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A Two Dimensional Adler-Manin Trace and Bi-Singular Operators

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Abstract

Motivated by the theory of bi-singular pseudodifferential operators, we introduce a two dimensional version of the Adler-Manin trace. Our construction is rather general in the sense that it involves a twist afforded by an algebra automorphism. That is, starting from an algebra equipped with an automorphism, two twisted derivations, and a twisted invariant trace, we construct an algebra of formal twisted pseudodifferential symbols and define a noncommutative residue. Also, we provide related examples.

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1 Introduction

The Adler-Manin trace was discovered for one dimensional pseudodifferential symbols [1, 23]. It was generalized by Wodzicki to the algebra of pseudodifferential operators on a manifold of arbitrary dimension [33]. A vast generalization of Wodzicki's noncommutative residue was obtained by Connes and Moscovici for spectral triples in the context of the local index formula in noncommutative geometry [7]. A pseudodifferential calculus was developed in [5] for C^*-dynamical systems. Noncommutative residues on the corresponding classical pseudodifferential operators for the canonical dynamical system defining noncommutative tori were studied in [18] (see also [17]). Also, noncommutatvive residues for pseudodifferential operators on noncommutative tori with toroidal symbols are built in [22]. Motivated by the notion of a twisted spectral triple [8], a twisted version of the Adler-Manin trace was studied in [14].

In this paper, motivated by the theory of bi-singular pseudodifferential operators [28], we find an algebraic setting for a noncommutative residue defined analytically in [25].

Fredholm properties, index and residue were studied for an algebra of pseudodifferential operators, denoted by $HL(X_1 \times X_2)$, on the product of two manifolds X_1 and X_2 in [25]. In the the case $X_1 = X_2 = \mathbb{S}^1$, an operator in $HL(X_1 \times X_2) = HL(\mathbb{T})$ (with $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$) can be defined by a global symbol $a(x_1, x_2, \xi_1, \xi_2)$ as

$$\begin{aligned} a(x_1,x_2,D_1,D_2)u &= \int e^{2\pi i(x_1\xi_1+x_2\xi_2)} a(x_1,x_2,\xi_1,\xi_2)\widehat{u}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\ &= \sum_{k_1,k_2\in\mathbb{Z}} e^{2\pi i(x_1k_1+x_2k_2)} a(x_1,x_2,k_1,k_2) c_{k_1,k_2}, \end{aligned} \tag{1.1}$$

where $c_{k_1,k_2} = \iint_{\mathbb{T}} u(x_1,x_2) e^{-2\pi i(x_1k_1+x_2k_2)}\, dx_1 dx_2$ are the Fourier coefficients of $u \in C^\infty(\mathbb{T})$ (in the first line u is regarded as a 1-periodic function and its Fourier transform is a Dirac comb, which gives the discrete sum). The symbol a is supposed to have an asymptotic expansion $a \sim \sum_{m=-\infty}^{M} \sum_{n=-\infty}^{N} \sigma_{m,n}(x_1,x_2,\xi_1,\xi_2)$, where $\sigma_{m,n}(x_1,x_2,\xi_1,\xi_2)$ is smooth in $\mathbb{T} \times (\mathbb{R} \setminus \{0\})^2$ and positively homogeneous of degree m,n with respect to ξ_1 and ξ_2 respectively. Hence

$$\sigma_{m,n}(x_1,x_2,\xi_1,\xi_2) = \begin{cases} a_{m,n}^{(1,1)}(x_1,x_2)\xi_1^m \xi_2^n & \text{if } \xi_1 > 0,\ \xi_2 > 0 \\ a_{m,n}^{(1,2)}(x_1,x_2)\xi_1^m \xi_2^n & \text{if } \xi_1 > 0,\ \xi_2 < 0 \\ a_{m,n}^{(2,1)}(x_1,x_2)\xi_1^m \xi_2^n & \text{if } \xi_1 < 0,\ \xi_2 > 0 \\ a_{m,n}^{(2,2)}(x_1,x_2)\xi_1^m \xi_2^n & \text{if } \xi_1 < 0,\ \xi_2 < 0 \end{cases}$$

for suitable smooth functions $a_{m,n}^{(s,t)}$, $s,t = 1,2$, on $\mathbb{T}$ (see [25, Section 2] for more details).

Now, writing I for the two sided ideal of the operators which are smoothing with respect to at least one of the variables on $\mathbb{T}$, we have an isomorphism

$$HL(\mathbb{T})/I \to \Psi(A, \delta_1, \delta_2), \tag{1.2}$$

where $\Psi(A, \delta_1, \delta_2)$ is an algebra of formal pseudodifferential operators with coefficients in the algebra A, given by the direct sum of 4 copies of the algebra $C^\infty(\mathbb{T})$, and δ_1, δ_2 are the standard partial derivatives. To be precise, each element in the right-hand side of (1.2) is by definition a formal series

$$\sum_{m=-\infty}^{M} \sum_{n=-\infty}^{N} a_{m,n}\xi_1^m \xi_2^n, \tag{1.3}$$

with $a_{m,n} = (a_{m,n}^{(s,t)})_{s,t=1,2} \in A$. We have 4 linearly independent traces $\tau_{s,t} : A \to \mathbb{C}$ on the algebra A, defined by

$$\tau_{s,t}(a) = \iint_{\mathbb{T}} a^{(s,t)}(x_1, x_2)\, dx_1\, dx_2, \quad s,t = 1,2, \quad a \in A.$$

Correspondingly, we will have 4 noncommutative residues on $\Psi(A,\delta_1,\delta_2)$, defined via $\tau_{s,t}, s,t = 1,2$, by considering the term $\tau_{s,t}(a_{-1,-1})$ for any $a \in A$. We will prove these results in Section 4 for the *abstract* algebra $\Psi(A,\delta_1,\delta_2)$, generalizing therefore those obtained for bi-singular operators on the torus in [25].

