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Abstract

In this paper, we show that the space of three-interval scaling functions with the induced metric of $L^2(\mathbb{R})$ consists of three path-components each of which is contractible and hence, the first fundamental group of these spaces is zero. One method to construct simple scaling sets for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ is described. Further, we obtain a characterization of a method to provide simple scaling sets for higher dimensions with the help of lower dimensional simple scaling sets and discuss scaling sets, wavelet sets and multiwavelet sets for a reducing subspace of $L^2(\mathbb{R}^n)$. The contractibility of simple scaling sets for different subspaces are also discussed.

AMS Subject Classification: 42C40.

Keywords: Wavelet; Multiresolution Analysis; Wavelet and Scaling sets; Pathconnectivity; Contractibility.

1 Introduction

Considering the set $\mathcal{W}$ of all orthonormal wavelets as a subspace formed by the induced metric of $L^2(\mathbb{R})$, the completeness property and the topological properties like connectedness and pathconnectedness for $\mathcal{W}$ and certain of its subsets have drawn attention of many contributors in the field of wavelets during the past one decade [2, 9-11, 13, 15-17]. This study has been carried over to higher dimensions as well [9-11, 16]. Different sets of other collections in $L^2(\mathbb{R})$, e.g. frame wavelets, Reisz wavelets, tight frame wavelets have also

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been considered. We denote the collection of all MRA wavelets by $\mathcal{W}_m$ and that of the collection of all MSF wavelets by $\mathcal{W}_s$ in $L^2(\mathbb{R})$. In Wutam Consortium [17], it has been obtained that $\mathcal{W}_m$ is pathconnected, and Speegle [16] has obtained that $\mathcal{W}_s$ is pathconnected. Later, Leon [9] obtains that $\mathcal{W}_m \cap \mathcal{W}_s$ is pathconnected. The pathconnectivity of $\mathcal{W}$ is still an open problem.

Recently, K. D. Merill introduces the term simple wavelet set and discuss about the expansive integer matrix dilations in $\mathbb{R}^2$ which have wavelet sets that are finite unions of convex sets [12]. By a *simple wavelet set*, we mean that it can be written as a finite union of convex sets, or equivalently, as a finite union of bounded convex polygons. The simple wavelet sets and simple scaling sets have been extensively studied in [7, 12-15]. By a *simple scaling set* in $\mathbb{R}^n$, we mean that it can be written as a finite union of convex sets, or equivalently, as a finite union of bounded convex polygons. Aimed at the pathconnectivity and contractibility of scaling sets in different subspaces, we provide a method to obtain simple scaling sets in case of one dimension and also, in higher dimensions for different closed subspaces of $L^2(\mathbb{R}^n)$.

Let $\mathbb{E}$ be a measurable subset of $\mathbb{R}$ such that $\mathbb{E} = 2\mathbb{E}$. Then the space $L^2_{\mathbb{E}}(\mathbb{R})$ defined by

$$L^2_{\mathbb{E}}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}\widehat{f} \subseteq \mathbb{E} \right\}$$

is a closed subspace of $L^2(\mathbb{R})$, where $\operatorname{supp}\widehat{f}$ denotes the support of Fourier transform of f. Hence, it is a Hilbert space under the induced inner product of $L^2(\mathbb{R})$. In case $\mathbb{E} = \mathbb{R}^+ \equiv (0, \infty)$, $L^2_{\mathbb{E}}(\mathbb{R})$ is the classical *Hardy space* $H^2(\mathbb{R})$. The study of subspace wavelets initiated by Dai and Lu [6] in one-dimension has been carried over to higher dimension by Dai, Larson and Speegle [5], and Dai, Diao, Gu and Han [3]. The existence of subspace MRA wavelets in one-dimension have been established in [6] while the existence of subspace wavelet sets for higher-dimension have been established in [5]. Furthermore, Gu and Han [8] discussed the existence of singly generated MRA and subspace MRA wavelets. In this paper, we are studying MSF wavelets on $L^2_{\mathbb{E}}(\mathbb{R}^n)$ and also, providing a class of n-interval scaling sets for $H^2(\mathbb{R})$.

The paper is organized as follows. After providing basic definitions and results in Section 2, we provide a method to obtain simple scaling sets for one dimension in Section 3.1. In order to discover more wavelet sets for different subspaces, we describe a method to obtain simple scaling sets for $H^2(\mathbb{R})$, in Section 3.2. In turn, we obtain a characterization of three-interval scaling sets in $H^2(\mathbb{R})$. Furthermore, we notice that the families of simple wavelet sets determined by these simple scaling sets includes some of the families of wavelet sets obtained by Dai and Larson [4], Arcozzi, Behera and Madan [1]. The Section 3.3 is devoted to establish the existence of simple scaling sets and development of multiwavelet sets through these scaling sets. In the last Section 3.4, we show that the space of three-interval scaling functions with the induced metric of $L^2(\mathbb{R})$ consists of three path-components each of which is contractible and hence, the first fundamental group of these spaces is zero. In addition, we discuss other contractible spaces. It is obvious that every contactible space is pathconnected.

2 Basic definitions and notation

Throughout, A denotes an $n \times n$ expansive matrix having integer entries and A^*, the transpose of A. By an expansive matrix, we mean a matrix for which the eigenvalue is greater than 1 in absolute value.

A family of functions $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\} \subset L^2(\mathbb{R}^n)$ is called an *A-multiwavelet* if the system $\{|detA|^{\frac{j}{2}} \psi_i(A^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, i = 1, 2, \cdots, L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Next, we provide some definitions and results which will be used in the sequel.

Definition 2.1 (2)**.** An MSF (*minimally supported frequency*) *A-multiwavelet* is an A-multiwavelet $\Psi = \{\psi_1, \psi_2, \cdots, \psi_L\}$ such that $|\widehat{\psi_i}| = \chi_{W_i}$ for some measurable sets $W_i \subset \mathbb{R}^n, i = 1, 2, \cdots, L$. For $L = 1$, an MSF A-multiwavelet $\Psi = \{\psi_1\}$ is simply referred as an MSF *A-wavelet*.

Definition 2.2 (2)**.** A measurable set $W \subset \mathbb{R}^n$ is an *A-multiwavelet set* of order L if $W = \dot{\bigcup}_{i=1}^{L} W_i$ ($\dot{\bigcup}$ denotes disjoint union) for some $W_1, W_2, \cdots, W_L$ satisfying

(i) $\sum_{i=1}^{L} \sum_{j \in \mathbb{Z}} \chi_{W_i}(A^{*j}\xi) = 1, \quad$ a.e. $\xi \in \mathbb{R}^n$,

(ii) $\sum_{k \in \mathbb{Z}^n} \chi_{W_i}(\xi + 2k\pi)\chi_{W_{i'}}(\xi + 2k\pi) = 1, \quad$ a.e. $\xi \in \mathbb{R}^n$, $i, i' = 1, 2, \cdots, L$.

The following theorem characterizes all A-multiwavelet sets.

