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Discrete Calculus of Variations for Quadratic Lagrangians

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Abstract

The intent of this paper is to develop a framework for discrete calculus of variations with action densities involving a new class of discretization operators. We introduce first the generalized scale derivatives, study their regularity and state some Leibniz formulas. Then, we deduce the discrete Euler-Lagrange equations for critical points of sampled actions that we compare to existing versions. Next, we investigate the case of general quadratic lagrangians and provide two examples of such lagrangians. At last, we find nontrivial properties for the discretization of a quadratic null-lagrangian.

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1 Introduction

Discrete calculus of variations is an active field of research and comes in at least three theories. The quantum framework is depicted for example by Cresson and coauthors in [4, 5]. Another context uses the Δ or/and ∇-derivatives on times scales [3, 8, 10]. Discrete programming and specific mechanic problems are investigated by Marsden and subsequent authors [12, 13, 14].

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In order to extend the principle of least action to the case of non-differentiable dynamical variables, a common trick (see [4, 5]) is to replace the usual derivative $\dot{\mathbf{x}}(t)$ of the dynamical variable $\mathbf{x}(t)$ with expressions such as:

$$\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon}, \frac{\mathbf{x}(t)-\mathbf{x}(t-\varepsilon)}{\varepsilon} \text{ or } \frac{1-i}{2\varepsilon}\mathbf{x}(t+\varepsilon)+\frac{i}{\varepsilon}\mathbf{x}(t)-\frac{1+i}{2\varepsilon}\mathbf{x}(t-\varepsilon). \tag{1.1}$$

Note that these operators are specific and require that t belongs to $\mathbb{R}$ to be properly defined. In contrast, in discrete mechanics, dynamic variables, derivatives and lagrangians are replaced at the very beginning of the theory with suitable sequences of numbers but not with functions [12, 13, 14].

In this paper we modify (1.1) and we use instead generalized scale derivatives. Let $[a,b]$ be some interval of time, $\varepsilon > 0$ a fixed delay, d the "physical" dimension, N the number of samples in $\mathbb{C}^d$, $(c_\ell)_{\ell\in\{-N,\dots,N\}} \in \mathbb{C}^{2N+1}$ and, lastly, χ the characteristic function of $[a,b]$. So we modify the formulas (1.1) as follows:

$$\Box\mathbf{x}(t) = \sum_{\ell=-N}^{N} c_\ell \mathbf{x}(t+\ell\varepsilon)\chi(t+\ell\varepsilon), \tag{1.2}$$

for all $\mathbf{x} : [a,b] \to \mathbb{C}^d$. The role of $\chi(t+\ell\varepsilon)$ is to prevent $t+\ell\varepsilon$ from belonging to an interval in which $\mathbf{x}$ is undefined.

Introducing generalized derivatives such as (1.2), with unspecified parameters $c_\ell \in \mathbb{C}$ and variable coefficients $\chi(t+\ell\varepsilon)$ seems to be new and leads to interesting problems. First, we may discuss the choice of the specific operator occuring in [4, 5]. Next, we may analyze the similarities and differences between the equations of motion for discretized dynamical variables and the classical Euler-Lagrange equations

$$\mathrm{grad}_{\mathbf{x}}\mathcal{L} - \frac{d}{dt}\mathrm{grad}_{\dot{\mathbf{x}}}\mathcal{L} = 0.$$

A third problem may consist in the generalization of some results of [4, 5] to operators $\Box$ not satisfying a Leibniz formula and to extremal curves $\mathbf{x}(t)$ which are not Hölder continuous.

The paper is organized as follows. Section 2 is preparatory and shows how to handle formula (1.2). In Section 3, we describe a class of operators $\Box$ satisfying an appropriate extension of Leibniz formula. In Section 4, we get the necessary first order condition for finding a minimizer of a discrete action $\mathcal{A}_d(\mathbf{x})$. The resulting equations of motion

$$\mathrm{grad}_{\mathbf{x}}\mathcal{L} + \Box^\star\mathrm{grad}_{\dot{\mathbf{x}}}\mathcal{L} = 0, \tag{1.3}$$

for all lagrangian $\mathcal{L}$, are higlighted. The operator $\Box^\star$ is obtained from $\Box$ by reversing its coefficients c_ℓ. In Section 5, we compare (1.3) with other versions of discrete Euler-Lagrange equations. In Section 6, we first introduce the classical and discrete models for the quadratic lagrangians and emphasize on their similarity. The Section 7 is dedicated to oscillatory solutions of discrete Euler-Lagrange equations for the harmonic oscillator. At last, in Section 8, we investigate the case of a null-lagrangian in the classical setting which behaves very differently in the discrete one.

Throughout this paper, C.E.L./D.E.L. stand for classical/discrete Euler-Lagrange equations. Moreover, we suppose for sake of conciseness that $0 < \varepsilon < \frac{b-a}{2N}$.

2 Regularity of generalized scale derivatives

The following two results intend to study the continuity of the generalized scale derivatives defined by (1.2) and the conditions under which they behave as ordinary derivatives.