Bi-singular operators P were originally studied, during the years 60's and 70's, under the form

$$\begin{aligned} Pf(z_1,z_2) &= b_0(z_1,z_2)f(z_1,z_2) \\ &\quad + \frac{1}{\pi i}\int_{\mathbb{S}^1} \frac{b_1(z_1,z_2,\zeta_1)}{\zeta_1 - z_1} f(\zeta_1,z_2)\, d\zeta_1 \\ &\quad + \frac{1}{\pi i}\int_{\mathbb{S}^1} \frac{b_2(z_1,z_2,\zeta_2)}{\zeta_2 - z_2} f(z_1,\zeta_2)\, d\zeta_2 \\ &\quad - \frac{1}{\pi^2}\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{b_{1,2}(z_1,z_2,\zeta_1,\zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)} f(\zeta_1,\zeta_2)d\zeta_1\, d\zeta_2. \end{aligned} \tag{1.4}$$

Here z_1, z_2 belong to $\mathbb{S}^1$, regarded as the unit circle in the complex plane, and we understood counter-clockwise integration in the principal value Cauchy sense. We assume the functions b_0, b_1, $b_2, b_{1,2}$ are C^∞ with respect to all of the variables. These operators, bounded on $L^2(\mathbb{T})$, can be regarded as operators of order zero in $HL(\mathbb{T})$, namely $a_{m,n} = 0$ for $(m,n) \neq (0,0)$ in (1.3). To respect the algebraic isomorphism in (1.2), one sets in (1.3) $a_{0,0} = (a_{0,0}^{(s,t)})_{s,t=1,2}$, where

$$\begin{aligned} a_{0,0}^{(s,t)} &= b_0(z_1,z_2) - (-1)^s b_1(z_1,z_2,z_1) \\ &\quad -(-1)^t b_2(z_1,z_2,z_2) + (-1)^{s+t} b_{1,2}(z_1,z_2,z_1,z_2), \end{aligned} \tag{1.5}$$

with $z_j = e^{2\pi i x_j}$, $j = 1,2$ (see Example 5.1 below for computations).

The literature in this connection is wide, involving problems of operator theory, harmonic analysis and several complex variables, see for example [34, 32, 19], coming from the school of A. Zygmund, and [30, 26, 27, 20], from the school of F. D. Gahov.

Starting from the 70's, bi-singular operators were reconsidered from the point of view of the theory of pseudodifferential operators, see [13] and [28], emphasizing the connection with the proof of the index theorem [3]. Roughly, the algebra $HL(\mathbb{T})$ is obtained by composing P in (1.4) with partial derivatives on $\mathbb{T}$ and their inverses. This originates terms of positive and negative orders in (1.3), and gives rise to non-trivial residues. Note that, arguing from a purely algebraic point of view, Fredholm property and index have no meaning for $\Psi(A,\delta_1,\delta_2)$, nevertheless a trace τ on $\Psi(A,\delta_1,\delta_2)$ in (1.2) can be obviously extended to $HL(\mathbb{T})$ by setting $\tau = 0$ on the ideal I.

A natural question is whether a related class of pseudodifferential operators exists, providing in (1.2) an isomorphism with $\Psi(A,\delta_1,\delta_2)$ where simply $A = C^\infty(\mathbb{T})$. To this end, we may address to the Toeplitz operators in a quadrant of [11, 12, 4, 31].

This paper is organized as follows. In Section 2, we recall some standard constructions and the Adler-Manin trace in the one dimensional case. In Section 3, we recall from [14] the extension of the Adler-Manin trace to a twisted set-up, where the twist is afforded by an algebra automorphism. This work was motivated by the notion of twisted spectral triples, introduced recently by Connes and Moscovici [8], which appear naturally in the study of type III examples of foliation algebras. They have shown that given a twisted spectral triple, the Dixmier trace induces a twisted trace on the base algebra, where lack of any non-trivial trace is the characteristic property of such algebras. Also, we briefly recall from [24] the construction of spectral triples twisted by scaling automorphisms, and the existence of a tracial noncommutative residue on the corresponding algebra of twisted pseudodifferential operators. In Section 4, we construct a two dimensional version of the algebra of twisted formal pseudodifferential symbols and introduce a noncommutative residue on this algebra. That is, starting from an algebra equipped with an automorphism and two twisted derivations, we construct an algebra of formal twisted pseudodifferential symbols. Then, we introduce a noncommutative residue on this algebra via an invariant twisted trace on the base algebra, and show that it is a trace functional. In Section 5, as an example of our two dimensional analogue of the Adler-Manin noncommutative residue, we shall consider the noncommutative residue of bi-singular operators.

2 The Adler-Manin Trace

Let A be a unital associative algebra over $\mathbb{C}$ which is not necessarily commutative. Recall that a *derivation* on A is a linear map $\delta : A \to A$ such that

$$\delta(ab) = a\delta(b) + \delta(a)b, \qquad a,b \in A,$$

Given a pair (A,δ) as above, the algebra of *formal differential symbols* $D(A,\delta)$ [21, 23] is, by definition, the algebra generated by A and a symbol ξ subject to the relations

$$\xi a = a\xi + \delta(a), \qquad a \in A. \tag{2.1}$$

For every element D of $D(A,\delta)$ there is a unique expression of the form

$$\sum_{i=0}^{N} a_i \xi^i,$$

with $N \geq 0, a_i \in A$. We think of D as a differential operator of order at most N. Using (2.1), one can prove by induction that

$$\xi^n a = \sum_{j=0}^{n} \binom{n}{j} \delta^j(a)\xi^{n-j}, \qquad a \in A, \qquad n \geq 0. \tag{2.2}$$

Using (2.2), we obtain the following multiplication formula in $D(A,\delta)$:

$$\Big(\sum_{i=0}^{M} a_i\xi^i\Big)\Big(\sum_{j=0}^{N} b_j\xi^j\Big) = \sum_{n=0}^{M+N}\Big(\sum_{i,j,k}\binom{i}{k}a_i\delta^k(b_j)\Big)\xi^n,$$

where the internal summation is over all $0 \le k \le i \le M$, and $0 \le j \le N$ such that $i+j-k = n$.

By formally inverting ξ in $D(A,\delta)$ and completing the resulting algebra, one obtains the algebra of *formal pseudodifferential symbols* of (A,δ) [1, 23, 21], denoted by $\Psi(A,\delta)$. More precisely, it is defined as follows. Elements of $\Psi(A,\delta)$ consist of formal sums

$$D = \sum_{i=-\infty}^{N} a_i\xi^i,$$

with $a_i \in A$, and $N \in \mathbb{Z}$. Its multiplication is defined by extending (2.2) to all $n \in \mathbb{Z}$ as follows. For any $a \in A$ and $n \in \mathbb{Z}$ one defines

$$\xi^n a = \sum_{j=0}^{\infty}\binom{n}{j}\delta^j(a)\xi^{n-j}.$$

Here the binomial coefficient $\binom{n}{j}$ for $n \in \mathbb{Z}$ and non-negative $j \in \mathbb{Z}$, is defined by $\binom{n}{j} := \frac{n(n-1)\cdots(n-j+1)}{j!}$. Notice that for $n < 0$, we have an infinite formal sum. It follows that for general $D_1 = \sum_{i=-\infty}^{M} a_i\xi^i$ and $D_2 = \sum_{j=-\infty}^{N} b_j\xi^j$ in $\Psi(A,\delta)$, the multiplication is given by

$$D_1D_2 = \sum_{n=-\infty}^{M+N}\Big(\sum_{i,j,k}\binom{i}{k}a_i\delta^k(b_j)\Big)\xi^n,$$

where the internal summation is over all integers $i \le M$, $j \le N$, and $k \ge 0$ such that $i+j-k = n$.