Theorem 2.3 (2)**.** *A measurable set* $W = \dot{\bigcup}_{i=1}^{L} W_i$ *is an A-multiwavelet set of order L if and only if*

(i) $\{A^{*j} W_i : j \in \mathbb{Z}, i = 1, 2, \cdots, L\}$ *is a measurable partition of* $\mathbb{R}^n$*, and*

(ii) for each $i = 1, 2, \cdots, L$*,* $\{W_i + 2k\pi : k \in \mathbb{Z}^n\}$ *is a measurable partition of* $\mathbb{R}^n$*.*

Perhaps, the most elegant method to construct orthonormal A-multiwavelets is based on multiresolution analysis which is a family of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying certain properties.

Definition 2.4. A pair $\left(\left\{V_j\right\}_{j \in \mathbb{Z}}, \varphi\right)$ consisting of a family $\left\{V_j\right\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ together with a function $\varphi \in V_0$ is called a *multiresolution analysis* (MRA) if it satisfies the following conditions:
(i) $V_j \subset V_{j+1}$, (ii) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$, (iii) $\overline{\cup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$, (iv) $f \in V_j$ if and only if $f(A(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$, and (v) $\{\varphi(\cdot - k) \ : \ k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0.

An A-multiwavelet defined by an MRA, is called an MRA *A-multiwavelet* and a function φ is called *scaling function* of the MRA. If Ψ is an MSF A-multiwavelet which arises from an MRA, then its scaling function φ satisfies $|\hat{\varphi}| = \chi_S$, for some measurable set $S \subset \mathbb{R}^n$. Such a set S is called an *A-scaling set* which provides A-multiwavelet sets of order L by $A^*S \setminus S = \dot{\bigcup}_{i=1}^{L} W_i$, where $L = |detA| - 1$. With the help of a characterization of scaling functions $\varphi_1, \varphi_2, \cdots, \varphi_d$ of multiplicity d in $L^2(\mathbb{R}^n)$ given by Calogero [A characterization of scaling functions of multiresolution analyses on general lattices, preprint, 1998], we illustrate a characterization of a scaling function φ such that $|\widehat{\varphi}| = \chi_S$, for some measurable set S of $\mathbb{R}^n$ which will be used in the sequel.

Theorem 2.5. *A function φ such that $|\widehat{\varphi}| = \chi_S$, for some measurable set S of $\mathbb{R}^n$ is a scaling function of an MRA if and only if*

(i) $\{S + 2k\pi : k \in \mathbb{Z}^n\}$ *is a measurable partition of* $\mathbb{R}^n$*,*

(ii) $\bigcup_{j\in\mathbb{Z}} A^{*-j} S = \mathbb{R}^n$*, and* *(iii)* $S \subset A^* S$.

The notion of orthonormal wavelets, minimally supported frequency wavelets, wavelet sets, multiresolution analysis, etc. introduced for $L^2(\mathbb{R}^n)$ are analogously carried over to $L^2_{\mathbb{E}}(\mathbb{R}^n)$, where $L^2_{\mathbb{E}}(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \operatorname{supp}\widehat{f} \subseteq \mathbb{E} \right\}$ together with $A^*\mathbb{E} = \mathbb{E}$ and $\mathbb{E} \subset \mathbb{R}^n$ [3, 5-7].

3 Contractibility of families of simple scaling sets

To discuss the topological behavior viz. path connectivity, contractibility, path-homotopy, etc., of scaling sets and wavelet sets, we concentrate on some classes of scaling sets. Regarding the further investigation, we provide a method to obtain simple wavelet sets associated with multiresolution analysis for closed subspaces of $L^2(\mathbb{R}^n)$.

3.1 A construction of simple scaling sets for $L^2(\mathbb{R})$

In this Section, we provide a method to obtain n-interval wavelet sets associated with multiresolution analysis by dilation 2. The same method can also apply to obtain simple scaling sets by dilation d and hence associated multiwavelet sets of order $d-1$, where $|d| \in \mathbb{N}\backslash\{1\}$. The families of wavelet sets obtained from the n-interval scaling sets by dilation 2 include the various families of wavelet sets obtained by Dai and Larson in [4], and also obtained by Arcozzi, Behera, and Madan in [1].

For $\alpha \in (0, 2\pi)$, the interval $[\alpha - 2\pi, \alpha)$ is a scaling set while there is no scaling set having two intervals. Following is a characterization of scaling sets having three-intervals:

Theorem 3.1 (14)**.** *There are precisely three kinds of three-interval scaling sets described as follows:*

(i) $S_3^1 = \{[\gamma - 2\pi, \alpha) \cup [\beta, \gamma) \cup [\alpha + 2\pi, \beta + 2\pi) : 2\alpha \ge \gamma, \alpha + 2\pi \ge 2\beta, 2\gamma \ge \beta + 2\pi\}$,

(ii) $S_3^2 = \{[\alpha - 2\pi, \beta - 2\pi) \cup [\gamma - 2\pi, \alpha) \cup [\beta, \gamma) : 2\alpha \ge \gamma, \alpha + 2\pi \ge 2\gamma\}$*, and*

(iii) $S_3^3 = \{[\beta - 4\pi, \gamma - 4\pi) \cup [\alpha - 2\pi, \beta - 2\pi) \cup [\gamma - 2\pi, \alpha) : 2\gamma \le \alpha + 2\pi, \gamma \le 2\beta, 2\alpha \le \beta\}$

where $0 < \alpha < \beta < \gamma < 2\pi$.

For an $n \in \mathbb{N}$, choose $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$. Then

$$[\alpha_1, \alpha_2) \cup [\alpha_2, \alpha_3) \cup ... \cup [\alpha_{n-1}, \alpha_n) \cup [\alpha_n, \alpha_1 + 2\pi) \qquad (\eta)$$

divided into n-parts is an interval of measure 2π. Notice that integral translates of any part by 2π keeps the measure of the resulting sets intact which remains 2π-translation congruent to (η).

First, assume that n is an odd natural number greater than 2. Translate the interval $[\alpha_n, \alpha_1 + 2\pi)$ by -2π and intervals $[\alpha_1, \alpha_2)$, $[\alpha_3, \alpha_4)$, $[\alpha_5, \alpha_6), ..., [\alpha_{n-2}, \alpha_{n-1})$, each by -2π, to have

$$S = \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1} - 2\pi, \alpha_{2m} - 2\pi) \cup [\alpha_n - 2\pi, \alpha_1) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m}, \alpha_{2m+1}).$$

By the construction, S is 2π-translation congruent to $[\alpha, \alpha + 2\pi)$. Hence, in order that S be a scaling set, we should have the requirement $S \subset 2S$, which holds when (i) $\alpha_n \leq 2\alpha_1$, and (ii) $2\alpha_n \leq \alpha_1 + 2\pi$, or equivalently, $\alpha_n \leq \min\left(2\alpha_1, \frac{\alpha_1}{2} + \pi\right)$.

