1. Let $d \in (1,2]$ and let Ω be a domain with d-summable boundary γ and suppose $\nu > d-1$. We define the hypercomplex Cauchy type integral of $f \in \mathcal{H}_\nu(\gamma)$ by the formula

$$(\mathbf{C}^*_\gamma f)(z) = f^\omega(z) - \int_\Omega e_z(\zeta)\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)d\xi d\eta,\ z \in \mathbb{C}\setminus\gamma, \tag{2.1}$$

with $\zeta = \xi + i\eta$.

Proposition 2.2. *[2, Proposition 1] The hypercomplex function* (2.1) *exists for any $z \in \mathbb{C}\setminus\gamma$ and its value does not depend on the particular choice of $\mathcal{E}_0(f)$.*

Let us also introduce the following fractal hypercomplex Hilbert transform

$$(H^*_\gamma f)(t) := 2(\mathbf{C}^{*+}_\gamma f)(t) - f(t), \quad t \in \gamma,$$

where $\mathbf{C}^{*+}_\gamma f$ denotes the trace on γ of the continuous extension of $\mathbf{C}^*_\gamma f$ to $\Omega \cup \gamma$. Note that, $H^*_\gamma f$ may be rewritten as

$$(H^*_\gamma f)(t) = f(t) - 2\int_\Omega e_t(\zeta)\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)d\xi d\eta, \quad t \in \gamma,$$

which is a natural generalization of the more conventional hypercomplex Hilbert transform defined as the following singular principal value operator

$$(H_\gamma f)(t) := \int_\gamma e_z(\zeta)(f(\zeta) - f(t))d\zeta + f(t), \quad t \in \gamma.$$

Remark 2.3. Notice that when γ being sufficiently smooth, the hypercomplex Borel Pompeiu formula [6, Theorem 1.3] give that H^*_γ and H_γ coincide.

Theorem 2.4. *Let γ be d−summable and $\nu > \frac{d}{2}$. Then the Cauchy type integral (2.1) has continuous extension to $\Omega \cup \gamma$. Futhermore, $H^*_\gamma f \in \mathcal{H}_\beta(\gamma)$ with $\beta < \dfrac{2\nu - d}{2 - d}$.*

Proof. We observe that if $\nu > \frac{d}{2}$, then $\dfrac{2-d}{1-\nu} > 2$. Then we can choose p such that $2 < p < \dfrac{2-d}{1-\nu}$. We now claim that $\partial^J_{\bar z}\mathcal{E}_0(f) \in L^p(\Omega)$. In fact,

$$\begin{aligned}\int_\Omega \|\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)\|^p d\xi d\eta &= \sum_{Q\in\mathcal{W}}\int_Q \|\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)\|^p d\xi d\eta \le c|f|^p_{\nu,\gamma}\sum_{Q\in\mathcal{W}}\int_Q (\text{dist}(\zeta,\gamma))^{p(\nu-1)}d\xi d\eta \\ &\le c|f|^p_{\nu,\gamma}\sum_{Q\in\mathcal{W}}|Q|^{2+p(\nu-1)}.\end{aligned}$$

Observe that the last sum above is finite since γ is d-summable and $2 + p(\nu - 1) > d$. Then the integral term in (2.1) represents a continuous function in $\mathbb{C}$, what forces $\mathbf{C}^*_\gamma f$ to admits a continuus extension to $\Omega \cup \gamma$, whence H^*_γ is well defined. Moreover,

$$T_\Omega[\partial^J_{\bar z}\mathcal{E}_0 f](z) := \int_\Omega e_z(\zeta)\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)d\xi d\eta \in \mathcal{H}_{(p-2)/p}(\mathbb{C}),$$

following [6, Theorem 1.25]. Hence it implies that $H^*_\gamma f \in \mathcal{H}_\beta(\gamma)$ for any β satisfying $\beta < \dfrac{2\nu - d}{2 - d}$. □

2.1 Hölder norm estimate for H^*_γ

From Theorem 2.4, we learn that the Hilbert transform H^*_γ acts from $\mathcal{H}_\nu(\gamma)$ into $\mathcal{H}_\beta(\gamma)$ whenever $0 < \beta < \dfrac{2\nu - d}{2 - d}$.

We will now show that H^*_γ is a bounded operator between these spaces and present an upper bound for its Hölder norm. The main result is of comparable strength to that of [1, Theorem 5] for the case of piecesmooth curves.

Theorem 2.5. *Let γ be d-summable and suppose $0<\beta<\dfrac{2\nu-d}{2-d}<1$. Then H^*_γ is bounded from $\mathcal{H}_\nu(\gamma)$ into $\mathcal{H}_\beta(\gamma)$ and*

$$||H^*_\gamma|| \leq 1+|\gamma|^{\nu-\beta}+c_1(s(\delta))^{\frac{1-\beta}{2}}|\gamma|^\beta+c_2(s(\delta))^{\frac{1-\beta}{2}}, \tag{2.2}$$

where $\delta := 2\dfrac{\nu-\beta}{1-\beta}$ and c_1, c_2 depend only on ν,β.