Now consider a *δ-invariant trace* $\tau : A \to \mathbb{C}$. By definition, τ is a linear functional such that

$$\tau(ab) = \tau(ba), \qquad \tau(\delta(a)) = 0, \qquad a,b \in A.$$

The *Adler-Manin noncommutative residue* [1, 23, 21] is the linear functional $\mathrm{res} : \Psi(A,\delta) \to \mathbb{C}$ defined by

$$\mathrm{res}\Big(\sum_{i=-\infty}^{N} a_i\xi^i\Big) = \tau(a_{-1}).$$

One checks that res is a trace, i.e.

$$\mathrm{res}([D_1,D_2]) = \mathrm{res}(D_1D_2 - D_2D_1) = 0, \qquad D_1,D_2 \in \Psi(A,\delta).$$

Equivalently, one shows that the map $\mathrm{res} : \Psi(A,\delta) \to A/([A,A]+\mathrm{im}\,\delta)$

$$D \mapsto a_{-1} \quad \mod \quad [A,A]+\mathrm{im}\,\delta$$

is a trace on $\Psi(A,\delta)$ with values in $A/([A,A]+\mathrm{im}\,\delta)$.

A relevant example is when $A = C^\infty(\mathbb{S}^1)$, the algebra of smooth functions on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, with

$$\tau(f) = \int_0^1 f(x)\,dx, \qquad \delta(f) = f', \qquad f \in C^\infty(\mathbb{S}^1).$$

In this case the noncommutative residue coincides with the Wodzicki residue on the algebra of classical pseudodifferential operators on the circle, see Section 5 below.

3 Twisted Symbols and Noncommutative Residues

Motivated by the notion of a twisted spectral triple [7], the Adler-Manin trace was extended to a twisted set-up in [14]. That is, an algebra of twisted formal pseudodifferential symbols was defined, where the twist is afforded by an automorphism of the base algebra, and a noncommutative residue was introduced. In this section we briefly recall this construction and highlight the tracial property of the corresponding noncommutative residue. We also recall some facts about noncommutative geometric spaces [6], namely spectral triples and the corresponding algebras of pseudodifferential operators, which admit noncommutative residues under some mild conditions [7]. We also briefly recall the twisted spectral spectral triples obtained from scaling automorphisms of a spectral triple in [24], and the corresponding algebras of twisted pseudodifferential operators and noncommutative residues.

3.1 Adler-Manin trace in a twisted set-up

We assume that A is a unital associative algebra over $\mathbb{C}$, and $\sigma : A \to A$ is an algebra automorphism. A linear map $\delta : A \to A$ is a *σ-derivation* or a *twisted derivation* if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b), \qquad a, b \in A.$$

Given such a triple (A, σ, δ), the algebra of *formal twisted differential symbols* $D(A, \sigma, \delta)$ is defined as follows. Its elements are polynomials in ξ with coefficients from A, which are of the form

$$\sum_{i=0}^{N} a_i \xi^i, \qquad N \geq 0, \qquad a_i \in A.$$

The multiplication of $D(A, \sigma, \delta)$ is defined by the relations

$$\xi a = \sigma(a)\xi + \delta(a), \qquad a \in A.$$

By induction, it follows that

$$\xi^n a = \sum_{i=0}^{n} P_{i,n}(\sigma, \delta)(a)\xi^i, \qquad n > 0, \qquad a \in A, \tag{3.1}$$

where $P_{i,n}(\sigma, \delta) : A \to A$ is the noncommutative polynomial in σ and δ with $\binom{n}{i}$ terms of total degree n such that the degree of σ is i. For example

$$P_{3,4}(\sigma, \delta) = \delta\sigma^3 + \sigma\delta\sigma^2 + \sigma^2\delta\sigma + \sigma^3\delta.$$

By formally inverting ξ and using induction, for $n = -1$ and $a \in A$, we have

$$\xi^{-1}a = \sum_{i=0}^{N}(-1)^i\sigma^{-1}(\delta\sigma^{-1})^i(a)\xi^{-1-i} + (-1)^{N+1}\xi^{-1}(\delta\sigma^{-1})^{N+1}(a)\xi^{-1-N}.$$

This suggests setting

$$\xi^{-1}a = \sum_{i=0}^{\infty}(-1)^i\sigma^{-1}(\delta\sigma^{-1})^i(a)\xi^{-1-i}.$$

Therefore for $n > 0, a \in A$, we set

$$\xi^{-n}a = \sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}(-1)^{i_1+\cdots+i_n}\sigma^{-1}(\delta\sigma^{-1})^{i_n}\cdots\sigma^{-1}(\delta\sigma^{-1})^{i_1}(a)\xi^{-n-i_1-\cdots-i_n}. \tag{3.2}$$

Thus, the elements of the algebra of *formal twisted pseudodifferential symbols* [14], denoted by $\Psi(A,\sigma,\delta)$, are of the form

$$\sum_{-\infty}^{N} a_i\xi^i, \qquad N \in \mathbb{Z}, \qquad a_i \in A,$$

and the multiplication in this algebra is defined as follows. For any

$$D_1 = \sum_{n=-\infty}^{N} a_n\xi^n, \quad D_2 = \sum_{m=-\infty}^{M} b_m\xi^m \quad \in \quad \Psi(A,\sigma,\delta),$$

we have

$$\begin{aligned} D_1D_2 \quad = \quad & \sum_{m=-\infty}^{M}\sum_{n<0}\sum_{i\geq 0}(-1)^{|i|}a_n\sigma^{-1}(\delta\sigma^{-1})^{i_{-n}}\cdots\sigma^{-1}(\delta\sigma^{-1})^{i_1}(b_m)\xi^{m+n-|i|} \\ & + \sum_{m=-\infty}^{M}\sum_{n=0}^{N}\sum_{j=0}^{n} a_nP_{j,n}(\sigma,\delta)(b_m)\xi^{m+j}, \end{aligned}$$

where for $n < 0$, $i = (i_1,\ldots,i_{-n})$ is an n-tuple of integers and $|i| = i_1 + \cdots + i_{-n}$.