Proposition 2.1. *For all operator* $\Box$ *of the shape (1.2) and for all* $\mathbf{x} \in C^0([a,b],\mathbb{C}^d)$*, the function* $\Box\mathbf{x} : \mathbb{R} \to \mathbb{C}^d$ *is piecewise continuous, bounded, compactly supported, admits everywhere two finite one-sided limits, and has at most* $4N+2$ *points of discontinuity belonging to* $\{a+k\varepsilon, b+k\varepsilon, |k| \le N\}$*.*

Proof. The main ingredient of the proof is the explicit formula for $\Box\mathbf{x}(t)$, for all function $\mathbf{x} : [a,b] \to \mathbb{C}^d$, that follows from (1.2):

$$\begin{cases} 0 & \text{if } t < a - N\varepsilon \\ c_N\mathbf{x}(t+N\varepsilon) & \text{if } a-N\varepsilon \le t < a-N\varepsilon+\varepsilon \\ c_{N-1}\mathbf{x}(t+N\varepsilon-\varepsilon)+c_N\mathbf{x}(t+N\varepsilon) & \text{if } a-N\varepsilon+\varepsilon \le t < a-N\varepsilon+2\varepsilon \\ \vdots & \vdots \\ c_{N-p}\mathbf{x}(t+N\varepsilon-p\varepsilon)+\ldots+c_N\mathbf{x}(t+N\varepsilon) & \begin{array}{c}\text{if } p \in \{0,\ldots,2N-1\} \text{ and} \\ a+(p-N)\varepsilon \le t < a+(p+1-N)\varepsilon\end{array} \\ \vdots & \vdots \\ c_{-N}\mathbf{x}(t-N\varepsilon)+\ldots+c_N\mathbf{x}(t+N\varepsilon) & \text{if } a+N\varepsilon \le t \le b-N\varepsilon \\ \vdots & \vdots \\ c_{-N}\mathbf{x}(t-N\varepsilon)+\ldots+c_{p-N}\mathbf{x}(t-N\varepsilon+p\varepsilon) & \begin{array}{c}\text{if } p \in \{0,\ldots,2N-1\} \text{ and} \\ b+(N-p-1)\varepsilon < t \le b+(N-p)\varepsilon\end{array} \\ \vdots & \vdots \\ c_{-N}\mathbf{x}(t-N\varepsilon)+c_{1-N}\mathbf{x}(t-N\varepsilon+\varepsilon) & \text{if } b+N\varepsilon-2\varepsilon < t \le b+N\varepsilon-\varepsilon \\ c_{-N}\mathbf{x}(t-N\varepsilon) & \text{if } b+N\varepsilon-\varepsilon < t \le b+N\varepsilon \\ 0 & \text{if } b+N\varepsilon < t \end{cases} \tag{2.1}$$

This formula shows that the function $\Box\mathbf{x}$ is piecewise continuous and its support is included in $[a-N\varepsilon, b+N\varepsilon]$. We note also that $\Box\mathbf{x}$ is bounded and $\sup_t \|\Box\mathbf{x}(t)\| \le (\sum |c_\ell|)\|\mathbf{x}(t)\|$. The function $\Box\mathbf{x}$ admits everywhere two one-sided limits, and may be not continuous only at the $4N+2$ points $\{a+k\varepsilon, b+k\varepsilon, |k| \le N\}$ depending on the $\{c_\ell\}_\ell$. $\Box$

Remark 2.2. In order to deal with discrete actions containing $\Box\mathbf{x}(t)$, we need to rewrite (2.1) under a matricial form. Let $S(\mathbf{x})$ denote the row vector-valued function

$$S(\mathbf{x})(t) = (x_1(t-N\varepsilon)\chi(t-N\varepsilon)\ldots x_j(t+\ell\varepsilon)\chi(t+\ell\varepsilon)\ldots x_d(t+N\varepsilon)\chi(t+N\varepsilon)), \tag{2.2}$$

where $\mathbf{x} = (x_1,\ldots,x_d)$. In $S(\mathbf{x})$ the ordering of variables is lexicographic, first on ℓ and next on j. The function $S(\mathbf{x})$ lies in a product of $d(2N+1)$ affine spaces modeled on $C^0([a,b],\mathbb{C})$ and, in this way, S is an injective continuous linear mapping. If $d=1$, for all $\mathbf{x} \in C^0([a,b],\mathbb{C})$, we have

$$\Box\mathbf{x} = (c_{-N}\ldots c_0\ldots c_N)^{\mathrm{T}}(S(\mathbf{x})).$$

When $d \ge 2$, the relationship between $\Box\mathbf{x}$ and $S(\mathbf{x})$ involves a $d\times d(2N+1)$ banded matrix and is given by

$$\square \mathbf{x} = \begin{pmatrix} c_{-N}\dots c_0\dots c_N & & & \\ & c_{-N}\dots c_0\dots c_N & & 0 \\ 0 & & \ddots & \\ & & & c_{-N}\dots c_0\dots c_N \end{pmatrix}^{\mathsf{T}}(S(\mathbf{x}))$$

and is equivalent to (2.1).