Next, assume that n is an even natural number greater than 2. Translate $[\alpha_n, \alpha_1 + 2\pi)$ by -2π, $[\alpha_{n-3}, \alpha_{n-2})$ by 6π and $[\alpha_1, \alpha_2)$, $[\alpha_3, \alpha_4)$, $[\alpha_5, \alpha_6), \ldots, [\alpha_{n-5}, \alpha_{n-4}), [\alpha_{n-2}, \alpha_{n-1})$, each by 2π, to have

$$\begin{aligned} S =& [\alpha_n - 2\pi, \alpha_1) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m}, \alpha_{2m+1}) \cup [\alpha_{n-1}, \alpha_n) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi) \\ & \cup [\alpha_{n-2} + 2\pi, \alpha_{n-1} + 2\pi) \cup [\alpha_{n-3} + 6\pi, \alpha_{n-2} + 6\pi). \end{aligned}$$

By the construction, S is 2π-translation congruent to $[\alpha, \alpha + 2\pi)$. Hence, in order that S be a scaling set, we should have the requirement $S \subset 2S$, which holds when

(i) $\alpha_n \leq 2\alpha_1$, (ii) $2\alpha_{n-1} = \alpha_{n-2} + 2\pi$, (iii) $\alpha_{n-1} + 2\pi \leq 2\alpha_n$,

(iv) $2\alpha_{2m} \leq \alpha_{2m-1} + 2\pi$, where $m \in \left\{1, 2, 3, ..., \frac{n-2}{2}\right\}$, and

(v) $\alpha_{2m} + 2\pi \leq 2\alpha_{2m+1}$, where $m \in \left\{1, 2, 3, ..., \frac{n-4}{2}\right\}$.

In case $n = 4$, we require (i), (ii), (iii) and (iv), as (v) does not arise. We sum up the above in the following:

Theorem 3.2. *Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ in S^1, or equivalently, in $[0, 2\pi)$ be such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$, where n is a natural number. Then*
(a) for odd $n \geq 3$,

$$S = \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1} - 2\pi, \alpha_{2m} - 2\pi) \cup [\alpha_n - 2\pi, \alpha_1) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m}, \alpha_{2m+1}),$$

is a scaling set under the condition $\alpha_n \leq \min\left(2\alpha_1, \frac{\alpha_1}{2} + \pi\right)$.
(b) for even $n \geq 4$,

$$\begin{aligned} S =& [\alpha_n - 2\pi, \alpha_1) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m}, \alpha_{2m+1}) \cup [\alpha_{n-1}, \alpha_n) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi) \\ & \cup [\alpha_{n-2} + 2\pi, \alpha_{n-1} + 2\pi) \cup [\alpha_{n-3} + 6\pi, \alpha_{n-2} + 6\pi), \end{aligned}$$

is a scaling set under the conditions:

(i) $\alpha_n \le 2\alpha_1$, *(ii)* $2\alpha_{n-1} = \alpha_{n-2} + 2\pi$, *(iii)* $\alpha_{n-1} + 2\pi \le 2\alpha_n$,

(iv) $2\alpha_{2m} \le \alpha_{2m-1} + 2\pi$, *where* $m \in \left\{1, 2, 3, ..., \frac{n-2}{2}\right\}$, *and*

(v) $\alpha_{2m} + 2\pi \le 2\alpha_{2m+1}$, *where* $m \in \left\{1, 2, 3, ..., \frac{n-4}{2}\right\}$.

In case $n = 4$, *we require (i), (ii), (iii) and (iv), as (v) does not arise.*

The set of all n-interval scaling sets for $n \ge 3$ obtained through the method being termed M as described in Theorem 3.2 is denoted by $\mathcal{S}_n^M$ whose corresponding the class of wavelet sets by $\mathcal{W}(M, n)$. Consequently, the following result follows immediately by noting that $W = 2S \setminus S$, for $S \in \mathcal{S}_n^M$.

Corollary 3.3. *A member* W *of* $\mathcal{W}(M, n)$, *where* $n \ge 3$ *is of the form:*

$$\text{(i) } W = \bigcup_{m=1}^{\frac{n-1}{2}} [2(\alpha_{2m-1} - 2\pi), 2(\alpha_{2m} - 2\pi)) \cup [2(\alpha_n - 2\pi), \alpha_1 - 2\pi) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m} - 2\pi, \alpha_{2m+1} - 2\pi)$$
$$\cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1}, \alpha_{2m}) \cup [\alpha_n, 2\alpha_1) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [2\alpha_{2m}, 2\alpha_{2m+1}),$$

where n *is odd, and* $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$, *together with* $\alpha_n \le \min\left(2\alpha_1, \frac{\alpha_1}{2} + \pi\right)$.

$$\text{(ii) } W = [2(\alpha_n - 2\pi), \alpha_n - 2\pi) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m-1}, \alpha_{2m}) \cup [\alpha_{n-3}, \alpha_{n-1}) \cup [\alpha_n, 2\alpha_1) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [2\alpha_{2m}, \alpha_{2m-1} + 2\pi)$$
$$\cup \bigcup_{m=1}^{\frac{n-4}{2}} [\alpha_{2m} + 2\pi, 2\alpha_{2m+1}) \cup [\alpha_{n-1} + 2\pi, 2\alpha_n) \cup \bigcup_{m=1}^{\frac{n-4}{2}} [2(\alpha_{2m-1} + 2\pi), 2(\alpha_{2m} + 2\pi))$$
$$\cup [2(\alpha_{n-2} + 2\pi), \alpha_{n-3} + 6\pi) \cup [2(\alpha_{n-3} + 6\pi), 2(\alpha_{n-2} + 6\pi)),$$

where n *is even, and* $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$, $\alpha_n \le 2\alpha_1$, $2\alpha_{n-1} = \alpha_{n-2} + 2\pi$, $\alpha_{n-1} + 2\pi \le 2\alpha_n$, $2\alpha_{2m} \le \alpha_{2m-1} + 2\pi$, $m \in \left\{1, 2, 3, ..., \frac{n-2}{2}\right\}$, *together with* $\alpha_{2m} + 2\pi \le 2\alpha_{2m+1}$, $m \in \left\{1, 2, 3, ..., \frac{n-4}{2}\right\}$.

The following shows that for each $n \ge 3$, $\mathcal{S}_n^M$ is nonempty.

Theorem 3.4. *For each integer* $n \ge 3$, $\mathcal{S}_n^M$ *is nonempty and hence* $\mathcal{W}(M, n)$ *is nonempty.*

Proof. For odd integer $n \ge 3$, $\mathcal{S}_n^M$ is nonempty. This follows by choosing suitable α_1 and α_n, we may have $\alpha_2, ..., \alpha_{n-1}$ as required.
For even integer $n \ge 3$, $\mathcal{S}_n^M$ is nonempty. This follows by writing

$$\alpha_n = 2\alpha_1, \quad 2\alpha_{n-1} = \alpha_{n-2} + 2\pi, \quad \alpha_{n-1} + 2\pi = 2\alpha_n,$$
$$2\alpha_{2m} = \alpha_{2m-1} + 2\pi, \text{ where } m \in \left\{1, 2, 3, ..., \frac{n-2}{2}\right\}, \text{ and}$$
$$\alpha_{2m} + 2\pi = 2\alpha_{2m+1}, \text{ where } m \in \left\{1, 2, 3, ..., \frac{n-4}{2}\right\},$$

we get a linear equation $AX = B$, where $X^* = (\alpha_1, \alpha_2, ..., \alpha_n)_{1\times n}$, $B^* = (2\pi, 2\pi, ..., 2\pi, 0)_{1\times n}$, and

$$A = \begin{pmatrix} -1 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \\ 2 & 0 & \dots & 0 & 0 & 0 & -1 \end{pmatrix}_{n\times n}.$$

Since $\det A = (-1)^{n-1} \cdot (2^n - 1) \neq 0$, the linear equation has a unique solution and we have the result. □

3.2 Construction of simple scaling sets and wavelet sets in $H^2(\mathbb{R})$

In this section, we provide a method to obtain H^2-wavelet sets associated with multiresolution analysis by dilation 2 while Arcozzi, Behera, and Madan [1] have been provided a characterization of all H^2-wavelet sets having finite intervals. The same can be seen for dilation d. By H^2-*wavelet set*, we mean that it is a wavelet set by dilation 2 in $H^2(\mathbb{R})$.