One of the main results proved in [14] is that given a twisted δ-invariant trace $\tau : A \to \mathbb{C}$, the linear functional $\mathrm{res}_\sigma : \Psi(A,\sigma,\delta) \to \mathbb{C}$ defined by

$$\mathrm{res}_\sigma\Big(\sum_{n=-\infty}^{N} a_n\xi^n\Big) = \tau(a_{-1})$$

is a trace. Here, by *twisted trace* we mean that

$$\tau(ab) = \tau(\sigma(b)a), \qquad a,b \in A.$$

Also, τ is said to be *δ-invariant* if $\tau \circ \delta = 0$.

3.2 Spectral triples twisted by scaling automorphisms

Geometric spaces are described by *spectral triples* $(\mathcal{A}, \mathcal{H}, D)$ in noncommutative geometry [6, 7]. Here, $\mathcal{A}$ is a $*$-algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D is an unbounded selfadjoint operator acting in $\mathcal{H}$, which interacts with $\mathcal{A}$ in a bounded fashion. That is, for any $a \in \mathcal{A}$, the commutator $[D, a] = Da - aD$ extends by continuity to a bounded linear operator on $\mathcal{H}$. A local index formula is proved for spectral triples in [7]. If a spectral triple is *n-summable*, the Dixmier trace Tr_ω induces a trace on the base algebra $\mathcal{A}$. This trace functional is given by

$$a \mapsto \mathrm{Tr}_\omega(a|D|^{-n}), \qquad a \in A.$$

Also, it is shown in [7] that if the spectral triple has a *simple dimension spectrum*, the analogue of Wodzicki's residue defines a trace on the corresponding algebra $\Psi(\mathcal{A}, \mathcal{H}, D)$ of pseudodifferential operators. This trace is defined by

$$P \mapsto \mathrm{Res}_{z=0}\mathrm{Trace}(P|D|^{-2z}), \qquad P \in \Psi(\mathcal{A}, \mathcal{H}, D).$$

Since the existence of a non-trivial trace is the characteristic of type II situations in Murray-von Neumann classification of rings of operators, in order to incorporate type III examples, the notion of twisted spectral triples was introduced recently by Connes and Moscovici [8]. In fact, this notion arises naturally in the study of type III examples of foliation algebras, and also in noncommutative conformal geometry [8, 10, 15, 9, 16]. For twisted spectral triples, the ordinary commutators $[D, a]$ are not necessarily bounded operators, however, there exists an algebra automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that the twisted commutators $[D, a]_\sigma = Da - \sigma(a)D$ extend to bounded operators for all $a \in \mathcal{A}$.

Twisted spectral triples also arise naturally in conformal geometry of Riemannian manifolds [8, 24]. Let (M, g) be a connected compact Riemannian spin manifold of dimension n and $D = D_g$ be the associated Dirac operator acting on the Hilbert space of L^2-spinors $\mathcal{H} = L^2(M, S^g)$. Let $SCO(M, [g])$ denote the Lie group of diffeomorphisms of M that preserve the conformal structure $[g]$ (consisting of all Riemannian metrics that are conformally equivalent to g), the orientation, and the spin structure. Also, let $G = SCO(M, [g])_0$ denote the connected component of the identity. In [24], using a suitable automorphism of the crossed product algebra $C^\infty(M) \rtimes G$, a twisted spectral triple of the form $(C^\infty(M) \rtimes G, L^2(M, S^g), D)$ is constructed.

Similarly, by endowing $\mathbb{R}^n$ with the Euclidean metric, and considering the group G of conformal transformations of $\mathbb{R}^n$, a twisted spectral triple is constructed over the crossed product algebra $C_c^\infty(\mathbb{R}^n) \rtimes G$ [24]. An abstract formulation of this class of twisted spectral triples leads to the idea of twisting an ordinary spectral triple by its *scaling automorphisms*. A local index formula is proved in [24] for these examples under a particular invariance property under the twist, which enforces the Selberg principle for classical examples.

The set of scaling automorphisms of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, denoted by $\mathrm{Sim}(\mathcal{A}, \mathcal{H}, D)$, consists of all unitary operators U on $\mathcal{H}$ such that

$$U\mathcal{A}U^* = \mathcal{A} \qquad \text{and} \qquad UDU^* = \mu(U)D \qquad \text{for some} \qquad \mu(U) > 0.$$

One can see that $\mathrm{Sim}(\mathcal{A}, \mathcal{H}, D)$ is a group and the map $\mu : \mathrm{Sim}(\mathcal{A}, \mathcal{H}, D) \to (0, \infty)$ is a character. Since the group $\mathrm{Sim}(\mathcal{A}, \mathcal{H}, D)$ acts by conjugation on $\mathcal{A}$, by fixing a subgroup

$G \subset \mathrm{Sim}(\mathcal{A}, \mathcal{H}, D)$, one can form the crossed product algebra $\mathcal{A}_G = \mathcal{A} \rtimes G$. By definition, as a vector space, $\mathcal{A}_G$ is equal to $\mathcal{A} \otimes \mathbb{C}G$ whose elements are finite sums of the form

$$\sum a_U U, \qquad U \in G, \qquad a_U \in A.$$

The multiplication in this algebra is defined by

$$Ua = (UaU^*)U, \qquad U \in G, \qquad a \in A.$$

It is shown in [24] that the formula

$$\sigma(aU) = \mu(U)^{-1} aU, \qquad a \in \mathcal{A}, \qquad U \in G,$$

defines an automorphism of $\mathcal{A}_G$, and $(\mathcal{A}_G, \mathcal{H}, D)$ is a twisted spectral triple. For the twisted commutators, one has

$$[D, aU]_\sigma = DaU - \sigma(aU)D = [D, a]U,$$

which are bounded operators for all $a \in \mathcal{A}, U \in G$.

For this class of twisted spectral triples, the group $\mathrm{Sim}(\mathcal{A}, \mathcal{H}, D)$ acts by conjugation on the algebra of pseudodifferential operators $\Psi(\mathcal{A}, \mathcal{H}, D)$ of the base spectral triple. In [24] the crossed product algebra

$$\Psi(\mathcal{A} \rtimes G, \mathcal{H}, D) := \Psi(\mathcal{A}, \mathcal{H}, D) \rtimes G$$

is considered, and it is shown that the residue functional given by

$$P \mapsto \mathrm{Res}_{z=0} \mathrm{Trace}(P|D|^{-2z}), \qquad P \in \Psi(\mathcal{A} \rtimes G, \mathcal{H}, D),$$

defines a trace functional, under a similar assumption to the untwisted case, namely the *extended simple dimension spectrum hypothesis*.