Let us define the vector space $O_{N,\varepsilon}$ as the set of operators $\square$ of the shape (1.2) with coefficients $c_\ell = \frac{\gamma_\ell}{\varepsilon}$, with $\gamma_\ell \in \mathbb{C}$ not depending on ε. Algebraically, $O_{N,\varepsilon} \simeq \mathbb{C}^{2N+1}$. Of particular interest is the following subclass of operators of $O_{1,\varepsilon}$ given by

$$\square^{[r,s]}\mathbf{x}(t) = -\frac{s}{\varepsilon}\mathbf{x}(t-\varepsilon)\chi(t-\varepsilon) + \frac{s-r}{\varepsilon}\mathbf{x}(t)\chi(t) + \frac{r}{\varepsilon}\mathbf{x}(t+\varepsilon)\chi(t+\varepsilon), \tag{2.3}$$

where $r, s \in \mathbb{C}$. The operators (2.3) are obtained from (1.2) by setting $N = 1$, $s = -c_{-1}\varepsilon$, $r = c_1\varepsilon$ and $c_0 = -(c_{-1} + c_1)$. Three examples of such operators $\square^{[r,s]}$ are related to the discrete Euler forward and backward difference operators Δ^+ and Δ^- respectively, and the symmetric difference operator equal to the mean of Δ^+ and Δ^-. Choosing the appropriate coefficients r, s, we may introduce respectively the analogs $\square^{[1,0]}$, $\square^{[0,1]}$, and lastly $\square^{[\frac{1}{2},\frac{1}{2}]}$. A fourth example is given by $\square^{[\frac{1-i}{2},\frac{1+i}{2}]}$, which has the same coefficients than the discretization operator $\square_q$ presented in (1.1) and occuring in [4, 5]. Nevertheless, $\square_q$ occurs without characteristic functions.

Proposition 2.3. *If $\square \in O_{N,\varepsilon}$, then the following three conditions are equivalent:*

(a) for all $\mathbf{x} \in C^2([a,b],\mathbb{C}^d)$, the function $\square\mathbf{x}(t)$ tends to $\dot{\mathbf{x}}(t)$ locally uniformly in $]a,b[$ as ε tends to 0,

(b) $\square 1 = 0$ and $\square t = 1$ in $[a+N\varepsilon, b-N\varepsilon]$,

(c) $\displaystyle\sum_{\ell=-N}^{N} \gamma_\ell = 0$ and $\displaystyle\sum_{\ell=-N}^{N} \ell\gamma_\ell = 1$.

Proof. Let us consider the following particular case. When $d = 1$ and $x(t) = \alpha t + \beta$, $\alpha, \beta \in \mathbb{C}$, the formula (1.2) shows that, for all $t \in [a+N\varepsilon, b-N\varepsilon]$,

$$\square x(t) = \frac{\alpha t + \beta}{\varepsilon} \sum_{\ell=-N}^{N} \gamma_l + \alpha \sum_{\ell=-N}^{N} \ell\gamma_l.$$

The only way that the functions $\square x$ of $t \in [a+N\varepsilon, b-N\varepsilon]$ and $\varepsilon \in \mathbb{R}^\star$ converge to some limit when ε tends to 0, is that $\sum_{\ell=-N}^{N} \gamma_l = 0$. This being noticed, the equivalence between (b) and (c) is obvious and furthermore, (a) implies (c) by choosing $(\alpha,\beta) = (0,1)$ and $(\alpha,\beta) = (1,0)$. Lastly, let us prove that (c) implies (a). Let $\delta > 0$ and $\varepsilon < \frac{\delta}{N}$. Let us show that the function $\square\mathbf{x}$ converges uniformly in $[a+\delta, b-\delta]$ to $\dot{\mathbf{x}}$ as ε tends to 0. For all $t \in [a+N\varepsilon, b-N\varepsilon]$ and η small enough, we use the Taylor-Lagrange inequality

$$\|\mathbf{x}(t+\eta) - \mathbf{x}(t) - \eta\dot{\mathbf{x}}(t)\| \le \frac{1}{2}\eta^2 \sup_{t\in[a,b]} \|\ddot{\mathbf{x}}(t)\|. \tag{2.4}$$

Since $t \in [a+N\varepsilon, b-N\varepsilon]$, the formulas (2.1) and (2.4) with $\eta = \ell\varepsilon$ yield the following inequality

$$\left\| \Box \mathbf{x}(t) - \left(\sum_{\ell=-N}^{N} c_\ell \right) \mathbf{x}(t) - \varepsilon \left(\sum_{\ell=-N}^{N} \ell c_\ell \right) \dot{\mathbf{x}}(t) \right\| \leq \frac{1}{2} \varepsilon^2 \left(\sum_{\ell=-N}^{N} \ell^2 |c_\ell| \right) \|\ddot{\mathbf{x}}\|_\infty.$$

Due to the assumptions (c), we obtain readily $\lim_{\varepsilon \to 0} \sup_{t \in [a+\delta, b-\delta]} \|\Box \mathbf{x}(t) - \dot{\mathbf{x}}(t)\| = 0$ for all δ, and the proof is complete. □

At last, we define $\tilde{O}_{N,\varepsilon}$ as the affine space of operators $\Box \in O_{N,\varepsilon}$ satisfying any condition of Proposition 2.3. Let us consider the case $N = 1$. The condition (c) of the previous Proposition is nothing but $\gamma_{-1} + \gamma_0 + \gamma_1 = 0$ and $\gamma_1 - \gamma_{-1} = 1$. So $\tilde{O}_{1,\varepsilon}$ is the affine straight line with elements $\Box^{[r,1-r]}$, $r \in \mathbb{C}$. In this space, we may excerpt the four operators $\Box^{[1,0]}$, $\Box^{[0,1]}$, $\Box^{[\frac{1}{2},\frac{1}{2}]}$ and $\Box^{[\frac{1-i}{2},\frac{1+i}{2}]}$ seen previously.