For an $n \in \mathbb{N}$, we choose $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \alpha_{n+1} = 2\pi$. Then

$$[0, \alpha_1) \cup [\alpha_1, \alpha_2) \cup [\alpha_2, \alpha_3) \cup \cdots \cup [\alpha_{n-1}, \alpha_n) \cup [\alpha_n, 2\pi),$$

divided into $(n+1)$-parts is an interval of measure 2π.

The interval $[0, 2\pi)$ is an H^2-scaling set having one interval follows from the Theorem 2.5 whose associated H^2-wavelet set is $[2\pi, 4\pi)$. By H^2-*scaling set*, we mean that it is a scaling set by dilation 2 in $H^2(\mathbb{R})$. In fact, S is a *scaling set* by dilation 2 for the classical Hardy space $H^2(\mathbb{R})$ if and only if

(i) $S \subset [0, \infty)$ contains a neighborhood of zero,

(ii) $S \subset 2S$, and (iii) S is 2π-translation congruent to $[\alpha, \alpha + 2\pi)$, where $\alpha \in \mathbb{R}$,

and associated H^2-wavelet set is $2S \setminus S$.

Next, we see that there is no 2-interval H^2-scaling set. For this, suppose $S = [0, a) \cup [b, c)$, where $0 < a < b < c$, is a 2-interval scaling set. Since S is 2π-translation congruent to an interval of measure 2π of $\mathbb{R}$, we have $a < 2\pi, c - b < 2\pi, a = b - 2m\pi$, and $2\pi = c - 2m\pi$, for an $m \in \mathbb{N}$. Also, $c \leq 2a$ because $S \subset 2S$. Since $c \leq 2a$, $a < 2\pi$ and $2\pi = c - 2m\pi$, therefore $c = 2(m+1)\pi \leq 2a$, i.e. $2(m+1)\pi < 4\pi$, and hence $m < 1$, which is a contradiction. Therefore, there is no H^2-scaling set having two intervals.

Next, we obtain a characterization of 3-interval H^2-scaling sets. The families of 3-interval wavelet sets have been characterized by Arcozzi, Behera, and Madan [1].

Theorem 3.5. *Three-interval H^2-scaling sets are precisely*

$$S \equiv [0, \alpha_1) \cup [\alpha_2, 2\pi) \cup [\alpha_1 + 2\pi, \alpha_2 + 2\pi),$$

under the conditions $\pi \leq \alpha_1$, $2\alpha_2 \leq \alpha_1 + 2\pi$, and $0 < \alpha_1 < \alpha_2 < 2\pi$. The associated H^2-wavelet set to S is given by
$W \equiv 2S \setminus S = [\alpha_1, \alpha_2) \cup [2\pi, 2\alpha_1) \cup [2\alpha_2, \alpha_1 + 2\pi) \cup [\alpha_2 + 2\pi, 4\pi) \cup [2(\alpha_1 + 2\pi), 2(\alpha_2 + 2\pi)).$

Proof. Suppose $S = [0,a) \cup [b,c) \cup [d,e)$, where $0 < a < b < c < d < e$, is a 3-interval H^2-scaling set. As a scaling set satisfies $S \subset 2S$, we shall have either of the following cases:
Case (i) $e \leq 2a$.
Case (ii) (1) $c \leq 2a$, (2) $2b \leq d$, and (3) $e \leq 2c$.

Since an H^2-scaling set is associated with an H^2-MRA, S is 2π-translation congruent to an interval of $\mathbb{R}$ of measure 2π, and hence $a < 2\pi, c-b < 2\pi$, and $e-d < 2\pi$. Suppose S is 2π-translation congruent to $[0, 2\pi)$. Then, we have either of the following situations, for $m, n \in \mathbb{N} \cup \{0\}$:

$$(A) \quad \begin{aligned} a &= b - 2m\pi \\ c - 2m\pi &= d - 2n\pi \\ e - 2n\pi &= 2\pi \end{aligned} \qquad , \qquad (B) \quad \begin{aligned} a &= d - 2n\pi \\ e - 2n\pi &= b - 2m\pi \\ c - 2m\pi &= 2\pi \end{aligned}$$

Consider Case (ii) with (B). Since $c \leq 2a$ and $a < 2\pi$, from $2\pi = c - 2m\pi$, we deduce that $m = 0$, and hence $c = 2\pi$ and $e = b + 2n\pi$. Also, $e \leq 2c$ gives $n = 0, 1$. If $n = 0$, then $e = b$, which contradicts the fact that $b < e$. Finally, $m = 0$ and $n = 1$ give

$$S = [0, a) \cup [b, 2\pi) \cup [a + 2\pi, b + 2\pi),$$

which is an H^2-scaling set under conditions $\pi \leq a$, $2b \leq a + 2\pi$, and $0 < a < b < 2\pi$. Now, we arrive at a similar contradiction by considering other cases. □

Next, we provide a method to obtain simple scaling sets and wavelet sets for the Hardy space which is analogous to given in Section 3.1.

Theorem 3.6. *Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be in $[0, 2\pi)$ such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \alpha_{n+1} = 2\pi$, where n is a natural number. Then*
(a) $[0, 2\pi)$ is an H^2-scaling set,

(b) for odd $n > 1$,

$$S \equiv \bigcup_{m=0}^{\frac{n-1}{2}} [\alpha_{2m}, \alpha_{2m+1}) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi) \cup [\alpha_n + 4\pi, 6\pi),$$

is an H^2-scaling set under the conditions:

(i) $\pi \leq \alpha_2$, (ii) $2\alpha_1 = \alpha_n$, (iii) $2\alpha_{2m} \leq \alpha_{2m-1} + 2\pi$, and
(iv) $\alpha_{2m} + 2\pi \leq 2\alpha_{2m+1}$, where $m \in \left\{1, 2, \cdots, \frac{n-1}{2}\right\}$,

(c) for even $n \geq 2$,

$$S \equiv \bigcup_{m=0}^{\frac{n}{2}} [\alpha_{2m}, \alpha_{2m+1}) \cup \bigcup_{m=1}^{\frac{n}{2}} [\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi),$$

is an H^2-scaling set under the conditions:

(i) $\pi \leq \alpha_1$, (ii) $2\alpha_{2m} \leq \alpha_{2m-1} + 2\pi$, where $m \in \left\{1, 2, \cdots, \frac{n}{2}\right\}$, and
(iii) $\alpha_{2m} + 2\pi \leq 2\alpha_{2m+1}$, where $m \in \left\{1, 2, \cdots, \frac{n-2}{2}\right\}$.
Hence, the associated wavelet sets can be obtained by $2S \setminus S$.