4 Two Dimensional Noncommutative Residue

In this section, motivated by the theory of bi-singular pseudodifferential operators [28], we find an algebraic setting for a noncommutative residue defined analytically in [25]. The setting is a higher-dimensional version of the one for the Adler-Manin trace, explained in Section 2. Our construction is rather general in the sense that, similar to the construction described in Section 3, it involves a twist afforded by an algebra automorphism.

As above, we assume that A is a unital associative algebra and $\sigma : A \to A$ is an algebra automorphism. We consider two σ-derivations $\delta_1, \delta_2 : A \to A$ such that $\delta_1\delta_2 = \delta_2\delta_1, \delta_1\sigma = \sigma\delta_1, \delta_2\sigma = \sigma\delta_2$. First, we associate to this data an algebra $D(A, \sigma, \delta_1, \delta_2)$ of twisted differential symbols whose elements are polynomials in ξ_1, ξ_2 with coefficients in A, which are of the form

$$\sum_{m=0}^{M} \sum_{n=0}^{N} a_{m,n} \xi_1^m \xi_2^n, \qquad M, N > 0, \qquad a_{m,n} \in A.$$

The multiplication in $D(A,\sigma,\delta_1,\delta_2)$ is essentially defined by the relations

$$\xi_i a = \sigma(a)\xi_i + \delta_i(a), \qquad i = 1,2, \qquad a \in A.$$

Since δ_i and σ commute, by induction, for any $n > 0$ we have

$$\xi_i^n a = \sum_{j=0}^{n} \binom{n}{j} \sigma^{n-j}\delta_i^j(a)\xi_i^{n-j}, \qquad i = 1,2, \qquad a \in A.$$

We also define an algebra of formal twisted pseudodifferential symbols $\Psi(A,\sigma,\delta_1,\delta_2)$ (in short $\Psi(A,\delta_1,\delta_2)$ if $\sigma = id$) formally inverting each $\xi_i, i = 1,2$. The elements of this algebra are of the form

$$\sum_{m=-\infty}^{M} \sum_{n=-\infty}^{N} a_{m,n}\xi_1^m\xi_2^n, \qquad M,N \in \mathbb{Z}, \qquad a_{m,n} \in A.$$

The multiplication of this algebra is defined by the following relations:

$$\xi_1\xi_2 = \xi_2\xi_1,$$

$$\xi_i\xi_i^{-1} = \xi_i^{-1}\xi_i = 1, \qquad i = 1,2,$$

$$\xi_i^n a = \sum_{j=0}^{\infty} \binom{n}{j} \sigma^{n-j}\delta_i^j(a)\xi_i^{n-j}, \qquad i = 1,2, \qquad n \in \mathbb{Z}, \qquad a \in A.$$

We note that these relations follow from (3.1) and (3.2) under the assumption that the automorphism and the derivations commute with each other, which is assumed in our construction as mentioned above. Therefore, for any

$$D_1 = \sum_{m=-\infty}^{M} \sum_{n=-\infty}^{N} a_{m,n}\xi_1^m\xi_2^n, \quad D_2 = \sum_{p=-\infty}^{M'} \sum_{q=-\infty}^{N'} b_{p,q}\xi_1^p\xi_2^q \quad \in \quad \Psi(A,\sigma,\delta_1,\delta_2),$$

we have:

$$D_1D_2 = \sum \sum_{j_1,j_2 \geq 0} \binom{m}{j_1}\binom{n}{j_2} a_{m,n}\sigma^{m+n-j_1-j_2}\delta_1^{j_1}\delta_2^{j_2}(b_{p,q})\xi_1^{m+p-j_1}\xi_2^{n+q-j_2},$$

where the first summation is over all integers m,n,p,q such that $m \leq M, n \leq N, p \leq M', q \leq N'$.

In the following, we define a noncommutative residue $\mathrm{Res} : \Psi(A,\sigma,\delta_1,\delta_2) \to \mathbb{C}$ and prove that it is a trace functional. For the definition, we start from a twisted trace $\tau : A \to \mathbb{C}$, which satisfies some invariance properties stated in the hypotheses of Theorem 4.3.

Definition 4.1. For any

$$D = \sum_{m=-\infty}^{M} \sum_{n=-\infty}^{N} a_{m,n}\xi_1^m\xi_2^n \quad \in \quad \Psi(A,\sigma,\delta_1,\delta_2),$$

we define its noncommutative residue by

$$\mathrm{Res}(D) = \tau(a_{-1,-1}).$$

In order to investigate the tracial property of Res, which is carried out in Theorem 4.3, first, we need to prove a lemma:

Lemma 4.2. Let $\delta : A \to A$ be a σ-derivation such that $\delta\sigma = \sigma\delta$. If $\tau : A \to \mathbb{C}$ is a linear functional such that $\tau \circ \delta = 0$, then for any $a, b \in A$ and non-negative integer i, we have

$$\tau(\delta^i(a)b) = (-1)^i \tau(\sigma^i(a)\delta^i(b)).$$

Proof. For any $a, b \in A$, we have

$$\begin{aligned} \tau(\delta(ab)) &= \tau(\delta(a)b + \sigma(a)\delta(b)) \\ &= \tau(\delta(a)b) + \tau(\sigma(a)\delta(b)) \\ &= 0. \end{aligned}$$

Therefore

$$\tau(\delta(a)b) = -\tau(\sigma(a)\delta(b)), \qquad a, b \in A.$$

So the statement holds for $i = 1$. By induction, for any non-negative integer i we have:

$$\begin{aligned} \tau(\delta^i(a)b) &= \tau(\delta^{i-1}(\delta(a))b) \\ &= (-1)^{i-1}\tau(\sigma^{i-1}(\delta(a))\delta^{i-1}(b)) \\ &= (-1)^{i-1}\tau(\delta(\sigma^{i-1}(a))\delta^{i-1}(b)) \\ &= (-1)^i \tau(\sigma^i(a)\delta^i(b)). \end{aligned}$$

$\square$

Now, we show that under suitable invariance properties on the twisted trace $\tau : A \to \mathbb{C}$, the noncommutative residue defined above gives a trace on the corresponding algebra of two dimensional formal twisted pseudodifferential symbols.