3 Leibniz formulas for □ operators in $O_{1,\varepsilon}$

In order to deduce his version of D.E.L., Cresson found in [4] a product formula for $\Box_q$ which is an analog of the classical Leibniz formula. When such a formula exists, a principle of discrete virtual works may be stated. In this section, we generalize Cresson's identity to the family of operators (2.3).

Theorem 3.1. *Let $\Box \in O_{1,\varepsilon}$ be of the shape (2.3) with $r, s \in \mathbb{C}^\star$ and $\frac{s}{r} \notin \mathbb{R}$. Then, for all continuous functions $\mathbf{f}, \mathbf{g} : [a,b] \to \mathbb{C}^d$, we get the generalized Leibniz formula*

$$\Box^{[r,s]}(\mathbf{f} \cdot \mathbf{g})(t) = \mathbf{f}(t) \cdot \Box^{[r,s]}\mathbf{g}(t) + \mathbf{g}(t) \cdot \Box^{[r,s]}\mathbf{f}(t) +$$
$$\frac{\varepsilon(r\bar{s}^2 - \bar{r}^2 s)}{(r\bar{s} - \bar{r}s)^2} \Box^{[r,s]}\mathbf{f}(t) \cdot \Box^{[r,s]}\mathbf{g}(t) - \frac{\varepsilon rs(r-s)}{(r\bar{s} - \bar{r}s)^2} \Box^{[\bar{r},\bar{s}]}\mathbf{f}(t) \cdot \Box^{[\bar{r},\bar{s}]}\mathbf{g}(t) +$$
$$\frac{\varepsilon rs(\bar{r} - \bar{s})}{(r\bar{s} - \bar{r}s)^2} \left(\Box^{[r,s]}\mathbf{f}(t) \cdot \Box^{[\bar{r},\bar{s}]}\mathbf{g}(t) + \Box^{[\bar{r},\bar{s}]}\mathbf{f}(t) \cdot \Box^{[r,s]}\mathbf{g}(t) \right). \tag{3.1}$$

Proof. Since the formula (3.1) is $\mathbb{C}$-bilinear w.r.t. $\mathbf{f}$ and $\mathbf{g}$ and □ acts component-wise, we may suppose without loss of generality that $d = 1$ and $\mathbf{f} = f, \mathbf{g} = g : [a,b] \to \mathbb{C}$. We slightly generalize the proof of Theorem 2.1 of [4]. Let W and $\tilde{W}$ two operators in $O_{1,\varepsilon}$. We study the existence of four complex numbers d_1, d_2, d_3, d_4 such that the following formula holds

$$W(fg) = W(f)g + fW(g) +$$
$$d_1 W(f)W(g) + d_2 \tilde{W}(f)W(g) + d_3 W(f)\tilde{W}(g) + d_4 \tilde{W}(f)\tilde{W}(g), \tag{3.2}$$

for all f and g in $C^0([a,b], \mathbb{C})$. We choose

$$W = \Box^{[r,s]} = r\Box^{[1,0]} + s\Box^{[0,1]} \text{ and } \tilde{W} = \Box^{[r',s']} = r'\Box^{[1,0]} + s'\Box^{[0,1]}$$

for some complex numbers r, s, r', s' such that $rs' - sr' \neq 0$. But we have also the two formulas

$$\square^{[1,0]}(fg) = f.\square^{[1,0]}g + g.\square^{[1,0]}f + \varepsilon\square^{[1,0]}f.\square^{[1,0]}g,$$
$$\square^{[0,1]}(fg) = f.\square^{[0,1]}g + g.\square^{[0,1]}f - \varepsilon\square^{[0,1]}f.\square^{[0,1]}g.$$

These two product formulas are merely generalizations of Cresson's ones [4]. Substituting the previous formulas in (3.2), the identity (3.2) holds for all f and g in $C^0([a,b],\mathbb{C})$ if and only if the coefficients d_1, d_2, d_3 and d_4 satisfy

$$\begin{pmatrix} r^2 & rr' & rr' & r'^2 \\ rs & sr' & rs' & r's' \\ rs & rs' & sr' & r's' \\ s^2 & ss' & ss' & s'^2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} r\varepsilon \\ 0 \\ 0 \\ -s\varepsilon \end{pmatrix}. \tag{3.3}$$

This matrix has a determinant equal to $-(rs' - sr')^4 \neq 0$. The coefficients d_ℓ, obtained by solving (3.3), are given by

$$d_1 = \frac{\varepsilon}{\delta}(rs'^2 - sr'^2),\ d_2 = d_3 = \frac{\varepsilon rs}{\delta}(r' - s'),\ d_4 = \frac{\varepsilon rs}{\delta}(s - r) \tag{3.4}$$

where $\delta = (rs' - sr')^2$. Since $\frac{s}{r} \notin \mathbb{R}$, we can choose $r' = \overline{r}$ and $s' = \overline{s}$ so that $\delta \neq 0$. We replace the preceding values in (3.2) and we get easily the formula (3.1). □

As an example of (3.2), we get $-d_1 = d_2 = d_3 = d_4 = -\frac{1}{2}i\varepsilon$ for the operator $\square^{[\frac{1-i}{2}, \frac{1+i}{2}]}$. Hence, the operator $\square_q$ chosen in [4, 5] satisfies the following Leibniz formula, for all continuous functions $f, g : [a,b] \to \mathbb{R}$

$$\square_q(fg) = \square_q(f)g + f\square_q(g) +$$
$$\frac{1}{2}i\varepsilon[\square_q(f)\square_q(g) - \square_q(f)\boxminus_q(g) - \boxminus_q(f)\square_q(g) - \boxminus_q(f)\boxminus_q(g)],$$

where $\boxminus_q$ stands for the complex conjugate operator of $\square_q$.