3.3 Construction of simple scaling sets and wavelet sets in higher dimension

In this section, we provide a method to construct scaling sets, wavelet sets, multiwavelet sets in higher dimensions. Let C denote an n-cube or a union of n-cubes in $\mathbb{R}^n$. By an n-cube in $\mathbb{R}^n$, we mean I^n, where I is a nondegenerate interval in $\mathbb{R}$. In this Section, a method to construct (L^2_C, A)-scaling sets for certain C has been described and thus (L^2_C, A)-wavelet sets for C are obtained. By an (L^2_C, A)-*scaling set*, we mean that it is a scaling set by dilation A in $L^2_C(\mathbb{R}^n)$. The set of all functions in $L^2(\mathbb{R}^n)$, the support of whose Fourier transforms is contained in a set $\mathbb{E}$ of $\mathbb{R}^n$ with positive Lebesgue measure is denoted by $L^2_{\mathbb{E}}(\mathbb{R}^n)$. In case $\mathbb{E} = [0, \infty)$, $L^2_{\mathbb{E}}(\mathbb{R})$ is the celebrated Hardy space $H^2(\mathbb{R})$ [3, 5-7].

For this, we begin with an $(L^2_{\mathbb{E}}, d)$-scaling set S in $\mathbb{E}$, where $|d| \in \mathbb{N}\backslash\{1\}$ and $\mathbb{E}$ is a measurable subset of $\mathbb{R}$. Let $\mathbb{E}_n$ denote the cartesian product $\underbrace{\mathbb{E}\times\mathbb{E}\times\cdots\times\mathbb{E}}_{n-\text{times}}$, and likewise S_n denote $\underbrace{S\times S\times\cdots\times S}_{n-\text{times}}$ whose elements are row-wise. We choose an $n \times n$ expansive matrix A with entries as integers satisfying $A^*\mathbb{E}_n = \mathbb{E}_n$ and $S_n \subset A^*S_n$. Since S contains a neighborhood of zero in $\mathbb{E}$, S_n will contain a neighborhood of zero in $\mathbb{E}_n$, and hence $\bigcup_{j\in\mathbb{Z}} A^{*j}S_n = \mathbb{E}_n$. To prove that S_n is an $(L^2_{\mathbb{E}_n}, A)$-scaling set, it suffices to show that $\{S_n + 2k\pi : k \in \mathbb{Z}^n\}$ is a measurable partition of $\mathbb{R}^n$. Since S is 2π-translation congruent to $[0, 2\pi)$, the map $\tau : S \longrightarrow [0, 2\pi)$ defined by $\tau(x) = x + 2m\pi$, for $m \in \mathbb{Z}$ is bijective, and hence the map $\tau_n : S_n \longrightarrow [0, 2\pi)^n$, defined by

$$\begin{aligned}\tau_n(x_1, x_2, \cdots, x_n) &= (\tau(x_1), \tau(x_2), \cdots, \tau(x_n)), \text{ for } x_i \in S,\ 1 \le i \le n\\ &= (x_1 + 2m_1\pi, x_2 + 2m_2\pi, \cdots, x_n + 2m_n\pi), \text{ for } m_i \in \mathbb{Z},\ 1 \le i \le n\\ &= (x_1, x_2, \cdots, x_n) + 2(m_1, m_2, \cdots, m_n)\pi,\end{aligned}$$

is also bijective. Thus, we obtain

Theorem 3.7. *Let S be an $(L^2_{\mathbb{E}}, d)$-scaling set in $\mathbb{E}$, where $|d| \in \mathbb{N}\backslash\{1\}$ and $\mathbb{E}$ is a measurable subset of $\mathbb{R}$. Choose an $n \times n$ expansive matrix A with integer entries such that $\mathbb{E}_n = A^*\mathbb{E}_n$, and $S_n \subset A^*S_n$. Then S_n is an $(L^2_{\mathbb{E}_n}, A)$-scaling set whose associated $(L^2_{\mathbb{E}_n}, A)$-multiwavelet set of order $|det(A)| - 1$ is $A^*S_n\backslash S_n$.*

Note that for scaling set S, the condition $S_n \subset A^*S_n$ does not necessarily hold. For example, consider the 2-scaling set S, where

$$S = \left[-\frac{2\pi}{7}, \frac{6\pi}{7}\right) \cup \left[\frac{10\pi}{7}, \frac{12\pi}{7}\right) \cup \left[\frac{20\pi}{7}, \frac{24\pi}{7}\right).$$

Then the condition $S_2 \not\subset A^*S_2$, for $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ which can be seen in Figures 1 and 2.

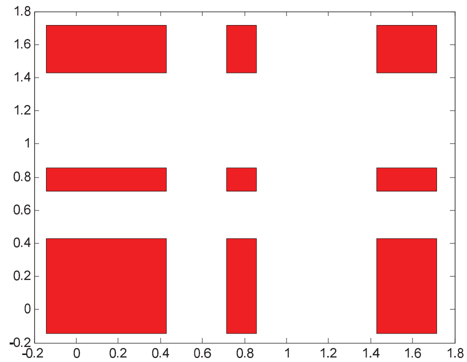

Figure 1. $S\times S$

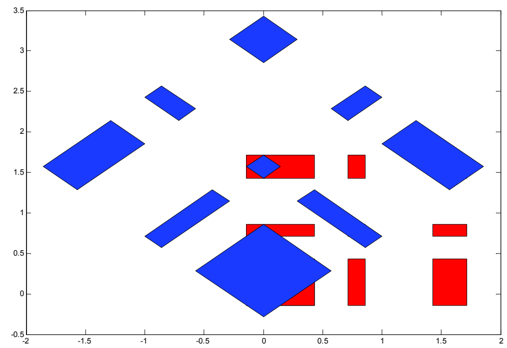

Figure 2. $S\times S \not\subset \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}(S\times S)$

Next, we provide a characterization.

Theorem 3.8. *Let $\mathbb{E}$ be a measurable subset of $\mathbb{R}$ and $A = \begin{pmatrix} 0_{1\times(n-1)} & d \\ I_{(n-1)\times(n-1)} & 0_{(n-1)\times 1}\end{pmatrix}_{n\times n}$, where $d\in\mathbb{N}\backslash\{1\}$. The set S is an $(L^2_{\mathbb{E}},d)$-scaling set if and only if S_n is an $(L^2_{\mathbb{E}_n},A)$-scaling set. Moreover, $S_{(n-1)}\times(dS\backslash S)$ is an $(L^2_{\mathbb{E}_n},A)$-multiwavelet set of order $(d-1)$ associated with an MRA.*

Proof. Suppose S is an $(L^2_{\mathbb{E}},d)$-scaling set. As $[S_{(n-1)}\times S]\subset[S_{(n-1)}\times dS]$, S_n is an $(L^2_{\mathbb{E}_n},A)$-scaling set by noting the above theorem. Conversely, assume that S_n is an $(L^2_{\mathbb{E}_n},A)$-scaling set. Then $[S_{(n-1)}\times S]\subset[S_{(n-1)}\times dS]$ and hence S should be contained in dS. Also, S should have a neighborhood of zero and 2π-translation congruent to $[0,2\pi)$, otherwise S_n will not have both.