Proof. By choosing $p=\dfrac{2}{1-\beta}$, the proof of Theorem 2.4 reveals that

$$\int_\Omega ||\partial^J_{\bar\zeta}\mathcal{E}_0(f)(\zeta)||^p d\xi d\eta \leq c|f|^p_{\nu,\gamma}\sum_{Q\in\mathcal{W}}|Q|^{2+p(\nu-1)} = c|f|^p_{\nu,\gamma}\sum_{Q\in\mathcal{W}}|Q|^{p(\nu-\beta)}$$
$$= c|f|^p_{\nu,\gamma}s(p(\nu-\beta)) = c|f|^p_{\nu,\gamma}s(\delta).$$

Since $\delta>d$, therefore

$$||\partial^J_{\bar\zeta}\mathcal{E}_0(f)||_{L^p} \leq c^{1/p}|f|_{\nu,\gamma}(s(\delta))^{1/p}.$$

The Hölder inequality leads to

$$||T_\Omega[\partial^J_{\bar\zeta}\mathcal{E}_0(f)](\zeta)|| \leq c||\partial^J_{\bar\zeta}\mathcal{E}_0(f)||_{L^p}\left(\int_\Omega \frac{d\xi d\eta}{||W(\zeta)-W(z)||^q}\right)^{\frac{1}{q}},$$

where $q=\frac{p}{p-1}$ as usual.

Using the basic property of W, see [6, inequality (1.4), pag. 12] we have

$$||T_\Omega[\partial^J_{\bar\zeta}\mathcal{E}_0(f)](\zeta)|| \leq c||\partial^J_{\bar\zeta}\mathcal{E}_0(f)||_{L^p}|\gamma|^{\frac{p-2}{p}} = ||\partial^J_{\bar\zeta}\mathcal{E}_0(f)||_{L^p}|\gamma|^\beta \leq c|f|_{\nu,\gamma}(s(\delta))^{1/p}|\gamma|^\beta,$$

and

$$|T_\Omega[\partial^J_{\bar\zeta}\mathcal{E}_0(f)]|_{\beta,\mathbb{C}} \leq c||\partial^J_{\bar\zeta}\mathcal{E}_0(f)||_{L^p} \leq c|f|_{\nu,\gamma}(s(\delta))^{1/p}.$$

Therefore, for every $\zeta\in\gamma$ we have

$$||H^*_\gamma f(\zeta)|| \leq ||f(\zeta)||+2||T_\Omega[\partial^J_{\bar\zeta}\mathcal{E}_0(f)](\zeta)|| \leq ||f(\zeta)||+2c_1|f|_{\nu,\gamma}(s(\delta))^{1/p}|\gamma|^\beta \leq \left(1+2c_1(s(\delta))^{1/p}|\gamma|^\beta\right)|f|_{\nu,\gamma},$$

$$|H^*_\gamma f|_{\beta,\gamma} \leq |f|_{\beta,\gamma}+2|T_\Omega[\partial^J_{\bar\zeta}\mathcal{E}_0(f)]|_{\beta,\mathbb{C}} \leq |\gamma|^{\nu-\beta}|f|_{\nu,\gamma}+2c_2|f|_{\nu,\gamma}(s(\delta))^{1/p} = \left(|\gamma|^{\nu-\beta}+2c_2(s(\delta))^{1/p}\right)|f|_{\nu,\gamma}.$$

And finally adding these inequalities we obtain

$$||H^*_\gamma f||_{\beta,\gamma} \leq \left(1+|\gamma|^{\nu-\beta}+2c_1(s(\delta))^{\frac{1-\beta}{2}}|\gamma|^\beta+2c_2(s(\delta))^{\frac{1-\beta}{2}}\right)||f||_{\nu,\gamma},$$

which finishes the proof. □

Theorem 2.6. *If γ is rectifiable, then the Hilbert transform H_γ is bounded from $\mathcal{H}_\nu(\gamma)$ into $\mathcal{H}_\beta(\gamma)$ whenever $0<\beta<2\nu-1<1$. Fuhermore,*

$$||H_\gamma|| \leq 1+c_1(length(\gamma))^{\nu-\beta}+c_2(length(\gamma))^{\frac{1-\beta}{2}}+c_3(length(\gamma))^{\frac{1+\beta}{2}}, \tag{2.3}$$

where c_1, c_2, c_3 depend only on ν,β.

Proof. Lemma 1.2 gives the following inequality:

$$s\left(\frac{2}{1-\beta}(\nu-\beta)\right) \leq c\frac{length(\gamma)}{1-\frac{2}{1-\beta}(1-\nu)},$$

by taking $\varepsilon = 1 - \frac{2}{1-\beta}(1-\nu) > 0$. Then the proof follows by direct application of Theorem 2.5. □

Remark 2.7. It is worth noting that the upper bound (2.3) offers certain improvement to that obtained in [3, Theorem 1].

Acknowledgments

The third author acknowledges the partial financial support via “Direction de la Recherche” (U. E. H.), Haiti.

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