Obvioulsy, the choice of $(r', s') = (\overline{r}, \overline{s})$ in the proof is made only to ensure that $\tilde{W}$ may be expressed conveniently through W. In that way, $\tilde{W}(\mathbf{f}) = \overline{W(\overline{\mathbf{f}})}$. Another choice may be dictacted by requiring that W and $\tilde{W}$ belong to $\tilde{O}_{1,\varepsilon}$. We obtain in that case a Leibniz formula for $W = \square^{[r,1-r]}$ and $\tilde{W} = \square^{[r-1,2-r]}$.

Theorem 3.1 shows that some operators $\square$ may not check a Leibniz formula. As a rule, integration by parts for $\int_a^b \mathbf{u}\square\mathbf{v}dt$ may not be readily performed but we shall see in the next section how to handle this difficulty.

4 Critical points of discrete actions

There exist many ways to define the discrete actions and lagrangians in terms of the classical ones. A first idea consists in discretizing the integral of action and working with finite sums. A second one, in time scale calculus, preserves the integral character of the actions by replacing $\dot{\mathbf{x}}(t)$ with $\mathbf{x}^\Delta(t)$ or $\mathbf{x}^\nabla(t)$, see [3, 8, 10] for definitions. In this paper, we proceed in another way.

Let $\mathcal{L} : [a,b] \times \mathbb{C}^{2d} \to \mathbb{C}$ be a continuous lagrangian depending on $2d+1$ variables. We set $L(t, S(\mathbf{x})(t)) = \mathcal{L}(t, \mathbf{x}(t), \Box\mathbf{x}(t))$ and we define $\mathcal{A}_c$ and $\mathcal{A}_d$ as

$$\mathcal{A}_c(\mathbf{x}) = \int_a^b \mathcal{L}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))dt \tag{4.1}$$

and

$$\mathcal{A}_d(\mathbf{x}) = \int_a^b L(t, S(\mathbf{x})(t))dt = \int_a^b \mathcal{L}(t, \mathbf{x}(t), \Box\mathbf{x}(t))dt. \tag{4.2}$$

These classical and discrete actions are well-defined, respectively on $C^1([a,b], \mathbb{C}^d)$ and $C^0([a,b], \mathbb{C}^d)$, in consideration of Proposition 2.1.

A fundamental problem is to minimize the action (4.1) under Dirichlet boundary conditions. Then we have to handle real-valued functions and parameters in order to deal with optima instead of critical points of the action (4.2). For all $\Box$, we define $\Box^\star$ as the operator obtained from (1.2) by substituting $c_{-\ell}$ for c_ℓ.

Theorem 4.1. *Suppose that $\mathcal{L}$ is C^1. Let $\mathbf{x} = (x_1, \ldots, x_d) \in C^0([a,b], \mathbb{C}^d)$ be a critical point of (4.2). Then $\mathbf{x}$ satisfies the following functional equation*

$$\forall j \in \{1, \ldots, d\}, \ \sum_{\ell=-N}^{N} \frac{\partial L}{\partial \xi_{j,\ell}}(t - \ell\varepsilon, S(\mathbf{x})(t - \ell\epsilon))\chi(t - \ell\varepsilon) = 0 \tag{4.3}$$

which is equivalent to

$$\frac{\partial \mathcal{L}}{\partial x_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) + \Box^\star \frac{\partial \mathcal{L}}{\partial \dot{x}_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) = 0, \tag{4.4}$$

for all $j \in \{1, \ldots, d\}$ and $t \in [a,b]$.

Proof. Let us remark first that for all $\mathbf{x} \in C^0([a,b], \mathbb{C}^d)$, the function $t \mapsto \mathcal{L}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) = L(t, S(\mathbf{x}(t)))$ is Riemann-integrable help to formulas (2.1) and (2.2). Since $\mathcal{L}$ is C^1, L is also C^1 and $\mathcal{A}_d$ admits everywhere a Gâteaux derivative. Indeed, let $\mathbf{h} \in C^0([a,b], \mathbb{C}^d)$ be such that $\mathbf{h}(a) = 0$ and $\mathbf{h}(b) = 0$, so we get

$$\begin{aligned}\mathcal{A}_d(\mathbf{x} + \eta\mathbf{h}) - \mathcal{A}_d(\mathbf{x}) &= \int_a^b (L(t, S(\mathbf{x} + \eta\mathbf{h})(t)) - L(t, S(\mathbf{x})(t)))dt \\ &= \eta \int_a^b \sum_{j=1}^{d} \sum_{\ell=-N}^{N} \frac{\partial L}{\partial \xi_{j,\ell}}(t, S(\mathbf{x})(t))\chi(t + \ell\varepsilon)h_j(t + \ell\varepsilon)dt + O(\eta^2).\end{aligned}$$