As S_n is an $(L^2_{\mathbb{E}_n},A)$-scaling set, we have $[S_{(n-1)}\times S]\subset[S_{(n-1)}\times dS]$ and hence $S_{(n-1)}\times(dS\backslash S)$ is an $(L^2_{\mathbb{E}_n},A)$-multiwavelet set of order $(d-1)$ associated with an MRA. □

Theorem 3.9. *Suppose S is a scaling set by dilation (-2) or symmetric scaling set by dilation 2 and $A = \begin{pmatrix} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ -2 & 0_{1\times(n-1)}\end{pmatrix}$. Then S_n is an A-scaling set whose associated A-wavelet set is $((-2S)\backslash S)\times S_{(n-1)}$.*

Next, we provide an example of scaling function which is not arising by tensor products.

Example 3.10. Suppose $A = \begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}$ and φ is a function in $L^2(\mathbb{R}^2)$ such that $\widehat{\varphi}=\chi_S$, where

$$S=\left[-\frac{3\pi}{2},-\pi\right)\times[-2\pi,-\pi)\cup[-\pi,\pi)\times[-\pi,0)\cup\left[-\pi,\frac{\pi}{2}\right)\times[0,\pi).$$

Then it is easy to check that φ is a scaling function of an MRA in $L^2(\mathbb{R}^2)$ which is not arising by cross products of two scaling sets. An associated multiwavelet is $\Psi=\{\psi_i:\widehat{\psi_i}=$

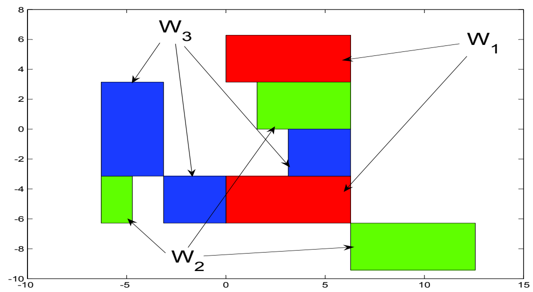

Figure 3: Multiwavelet Sets $\{W_1, W_2, W_3\}$

$\chi_{W_i}, i = 1,2,3\}$ (Figure 3), where

$$
\begin{aligned}
W_1 =& [0,2\pi)\times([\pi,2\pi)\cup[-2\pi,-\pi)),\\
W_2 =& \left[-2\pi,-\frac{3\pi}{2}\right)\times[-2\pi,-\pi)\cup\left[\frac{\pi}{2},2\pi\right)\times[0,\pi)\cup[2\pi,4\pi)\times[-3\pi,-2\pi),\\
W_3 =& [-\pi,0)\times[-2\pi,-\pi)\cup[-2\pi,-\pi)\times[-\pi,\pi)\cup[\pi,2\pi)\times[-\pi,0).
\end{aligned}
$$

With the help of Theorems 3.6 and 3.7, we provide scaling sets in higher dimensions having finite components.

Theorem 3.11. *Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be in $[0, 2\pi)$ such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \alpha_{n+1} = 2\pi$, where n is a natural number and $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Then*
(a) for odd $n > 1$,

$$
\begin{aligned}
S\times S \equiv& \bigcup_{k=1}^{\frac{n-1}{2}}[\alpha_n+4\pi,\, 6\pi)\times[\alpha_{2k-1}+2\pi,\, \alpha_{2k}+2\pi)\cup\\
&\bigcup_{k,m=0}^{\frac{n-1}{2}}[\alpha_{2m},\, \alpha_{2m+1})\times[\alpha_{2k},\, \alpha_{2k+1})\cup\bigcup_{k=1,m=0}^{\frac{n-1}{2}}[\alpha_{2m},\, \alpha_{2m+1})\times[\alpha_{2k-1}+2\pi,\, \alpha_{2k}+2\pi)\cup\\
&\bigcup_{m=0}^{\frac{n-1}{2}}[\alpha_{2m},\, \alpha_{2m+1})\times[\alpha_n+4\pi,\, 6\pi)\cup\bigcup_{k=0,m=1}^{\frac{n-1}{2}}[\alpha_{2m-1}+2\pi,\, \alpha_{2m}+2\pi)\times[\alpha_{2k},\, \alpha_{2k+1})\cup\\
&\bigcup_{k,m=1}^{\frac{n-1}{2}}[\alpha_{2m-1}+2\pi,\, \alpha_{2m}+2\pi)\times[\alpha_{2k-1}+2\pi,\, \alpha_{2k}+2\pi)\cup[\alpha_n+4\pi,\, 6\pi)\times[\alpha_n+4\pi,\, 6\pi)\cup\\
&\bigcup_{m=1}^{\frac{n-1}{2}}[\alpha_{2m-1}+2\pi,\, \alpha_{2m}+2\pi)\times[\alpha_n+4\pi,\, 6\pi)\cup\bigcup_{k=0}^{\frac{n-1}{2}}[\alpha_n+4\pi,\, 6\pi)\times[\alpha_{2k},\, \alpha_{2k+1})
\end{aligned}
$$

is an $(L^2_{[0,\infty)^2}, A)$-scaling set under the conditions:

(i) $\pi \le \alpha_2$, *(ii) $2\alpha_1 = \alpha_n$,* *(iii) $2\alpha_{2i} \le \alpha_{2i-1} + 2\pi$,* *and*

(iv) $\alpha_{2i} + 2\pi \le 2\alpha_{2i+1}$, where $i = m, k$; $i \in \left\{1, 2, \cdots, \frac{n-1}{2}\right\}$,

(b) for even $n \geq 2$,

$$S \times S \equiv \bigcup_{k,m=1}^{\frac{n}{2}} [\alpha_{2m-1}+2\pi, \alpha_{2m}+2\pi) \times [\alpha_{2k-1}+2\pi, \alpha_{2k}+2\pi) \cup$$
$$\bigcup_{k,m=0}^{\frac{n}{2}} [\alpha_{2m}, \alpha_{2m+1}) \times [\alpha_{2k}, \alpha_{2k+1}) \cup \bigcup_{k=1,m=0}^{\frac{n}{2}} [\alpha_{2m}, \alpha_{2m+1}) \times [\alpha_{2k-1}+2\pi, \alpha_{2k}+2\pi) \cup$$
$$\bigcup_{k=0,m=1}^{\frac{n}{2}} [\alpha_{2m-1}+2\pi, \alpha_{2m}+2\pi) \times [\alpha_{2k}, \alpha_{2k+1})$$

is an $(L^2_{[0,\infty)^2}, A)$-scaling set under the conditions:

(i) $\pi \leq \alpha_1$, *(ii)* $2\alpha_{2i} \leq \alpha_{2i-1}+2\pi$, *where* $i = m,\ k;\ i \in \left\{1, 2, \cdots, \frac{n}{2}\right\}$, *and*

(iii) $\alpha_{2i}+2\pi \leq 2\alpha_{2i+1}$, *where* $i = m,\ k;\ i \in \left\{1, 2, \cdots, \frac{n-2}{2}\right\}$.