Theorem 4.3. Let $\tau : A \to \mathbb{C}$ be a σ^2-trace such that $\tau \circ \delta_i = 0$ for $i = 1, 2$, and $\tau \circ \sigma = \tau$. Then the noncommutative residue Res gives a trace functional on $\Psi(A, \sigma, \delta_1, \delta_2)$.

Proof. Since Res is clearly a linear functional, in order to prove that it is a trace, it suffices to show that for any $a, b \in A$ and $m, n, p, q \in \mathbb{Z}$, we have:

$$\mathrm{Res}(a\xi_1^m\xi_2^n b\xi_1^p\xi_2^q) = \mathrm{Res}(b\xi_1^p\xi_2^q a\xi_1^m\xi_2^n).$$

First, we observe that

$$\begin{aligned} a\xi_1^m\xi_2^n b\xi_1^p\xi_2^q &= a\xi_1^m \sum_{j=0}^{\infty} \binom{n}{j} \sigma^{n-j}\delta_2^j(b)\xi_1^p\xi_2^{n-j+q} \\ &= a\sum_{j=0}^{\infty} \binom{n}{j} \xi_1^m \sigma^{n-j}\delta_2^j(b)\xi_1^p\xi_2^{n-j+q} \\ &= a\sum_{j=0}^{\infty} \binom{n}{j} \sum_{i=0}^{\infty} \binom{m}{i} \sigma^{m-i}\delta_1^i\sigma^{n-j}\delta_2^j(b)\xi_1^{m-i+p}\xi_2^{n-j+q} \\ &= \sum_{i,j=0}^{\infty} \binom{m}{i}\binom{n}{j} a\,\sigma^{m-i+n-j}\delta_1^i\delta_2^j(b)\xi_1^{m+p-i}\xi_2^{n+q-j}. \end{aligned}$$

Therefore

$$\mathrm{Res}(a\xi_1^m\xi_2^n b\xi_1^p\xi_2^q) = \tau\Big(\sum \binom{m}{i}\binom{n}{j} a\,\sigma^{m-i+n-j}\delta_1^i\delta_2^j(b)\xi_1^{m+p-i}\xi_2^{n+q-j}\Big), \tag{4.1}$$

where the summation is over all non-negative integers i, j such that $m+p-i=-1$ and $n+q-j=-1$.

Similarly, we have:

$$b\xi_1^p\xi_2^q a\xi_1^m\xi_2^n = \sum_{i,j=0}^{\infty} \binom{p}{i}\binom{q}{j} b\,\sigma^{p-i+q-j}\delta_1^i\delta_2^j(a)\xi_1^{m+p-i}\xi_2^{n+q-j}.$$

Therefore

$$\mathrm{Res}(b\xi_1^p\xi_2^q a\xi_1^m\xi_2^n) \;=\; \tau\Big(\sum \binom{p}{i}\binom{q}{j} b\,\sigma^{p-i+q-j}\delta_1^i\delta_2^j(a)\Big),$$

where the summation is again over all non-negative integers i, j such that $m+p-i=-1$ and $n+q-j=-1$. Since τ is a σ^2-trace, for each term in (4.2) we have

$$\begin{aligned}\tau\Big(\binom{p}{i}\binom{q}{j} b\,\sigma^{p-i+q-j}\delta_1^i\delta_2^j(a)\Big) &= \binom{p}{i}\binom{q}{j}\tau(\sigma^{p-i+q-j+2}\delta_1^i\delta_2^j(a)b)\\ &= \binom{p}{i}\binom{q}{j}\tau(\delta_1^i\delta_2^j\sigma^{p-i+q-j+2}(a)b),\end{aligned}$$

which, using Lemma 4.2, is equal to

$$\binom{p}{i}\binom{q}{j}(-1)^{i+j}\tau(\sigma^{p+q+2}(a)\delta_1^i\delta_2^j(b)).$$

Using $\tau\circ\sigma=\tau$, the latter is equal to

$$\binom{p}{i}\binom{q}{j}(-1)^{i+j}\tau(a\sigma^{-p-q-2}\delta_1^i\delta_2^j(b)).$$

Since $m+p-i=n+q-j=-1$ in the above sum, we have $-p-q-2=m+n-i-j$. Thus we have

$$\mathrm{Res}(b\xi_1^p\xi_2^q a\xi_1^m\xi_2^n) = \tau\Big(\sum \binom{p}{i}\binom{q}{j}(-1)^{i+j}\tau(a\sigma^{m+n-i-j}\delta_1^i\delta_2^j(b))\Big), \tag{4.2}$$

where the summation is again over all non-negative integers i, j such that $m+p-i=-1$ and $n+q-j=-1$.

By comparing (4.1) and (4.2), in order to prove that they are identical, we show that

$$\binom{m}{i}\binom{n}{j} = (-1)^{i+j}\binom{p}{i}\binom{q}{j},$$

for any fixed non-negative integers i, j such that $m+p-i=-1$ and $n+q-j=-1$. This can be proved by showing that $(-1)^i\binom{p}{i}=\binom{m}{i}$ and similarly $(-1)^j\binom{q}{j}=\binom{n}{j}$ as follows. We have:

$$\begin{aligned}
(-1)^i\binom{p}{i} &= (-1)^i\binom{-m+i-1}{i} \\
&= (-1)^i\frac{(-m+i-1)(-m+i-1-1)\cdots(-m+i-1-i+1)}{i!} \\
&= \frac{(m-i+1)(m-i+2)\cdots(m)}{i!} \\
&= \binom{m}{i}.
\end{aligned}$$

□

Example 4.1. Let $A = C^\infty(\mathbb{T})$, $\sigma = id$, $\delta_1 = \frac{\partial}{\partial\theta_1}$, $\delta_2 = \frac{\partial}{\partial\theta_2}$, and

$$\tau(f) = \int_0^1\int_0^1 f(\theta_1,\theta_2)\,d\theta_1\,d\theta_2, \qquad f \in A.$$

Then

$$\operatorname{Res}\Big(\sum_{m=-\infty}^{M}\sum_{n=-\infty}^{N} a_{m,n}\xi_1^m\xi_2^n\Big) = \int_0^1\int_0^1 a_{-1,-1}(\theta_1,\theta_2)\,d\theta_1\,d\theta_2.$$

5 Singular and Bi-Singular Operators

We add here some detail concerning the bi-singular operators mentioned in the introduction. To be definite, we consider first the one dimensional case.