In this formula, the coefficient of η is nothing but the Gâteaux derivative $\mathcal{D}\mathcal{A}_d(\mathbf{x})(\mathbf{h})$. By considering the change of variable $t = \tau - \ell\varepsilon$ and by setting

$$Z_{j,\ell}(\tau) = \frac{\partial L}{\partial \xi_{j,\ell}}(\tau - \ell\varepsilon, S(\mathbf{x})(\tau - \ell\varepsilon)),$$

we obtain

$$D\mathcal{A}_d(\mathbf{x})(\mathbf{h}) = \sum_{j=1}^{d}\sum_{\ell=-N}^{N}\int_{a+\ell\varepsilon}^{b+\ell\varepsilon} Z_{j,\ell}(\tau)\chi(\tau)h_j(\tau)d\tau.$$

We may extend the interval of integration to $[a-N\varepsilon, b+N\varepsilon]$ for each summand by observing that $\int_{a+\ell\varepsilon}^{b+\ell\varepsilon} f(\tau)d\tau = \int_{a-N\varepsilon}^{b+N\varepsilon} f(\tau)\chi(\tau-\ell\varepsilon)d\tau$, for all function f. Thus, for all $\mathbf{h}$ we have

$$\begin{aligned} D\mathcal{A}_d(\mathbf{x})(\mathbf{h}) &= \sum_{j=1}^{d}\int_{a-N\varepsilon}^{b+N\varepsilon}\sum_{\ell=-N}^{N} Z_{j,\ell}(\tau)\chi(\tau)\chi(\tau-\ell\varepsilon)h_j(\tau)d\tau \\ &= \sum_{j=1}^{d}\int_{a}^{b} Z_j(\tau)h_j(\tau)d\tau, \end{aligned}$$

where $Z_j(\tau) = \sum_{\ell=-N}^{N} Z_{j,\ell}(\tau)\chi(\tau-\ell\varepsilon)$. Since $\mathbf{x}$ is a critical point of (4.2), the previous sum vanishes for all $\mathbf{h}$. Help to the fundamental lemma of the calculus of variations, we get $Z_j(t) = 0$ for all $j \in \{1,\ldots,d\}$ and $t \in [a,b]$, and this amounts to (4.3).
Let us show that (4.3) may be rewritten as (4.4). By definition of $L(t, S(\mathbf{x})(t))$, we have for all $\ell \neq 0$,

$$\frac{\partial L}{\partial \xi_{j,\ell}}(t, S(\mathbf{x})(t)) = \frac{\partial \mathcal{L}}{\partial \dot{x}_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t))c_\ell\chi(t+\ell\varepsilon)$$

and for $\ell = 0$,

$$\frac{\partial L}{\partial \xi_{j,0}}(t, S(\mathbf{x})(t)) = \frac{\partial \mathcal{L}}{\partial \dot{x}_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t))c_0\chi(t) + \frac{\partial \mathcal{L}}{\partial x_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t)).$$

Thus, (4.3) is equivalent to

$$\sum_{\ell=-N}^{N}\frac{\partial \mathcal{L}}{\partial \dot{x}_j}(t-\ell\varepsilon, \mathbf{x}(t-\ell\varepsilon), \Box\mathbf{x}(t-\ell\varepsilon))c_\ell\chi(t)\chi(t-\ell\varepsilon) + \frac{\partial \mathcal{L}}{\partial x_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) = 0.$$

Replacing ℓ with $-\ell$ and having in mind formula (1.2), we notice that the first term is equal to $\Box^\star\frac{\partial \mathcal{L}}{\partial \dot{x}_j}(t, \mathbf{x}(t), \Box\mathbf{x}(t))$, and we prove the result. $\Box$

5 Comparison of some necessary optimality conditions of Euler-Lagrange type

In this section, we quote some necessary Euler-Lagrange equations of the first order which are derived by many authors and which look like (1.3).

Let us first discuss in details distinctive features of equations (1.3) and Cresson's version. In [4], Cresson deals with the case $N = 1$ and the so-called quantum derivative operator $\Box_q$. The fact that $\Box_q$ has a Leibniz formula is central in the deduction of his version of D.E.L. For any function $f(\mathbf{x},\varepsilon)$, Cresson defines the ε-dominant part $[f]_\varepsilon$ with a limiting process. He proved that if

$$\lim_{\varepsilon\to 0} D\mathcal{A}_d(\mathbf{x}) = 0,$$

then

$$\left[\frac{\partial\mathcal{L}}{\partial x_j}-\Box_q\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\right]_\varepsilon=\lim_{\varepsilon\to 0}\left(\frac{\partial\mathcal{L}}{\partial x_j}-\Box_q\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\right)=0 \tag{5.1}$$

which is an asymptotic alternative form of (1.3). Equation (1.3) is different from (5.1) at least in three respects. First, we do not use any property of $\Box$. In contrast, $\Box_q$ is such that

$$[\Im(\Box_q f)]_\varepsilon=0,\ \Re(\Box_q^\star f)=-\Re(\Box_q f),\text{ and }[\Box_q f(t)]_\varepsilon=[-\Box_q^\star f(t)]_\varepsilon$$

for all real-valued function f and all $t\in[a+\varepsilon,b-\varepsilon[$. Second, the characteristic functions $\chi(t+\ell\varepsilon)$ of the various intervals appear in the sampling process (1.2) as well as in the action (4.2). Third, (1.3) does not depend on the coefficients nor on the length $2N+1$ of formula (1.2).