To further study of topological behaviour of scaling sets, we obtain that S^1_3, S^2_3 and S^3_3 [cf. Theorem 3.1] are contractible spaces and hence pathconnected but unions of these sets are not. We call the inverse Fourier transform of a characteristic function whose support is a three-interval scaling set to be a *three-interval scaling function*, and its collection is denoted by Φ_3. For each $i = 1, 2, 3$, we denote $\{\chi^\vee_s : s \in S^i_3\}$ by Φ^i_3, and $S_3 = \cup^3_{i=1} S^i_3$. The set of all $(n+1)$-interval H^2-scaling sets obtained in Theorem 3.6 is denoted by $\mathcal{S}_n$ and the set of all $(L^2_{\mathbb{B}}, A)$-scaling sets $S \times S$ [cf. Theorem 3.11], where $S \in \mathcal{S}_n$, by $P(\mathcal{S}_n)$.

3.4 Contractibility of families of simple scaling sets

In this Section, we show that Φ^1_3, Φ^2_3 and Φ^3_3 are three path-components of Φ_3 by obtaining each one of these to be pathconnected and closed in Φ_3. This follows by obtaining that S_3 has three path-components namely S^1_3, S^2_3 and S^3_3, which are closed in S_3. Further, the contractibility of families of n-interval scaling sets described in Sections 3.1 and 3.2, and that of $(L^2_{[0,\infty)^2}, A)$-scaling sets obtained in the Theorem 3.11 are discussed, when considered as topological spaces. Let $\mathcal{M}$ denote the collection of all measurable sets of $\mathbb{R}$ having finite measure equipped with the symmetric difference metric. We obtain that $\mathcal{S}^M_n$ and $\mathcal{S}_n$ are contractible spaces under induced metric of $\mathcal{M}$, and also that $P(\mathcal{S}_n)$ is a contractible space under induced product metric.

Theorem 3.12. *For each $i \in \{1, 2, 3\}$, S^i_3 is pathconnected and closed in S_3.*

Proof. It suffices to show for $i = 1$ while for $i = 2$ and $i = 3$, the proof is similar. For this, we choose a standard three-interval scaling set in S^1_3 by assuming $2\alpha = \gamma, \alpha + 2\pi = 2\beta$, and $2\gamma = \beta + 2\pi$ which provide $\alpha = \frac{6\pi}{7}$, $\beta = \frac{10\pi}{7}$ and $\gamma = \frac{12\pi}{7}$. We denote it by s_1. Thus

$$s_1 = \left[-\frac{2\pi}{7}, \frac{6\pi}{7}\right) \cup \left[\frac{10\pi}{7}, \frac{12\pi}{7}\right) \cup \left[\frac{20\pi}{7}, \frac{24\pi}{7}\right).$$

To show that S_3^1 is pathconnected, we join s_1 to an arbitrary member s in S_3^1, given by

$$s = [\gamma - 2\pi, \alpha) \cup [\beta, \gamma) \cup [\alpha + 2\pi, \beta + 2\pi),$$

where $\alpha, \beta, \gamma \in (0, 2\pi)$ satisfy $\alpha < \beta < \gamma$, $\gamma \leq 2\alpha, 2\beta \leq \alpha + 2\pi$ and $\beta + 2\pi \leq 2\gamma$.

Define $\theta : [0,1] \to S_3^1$ by

$$\begin{aligned}\theta(t) = & \left[-(1-t)\frac{2\pi}{7} + t(\gamma - 2\pi), (1-t)\frac{6\pi}{7} + t\alpha\right) \cup \left[(1-t)\frac{10\pi}{7} + t\beta, (1-t)\frac{12\pi}{7} + t\gamma\right) \cup \\ & \left[(1-t)\frac{20\pi}{7} + t(\alpha + 2\pi), (1-t)\frac{24\pi}{7} + t(\beta + 2\pi)\right) \\ \equiv & I_t^1 \cup I_t^2 \cup I_t^3, \qquad (say)\end{aligned}$$

where $t \in [0,1]$. That θ is a path joining $\theta(0) = s_1$ to $\theta(1) = s$ follows by obtaining (i) for each t, $\theta(t) \in S_3^1$, and (ii) θ is continuous. Since

(a) $0 < \alpha < \beta < \gamma < 2\pi, \gamma \leq 2\alpha, 2\beta \leq \alpha + 2\pi$ and $\beta + 2\pi \leq 2\gamma$, $\theta(t) \subset 2\theta(t)$, for each $t \in [0,1]$, and

(b) $I_t^1 \cup I_t^2 \cup (I_t^3 - 2\pi) = \left[(-(1-t)\frac{2\pi}{7} + t(\gamma - 2\pi)), (-(1-t)\frac{2\pi}{7} + t(\gamma - 2\pi)) + 2\pi\right)$, $\theta(t)$ is 2π -translation congruent to $[\delta, \delta + 2\pi)$, where $\delta = -(1-t)\frac{2\pi}{7} + t(\gamma - 2\pi)$,

(i) follows. For (ii)-the continuity of θ, choose a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,1]$ converging to t in it. Then, we have $\lim_{n\to\infty} I_{t_n}^i = I_t^i$, for $i = 1,2,3$ and hence, we obtain that

$$\lim_{n\to\infty}\theta(t_n) = \theta(t).$$

Here, we observe that if $(s_n)_{n\in\mathbb{N}}$ is a sequence in S_3^1, where

$$s_n = [\gamma_n - 2\pi, \alpha_n) \cup [\beta_n, \gamma_n) \cup [\alpha_n + 2\pi, \beta_n + 2\pi), \text{ and } s = [\gamma - 2\pi, \alpha) \cup [\beta, \gamma) \cup [\alpha + 2\pi, \beta + 2\pi)$$

is in S_3^1, then $\lim_{n\to\infty} s_n = s$ if and only if $\lim_{n\to\infty}[\gamma_n - 2\pi, \alpha_n) = [\gamma - 2\pi, \alpha)$,

$$\lim_{n\to\infty}[\beta_n, \gamma_n) = [\beta, \gamma), \text{ and } \lim_{n\to\infty}[\alpha_n + 2\pi, \beta_n + 2\pi) = [\alpha + 2\pi, \beta + 2\pi) \text{ in } \mathcal{M}.$$

As there are two intervals lying entirely in $[0, \infty)$ in case of three-interval scaling sets in S_3^1 while in case of three-interval scaling sets in S_3^2 there is just one such interval and that in case of S_3^3 there is none, in view of above observation, a convergent sequence in S_3 having terms in S_3^1 converges in S_3^1. Thus S_3^1 is a closed set in S_3. □

Corollary 3.13. *The space of scaling functions Φ_3 has three path-components.*

Proof. It follows by noting that each of the $\Phi_3^i = \{\chi_s^\vee : s \in S_3^i\}$, $i = 1,2,3$ is pathconnected and closed in Φ_3. □

Theorem 3.14. *All the three path-components S_3^1, S_3^2 and S_3^3 of S_3 are contractible and hence each path-component of Φ_3 is contractible.*