5.1 Singular integral operators

Classical pseudodifferential operators [2, 29] of order $N \in \mathbb{Z}$ on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ can be defined by a global symbol $a(x,\xi)$, $x \in \mathbb{R}/\mathbb{Z}$, $\xi \in \mathbb{R}$, with an asymptotic expansion in homogeneous terms. Modulo regularizing operators, i.e. mappings $\mathcal{D}'(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$, this algebra can therefore be identified with that of formal series

$$\sigma = \sum_{n=-\infty}^{N} a_{n,1}(x)\xi_+^n + a_{n,2}(x)\xi_-^n, \tag{5.1}$$

where $\xi_+ = \max\{\xi,0\}$, $\xi_- = \min\{\xi,0\}$.

This corresponds to the Adler-Manin algebra

$$\Psi(A,\delta) = \Big\{\sum_{n=-\infty}^{N} a_n\xi^n; \quad N \in \mathbb{Z}, \quad a_n \in A\Big\},$$

with $A = C^\infty(\mathbb{S}^1)\oplus C^\infty(\mathbb{S}^1)$, $\delta = \frac{d}{dx}$, $a_n = (a_{n,1},a_{n,2})$. There are two linearly independent traces on A, namely

$$\tau_s(a) = \int_0^1 a^{(s)}(x)\,dx, \qquad s = 1,2, \qquad a = (a^{(1)},a^{(2)}) \in A.$$

Therefore, there are two noncommutative residues

$$\mathrm{Res}_s\Big(\sum_{n=-\infty}^{N} a_n\xi^n\Big) = \tau_s(a_{-1}) = \int_0^1 a_{n,s}(x)\,dx, \qquad s = 1,2,$$

which are known as Wodzicki's residues.

Introducing on $\Psi(A,\delta)$ the Toeplitz projection

$$P_+\sigma = \sum_{n=-\infty}^{N} a_{n,1}(x)\xi_+^n,$$

i.e. setting $a_{n,2} = 0$ for all n in (5.1), we may define the quotient subalgebra of the asymptotic expansions of the classical Toeplitz operators, cf. the example at the end of Section 2, where A is simply given by $C^\infty(\mathbb{S}^1)$.

5.2 Bi-singular integral operators

Consider the algebra $HL(\mathbb{T})$ of bi-singular integral operators on $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, which are defined by a global symbol

$$a(x_1,x_2,\xi_1,\xi_2),$$

as in (1.1), where $x_1,x_2 \in \mathbb{R}/\mathbb{Z}$, $\xi_1,\xi_2 \in \mathbb{R}$, with an asymptotic expansion in terms which are positively homogeneous with respect to ξ_1 and ξ_2 separately. As explained in the introduction, modulo sums of operators regularizing in x_1 or x_2, they are identified with formal series of the form

$$\sigma = \sum_{m=-\infty}^{M}\sum_{n=-\infty}^{N} \sigma_{m,n},$$

where each $\sigma_{m,n}$ is given by an expansion of the form

$$\sigma_{m,n} = \sum_{s,t=1,2} a_{m,n}^{(s,t)}(x_1,x_2)\xi_{1,s}^m\xi_{2,t}^n,$$

with $\xi_{j,1} = \max\{\xi_j,0\}$, $\xi_{j,2} = \min\{\xi_j,0\}$, $j = 1,2$.

Therefore, the above operators correspond to the algebra

$$\Psi(A,\delta_1,\delta_2) = \Psi(A,\sigma,\delta_1,\delta_2) = \Big\{\sum_{m=-\infty}^{M}\sum_{n=-\infty}^{N} a_{m,n}\xi_1^m\xi_2^n;\quad M,N \in \mathbb{Z},\quad a_{m,n} = (a_{m,n}^{(s,t)})_{s,t=1,2} \in A\Big\},$$

considered in Section 4, with

$$A = C^\infty(\mathbb{T})^4, \qquad \sigma = id, \qquad \delta_1 = \frac{\partial}{\partial x_1}, \qquad \delta_2 = \frac{\partial}{\partial x_2}.$$

Now we have four linearly independent traces on A, namely

$$\tau_{s,t}(a) = \int_0^1\int_0^1 a^{(s,t)}(x_1,x_2)\,dx_1\,dx_2, \qquad a = (a^{(s,t)}) \in A, \qquad s,t = 1,2.$$

Therefore, in this case, we have four noncommutative residues on $\Psi(A,\sigma,\delta_1,\delta_2)$, which are given by

$$\mathrm{Res}_{s,t}\Big(\sum_{m=-\infty}^{M}\sum_{n=-\infty}^{N} a_{m,n}\xi_1^j\xi_2^k\Big) = \tau_{s,t}(a_{-1,-1}), \qquad s,t=1,2.$$

They correspond to those found in [25] for general bi-singular operators on the product of two manifolds.

Finally, we note that the projection

$$P_{++}\sigma = \sum_{m=-\infty}^{M}\sum_{n=-\infty}^{N} a_{m,n}^{(1,1)}(x_1,x_2)\xi_{1,1}^m\xi_{2,1}^n$$

gives the algebra of the asymptotic expansions for Toeplitz operators in the quarter plane, corresponding to Example 4.1.

Example 5.1. We finally verify that the operator P in (1.4) falls in the class $HL(\mathbb{T})$ and prove the formula (1.5) for its symbol.

To this end, in (1.4) we perform a 0th order Taylor expansion of b_1, b_2, $b_{1,2}$ at $\zeta_1 = z_1$, $\zeta_2 = z_2$ and $\zeta_1 = z_1$, $\zeta_2 = z_2$ respectively. Ignoring operators whose integrals kernels are smooth with respect to a couple of variables, which belong to the ideal I, we get

$$\begin{aligned} Pf(z_1,z_2) &= b_0(z_1,z_2)f(z_1,z_2)+\frac{b_1(z_1,z_2,z_1)}{\pi i}\int_{\mathbb{S}^1}\frac{f(\zeta_1,z_2)}{\zeta_1-z_1}d\zeta_1 \\ &\quad +\frac{b_2(z_1,z_2,z_2)}{\pi i}\int_{\mathbb{S}^1}\frac{f(z_1,\zeta_2)}{\zeta_2-z_2}d\zeta_2 \\ &\quad -\frac{b_{1,2}(z_1,z_2,z_1,z_2)}{\pi^2}\iint_{\mathbb{S}^1\times\mathbb{S}^1}\frac{f(\zeta_1,\zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)}d\zeta_1\,d\zeta_2 \bmod I. \end{aligned}$$

The formula (1.5) then follows by expressing the Hilbert transform on a circe as a pseudodifferential operator:

$$\frac{1}{\pi i}\int_{\mathbb{S}^1}\frac{u(\zeta)}{\zeta-z}d\zeta = \sum_{k\geq 0} e^{2\pi i x k}a(x,k)c_k - \sum_{k<0} e^{2\pi i x k}a(x,k)c_k = a(x,D)u+Ru,$$

where $c_k = \int_{\mathbb{S}^1} u(x)e^{-2\pi i x k}\,dx$, R is a regularizing operator, and $a(x,\xi) = (1-\chi(\xi))\mathrm{sign}\,\xi$, χ being a smooth cut off function, $\chi(\xi) = 1$ in a neighborhood of the origin.