Let us compare now (1.3) with two other versions of the first order optimality condition. We use the framework of time scale calculus [3]. In [3] and [1], a time scale $\mathbb{T}$ is arbitrarily chosen and the scale derivative is either the Δ-derivative or the ∇-derivative. So the variational problems consist in minimizing the actions with lagrangians $\mathrm{grad}_{\mathbf{x}}\mathcal{L}(t,\mathbf{x}\circ\sigma,\Delta\mathbf{x})$ in [3] and $\mathrm{grad}_{\mathbf{x}}\mathcal{L}(t,\mathbf{x}\circ\rho,\nabla\mathbf{x})$ in [1]. The resulting Euler-Lagrange equations are

$$\begin{aligned}\mathrm{grad}_{\mathbf{x}}\mathcal{L}(t,\mathbf{x}\circ\sigma,\Delta\mathbf{x})-\Delta\mathrm{grad}_{\mathbf{v}}\mathcal{L}(t,\mathbf{x}\circ\sigma,\Delta\mathbf{x})=0,\\ \mathrm{grad}_{\mathbf{x}}\mathcal{L}(t,\mathbf{x}\circ\rho,\nabla\mathbf{x})-\nabla\mathrm{grad}_{\mathbf{v}}\mathcal{L}(t,\mathbf{x}\circ\rho,\nabla\mathbf{x})=0.\end{aligned} \tag{5.2}$$

Let us note that the duality between the Δ and the ∇ approaches may lead to inconsistences (see [7, 11]). Although being obviously similar, the equations (5.2) and (1.3) are of a distinct nature. First, (5.2) are ordinary differential equations in the intervals with zero graininess. In constrast, (1.3) consist in functional difference equations. Second, the required regularity of the minimizers is not the same, C^1_{rd} or C^1_{ld} for (5.2) and C^0 for (1.3). Third, Leibniz formula plays an important role in the derivation of (5.2) and not in the proof of (1.3).

Unified Euler-Lagrange necessary optimality conditions are provided in [6, 8] and pose the problem of minimizing or maximizing actions with lagrangians involving Δ and ∇-derivatives. The authors of [6, 8] consider a linear combination of lagrangians $\mathcal{L}=k_1\mathcal{L}_1+k_2\mathcal{L}_2$ that they discretize by replacing $\dot{\mathbf{x}}(t)$ in some parts of the lagrangian with $\Delta\mathbf{x}$ and in other parts with $\nabla\mathbf{x}$, that is to say

$$\mathcal{L}_d(\mathbf{x})=k_1\mathcal{L}_1(t,\mathbf{x}\circ\sigma,\Delta\mathbf{x})+k_2\mathcal{L}_2(t,\mathbf{x}\circ\rho,\nabla\mathbf{x}). \tag{5.3}$$

If $t\in\mathbb{T}$ is left and right-scattered, (5.3) embodies in some way the three terms of the usual Runge-Kutta operator. The delta-nabla Euler-Lagrange equations on time scales are no more "local" but integral equations. Another extension of Euler-Lagrange equations on time scales to set-valued functions has been developed in [9]. Both works lead to necessary first order conditions which are far different from (1.3).

Lastly, we may cite the Euler-Lagrange equations obtained in [12]. A discrete lagrangian $\mathcal{L}_d$ is associated to the lagrangian $\mathcal{L}$. The construction of $\mathcal{L}_d$ from $\mathcal{L}$ is based on some approximation to the action integral using for instance the rectangle rule or the Newmark method. Accordingly, the discrete action $\mathcal{A}_d = \sum_{k=0}^{M-1} \mathcal{L}_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \varepsilon)$ with $\varepsilon = \frac{b-a}{M}$ is defined on a suitable discrete path space. The first order necessary condition is

$$\mathrm{grad}_1 \mathcal{L}_d(\mathbf{q}_k, \mathbf{q}_{k+1}, \varepsilon) + \mathrm{grad}_2 \mathcal{L}_d(\mathbf{q}_{k-1}, \mathbf{q}_k, \varepsilon) = 0, \tag{5.4}$$

where grad_1 and grad_2 denote the gradients w.r.t. the first and the second variable respectively. For each integer k, the vector $\mathbf{q}_k \in \mathbb{R}^d$ may be thought as an approximation of $\mathbf{x}(a+k\varepsilon)$. Among the differences between (5.4) and (1.3), let us emphasize on the fact that (1.3) are functional delayed equations while (5.4) are recurrence equations.