Proof. Consider the standard three-interval scaling set s_1 in S_3^1 as described in Theorem 3.1. Then the map $F : S_3^1 \times [0,1] \to S_3^1$ defined by $F(s,t) = \theta(t)$, where $s \in S_3^1$ and $t \in [0,1]$, describes the homotopy between the identity map and the constant map on S_3^1 collapsing S_3^1 to $\theta(0) = s_1$. For the continuity of F, we employ observations used in Theorem 3.12. This shows that S_3^1 is contractible. Similarly, we obtain that S_3^2 and S_3^3 are contractible. □

Theorem 3.15. *$\mathcal{S}_n^M$, where $n \geq 3$ is an odd integer, is a contractible space. Also, $\mathcal{S}_n^M$, where $n \geq 4$ is an even integer, is a contractible space.*

Proof. Choose an element t in $\mathcal{S}_n^M$, where $n \geq 3$ is an odd integer, by considering β_1 and β_n in $(0,2\pi)$ satisfying $\beta_1 < \beta_n$ and $\beta_n \leq \min\left(2\beta_1, \frac{\beta_1}{2} + \pi\right)$, and selecting $\beta_2, \beta_3, ..., \beta_{n-1}$ such that $0 < \beta_1 < \beta_2 < ... < \beta_{n-1} < \beta_n < 2\pi$. Write

$$\begin{aligned} t &= \bigcup_{m=1}^{\frac{n-1}{2}} [\beta_{2m-1} - 2\pi, \beta_{2m} - 2\pi) \cup [\beta_n - 2\pi, \beta_1) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\beta_{2m}, \beta_{2m+1}) \\ &\equiv \bigcup_{m=1}^{\frac{n-1}{2}} (t_1)_m \cup t_2 \cup \bigcup_{m=1}^{\frac{n-1}{2}} (t_3)_m. \qquad (say) \end{aligned}$$

Let s be an arbitrary element of $\mathcal{S}_n^M$, where $n \geq 3$ is an odd integer.
Define $F : \mathcal{S}_n^M \times [0,1] \longrightarrow \mathcal{S}_n^M$ by

$$F(s,r) = \bigcup_{m=1}^{\frac{n-1}{2}} [r(t_1)_m + (1-r)(s_1)_m] \cup [rt_2 + (1-r)s_2] \cup \bigcup_{m=1}^{\frac{n-1}{2}} [r(t_3)_m + (1-r)(s_3)_m],$$

where $r \in [0,1]$. Note that $F(s,0) = s$ and $F(s,1) = t$.
It is easy to see that $F(s,r) \in \mathcal{S}_n^M$, for every $s \in \mathcal{S}_n^M$ and $r \in [0,1]$. For the continuity of F, we choose a sequence $(s^k, r^k)_{k \in \mathbb{N}}$ in $\mathcal{S}_n^M \times [0,1]$ converging to (s,r) in it, where s^k and s are given below:

$$\begin{aligned} s^k &= \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1}^k - 2\pi, \alpha_{2m}^k - 2\pi) \cup [\alpha_n^k - 2\pi, \alpha_1^k) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m}^k, \alpha_{2m+1}^k) \\ &\equiv \bigcup_{m=1}^{\frac{n-1}{2}} (s_1^k)_m \cup s_2^k \cup \bigcup_{m=1}^{\frac{n-1}{2}} (s_3^k)_m, \quad \text{and} \\ s &= \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m-1} - 2\pi, \alpha_{2m} - 2\pi) \cup [\alpha_n - 2\pi, \alpha_1) \cup \bigcup_{m=1}^{\frac{n-1}{2}} [\alpha_{2m}, \alpha_{2m+1}) \\ &\equiv \bigcup_{m=1}^{\frac{n-1}{2}} (s_1)_m \cup s_2 \cup \bigcup_{m=1}^{\frac{n-1}{2}} (s_3)_m. \end{aligned}$$

Observe that $\lim_{k\to\infty} s^k = s$ if and only if $\lim_{k\to\infty} s_2^k = s_2$, and

$$\lim_{k\to\infty} (s_i^k)_m = (s_i)_m, \text{ for } i = 1,3; \ m = 1,2,..., \frac{n-1}{2} \ \text{ in } \ \mathcal{M}.$$

Then, we find that $\lim_{k\to\infty} F(s^k, r^k) = F(s, r)$. Thus F describes a homotopy between the identity map and the constant map on $\mathcal{S}_n^M$ collapsing it to t. □

The following theorems show that $\mathcal{S}_n$ is a contractible space under induced metric of $\mathcal{M}$, while $P(\mathcal{S}_n)$ is a contractible space under induced product metric.

Theorem 3.16. *$\mathcal{S}_n$, where $n \in \mathbb{N}$, is a contractible space.*

Proof. We prove the theorem when n is even. For odd n, the proof is similar. Choose an element t in $\mathcal{S}_n$, where $n \in 2\mathbb{N}$, and

$$\begin{aligned} t &= \bigcup_{m=0}^{\frac{n}{2}} [\beta_{2m}, \beta_{2m+1}) \cup \bigcup_{m=1}^{\frac{n}{2}} [\beta_{2m-1} + 2\pi, \beta_{2m} + 2\pi) \\ &\equiv \bigcup_{m=0}^{\frac{n}{2}} (t_1)_m \cup \bigcup_{m=1}^{\frac{n}{2}} (t_2)_m. \qquad \text{(say)} \end{aligned}$$

Let s be an arbitrary element of $\mathcal{S}_n$, where

$$\begin{aligned} s &= \bigcup_{m=0}^{\frac{n}{2}} [\alpha_{2m}, \alpha_{2m+1}) \cup \bigcup_{m=1}^{\frac{n}{2}} [\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi) \\ &\equiv \bigcup_{m=0}^{\frac{n}{2}} (s_1)_m \cup \bigcup_{m=1}^{\frac{n}{2}} (s_2)_m. \qquad \text{(say)} \end{aligned}$$

Define $F : \mathcal{S}_n \times [0,1] \longrightarrow \mathcal{S}_n$ by

$$F(s,r) = \bigcup_{m=0}^{\frac{n}{2}} [r(t_1)_m + (1-r)(s_1)_m] \cup \bigcup_{m=1}^{\frac{n}{2}} [r(t_2)_m + (1-r)(s_2)_m],$$

where $r \in [0,1]$. Note that F describes a homotopy between the identity map and the constant map on $\mathcal{S}_n$ collapsing it to t. □

Theorem 3.17. *$P(\mathcal{S}_n)$, where $n \in \mathbb{N}$, is a contractible space.*

Proof. Let $T \in \mathcal{S}_n$. Then, by Theorem 3.16, there is a homotopy $F : \mathcal{S}_n \times [0,1] \longrightarrow \mathcal{S}_n$ such that $F(S,0) = S$ and $F(S,1) = T$, where $S \in \mathcal{S}_n$.

Now, it is straightforward to check that $G : P(\mathcal{S}_n) \times [0,1] \longrightarrow P(\mathcal{S}_n)$ defined by

$$G(S \times S, x) = (F,F)(S,x) = F(S,x) \times F(S,x), \text{ for } S \in \mathcal{S}_n \text{ and } x \in [0,1],$$

is a homotopy between the identity map and the constant map on $P(\mathcal{S}_n)$ collapsing it to $T \times T$. □

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