References

[1] M. Adler, On a trace functional for formal pseudo differential operators and the symplectic structure of the Korteweg-de Vries type equations. *Invent. Math.* **50** (1978/79), pp 219-248.

[2] M.S. Agranovich, Spectral properties of elliptic pseudodifferential operators on a closed curve. (Russian). *Funktsional Anal. i Priložen* **13** (1979), pp 54-56. (English translation in *Functional Analysis and Its applications* **13**, pp 279-281).

[3] M. F. Atiyah and I. M. Singer, The index of elliptic operators. *Ann. of Math.* **87** (1968), pp 484-530.

[4] L. A. Coburn, R. G. Douglas and I. Singer, An index theorem for Wiener-Hopf operators on the discrete quarter-plane. *J. Differential Geometry* **6** (1972), pp 587-593.

[5] A. Connes, C^*-algèbres et géométrie différentielle. *C.R. Acad. Sc. Paris* t. **290**, Série A (1980), pp 599-604.

[6] A. Connes, *Noncommutative Geometry.* Academic Press, 1994.

[7] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, *Geom. Funct. Anal.* **5** (1995), pp 174-243.

[8] A. Connes and H. Moscovici, Type III and spectral triples. *Traces in number theory, geometry and quantum fields*, pp 57-71, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008.

[9] A. Connes and H. Moscovici, Modular curvature for noncommutative two-tori. *arXiv:1110.3500.*

[10] A. Connes and P. Tretkoff, The Gauss-Bonnet theorem for the noncommutative two torus. *Noncommutative geometry, arithmetic, and related topics*, pp 141-158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[11] R. G. Douglas and R. Howe, On the C^*-algebras of Toeplitz operators on the quarter plane. *Trans. Amer. Math. Soc.* **158** (1971), pp 203-217.

[12] R. G. Douglas, On the invertibility of a class of Toeplitz operators on the quarter-plane. *Indiana Univ. Math. J.* **21** (1971/1972), pp 1031-1035.

[13] R. V. Dudučava, On bisingular integral operators with discontinuous coefficients. *Mat. Sbornik* **101** (1976); transl. *Math. USSR Sbornik* **30** (1976), pp 515-537.

[14] F. Fathizadeh and M. Khalkhali, The algebra of formal twisted pseudodifferential symbols and a noncommutative residue. *Letters in Mathematical Physics* **94** (2010), pp 41-61.

[15] F. Fathizadeh and M. Khalkhali, The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. *J. Noncommut. Geom.* **6** (2012), pp 457-480.

[16] F. Fathizadeh and M. Khalkhali, Scalar curvature for the noncommutative two torus. To appear in the *J. Noncomm. Geom*, arXiv:1110.3511.

[17] F. Fathizadeh and M. Khalkhali, Scalar Curvature for Noncommutative Four-Tori. *arXiv:1301.6135*.

[18] F. Fathizadeh and M. W. Wong, Noncommutative residues for pseudo-differential operators on the noncommutative two-torus. *Journal of Pseudo-Differential Operators and Applications* **2** (2011), pp 289-302.

[19] C. Fefferman, Estimates for double Hilbert transforms. *Studia Math* **44** (1972), pp 1-15.

[20] V.A. Kakičev. Regularization and pseudoregularization of complete bisingular operators with Cauchy kernels and normal characteristic part. *Dokl. Akad. Nauk SSSR* **229** (1976), pp 19-22; transl. *Soviet Math Dokl.* **17** (1976), pp 952-956.

[21] Ch. Kassel, Le résidu non commutatif (d'après M. Wodzicki). (French). [The noncommutative residue (after M. Wodzicki)] Séminaire Bourbaki, Vol. 1988/89. *Astérisque* No. **177-178** (1989), Exp. No. 708, pp 199-229.

[22] C. Lévy, C. Neira Jiménez and S. Paycha, The canonical trace and the noncommutative residue on the noncommutative torus. *arXiv:1303.0241*.

[23] Ju. I. Manin, Algebraic aspects of nonlinear differential equations. (Russian). *Current problems in mathematics* **11** pp 5-152. (errata insert) Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1978.

[24] H. Moscovici, Local index formula and twisted spectral triples. *Quanta of maths*, pp 465-500, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010.

[25] F. Nicola and L. Rodino, Residue index for bisingular operators. *C^* algebras and Elliptic Theory*, pp 187-202, Trends in Mathematics (2006).

[26] V. S. Pilidi, The index of bisingular operators. *Funkcional Anal. i Priložen* **7** (1973), pp 93-94; transl. *Functional Anal. Appl.* **7** (1973), pp 337-338.

[27] V. S. Pilidi and L. I. Sazonov, A priori estimates for characteristic bisingular integral operators. *Dokl. Akad. Nauk SSSR* **217** (1974), pp 285-287; transl. *Soviet Math. Dokl.* **15** (1974), pp 1064-1067.

[28] L. Rodino, A class of pseudodifferential operators on the product of two manifolds and applications. *Ann. Scuola Norm. Sup. Pisa* 2 ser. IV (1975), pp 287-302.

[29] M. Ruzhansky and V. Turunen, *Pseudodifferential operators and symmetries.* Pseudodifferential Operators, Theory and Applications, Birkhäuser, Basel, 2010.

[30] I. B. Simonenko, On the question of solvability of bisingular and polysingular equations. *Funkcional Anal. i Priložen* **5** (1971), pp 93-94; transl. *Functional Anal. Appl.* **5** (1971), pp 81-83.

[31] G. Strang, Toeplitz operators in a quarter-plane. *Bull. Amer. Math. Soc.* **76** (1970), pp 1303-1307.

[32] T. Walsh, On the existence of double singular integrals for kernels without smoothness. *Proc. Amer. Math. Soc.* **28** (1971), pp 439-445.

[33] M. Wodzicki, Noncommutative residue. I. Fundamentals. *K-theory, arithmetic and geometry* (Moscow, 1984–1986), pp 320-399, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[34] A. Zygmund, On the boundary values of functions of several complex variables I. *Fund. Math.* **36** (1949), pp 207-235.