6 Quadratic lagrangians in discrete and classical settings

In this section we deal with a system of d ordinary differential equations of the second order arising from the following lagrangian

$$\mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\,{}^{\mathrm{T}}\dot{\mathbf{x}} P \dot{\mathbf{x}} + \frac{1}{2}\,{}^{\mathrm{T}}\mathbf{x} Q \mathbf{x} + {}^{\mathrm{T}}\mathbf{x} R \dot{\mathbf{x}} + {}^{\mathrm{T}}J_1 \dot{\mathbf{x}} + {}^{\mathrm{T}}J_2 \mathbf{x} + J_3, \tag{6.1}$$

where $P(t), Q(t), R(t) \in \mathbb{R}^{d\times d}$, $J_1(t), J_2(t) \in \mathbb{R}^d$ and $J_3(t)$ is a scalar function. Many physical systems are modelized by such lagrangians, in electromagnetism, quantum mechanics, material science, regulators models and so on.

We assume from now on that the coefficients in (6.1) are real and smooth, and that for all $t \in [a,b]$, $P(t)$ and $Q(t)$ are symmetric. Although the symmetric part of $R(t)$ gives rise to a null lagrangian, we do not assume that $R(t)$ is skew-symmetric.

Theorem 6.1. *Let $\mathcal{L}$ be a quadratic lagrangian such that ${}^{\mathrm{T}}P = P$, ${}^{\mathrm{T}}Q = Q$, and L associated to $\mathcal{L}$ as in (4.2). The Euler-Lagrange equations associated to (6.1) can be written as*

$$-P\ddot{\mathbf{x}} + (-\dot{P} + R - {}^{\mathrm{T}}R)\dot{\mathbf{x}} + (Q - {}^{\mathrm{T}}\dot{R})\mathbf{x} - \dot{J}_1 + J_2 = 0. \tag{6.2}$$

The equations (6.2) may be discretized a posteriori to give

$$-P\Box(\Box\mathbf{x}) + (-\dot{P} + R - {}^{\mathrm{T}}R)\Box\mathbf{x} + (Q - {}^{\mathrm{T}}\dot{R})\mathbf{x} - \dot{J}_1 + J_2 = 0. \tag{6.3}$$

If $\mathbf{x} \in C^0([a,b], \mathbb{C}^d)$ is a critical point of the action (4.2) under the Dirichlet boundary conditions, then it must satisfy

$$\Box^\star(P\Box\mathbf{x}) + \Box^\star({}^{\mathrm{T}}R\mathbf{x}) + R\Box\mathbf{x} + Q\mathbf{x} + \Box^\star J_1 + J_2 = 0. \tag{6.4}$$

Proof. First, (6.2) is straightforward, since we get

$$\frac{\partial \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}})}{\partial \mathbf{x}} = Q\mathbf{x} + R\dot{\mathbf{x}} + J_2 \text{ and } \frac{\partial \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}} = P\dot{\mathbf{x}} + {}^{\mathrm{T}}R\mathbf{x} + J_1.$$

Next, (6.3) is obtained by discretizing the derivatives in (6.2), i.e. by replacing $\dot{\mathbf{x}}, \ddot{\mathbf{x}}$ by $\Box\mathbf{x}, \Box(\Box\mathbf{x})$ respectively and the result holds. At last, (6.4) is a consequence of (4.4). Indeed, using the previous derivatives, (4.4) gives

$$\square^\star(P\square\mathbf{x} + {}^{\mathrm{T}}R\mathbf{x} + J_1) + Q\mathbf{x} + R\square\mathbf{x} + J_2 = 0,$$

which ends the proof. $\square$

A straightforward computation, using the formula (1.2) for $\square$ and $\square^\star$, shows that (6.4) may be written, for all $t \in [a,b]$, as

$$\sum_{\substack{-2N \le \ell \le 2N \\ -N \le j \le N \\ |\ell + j| \le N}} c_{\ell+j}c_j\chi(t - j\varepsilon)\chi(t + \ell\varepsilon)P(t - j\varepsilon)\mathbf{x}(t + \ell\varepsilon) + Q(t)\mathbf{x}(t) +$$

$$\sum_{\ell=-N}^{N}\chi(t + \ell\varepsilon)(c_\ell R(t) + c_{-\ell}\,{}^{\mathrm{T}}R(t + \ell\varepsilon))\mathbf{x}(t + \ell\varepsilon) + \square^\star J_1(t) + J_2(t) = 0. \tag{6.5}$$

This equality is suitable to solve effectively the functional difference equations (1.3) in the case of quadratic lagrangians as we sall see in the following two sections.

7 A first example: the harmonic oscillator

Let us investigate now the harmonic oscillator. We have in this case $d = 1$, $R = 0$, $J_1 = J_2 = 0$ and we set moreover $P(t) = p$, $Q(t) = q$ with $pq < 0$. Solutions of C.E.L. $-p\ddot{x} + qx = 0$ are periodic. Let us study the periodicity of solutions of D.E.L. for $N = 1$ and ε small enough. If t lies in $[a + 2\varepsilon, b - 2\varepsilon]$, (6.5) may be simplified into

$$pc_{-1}c_1[x(t - 2\varepsilon) + x(t + 2\varepsilon)] + pc_0(c_{-1} + c_1)[x(t - \varepsilon) + x(t + \varepsilon)] + \\ (q + p(c_{-1}^2 + c_0^2 + c_1^2))x(t) = 0. \tag{7.1}$$

The result below presents an additional characterization of suitable $\square$ for getting pseudo-periodic solutions.

Proposition 7.1. *Let $\square \in \tilde{O}_{1,\varepsilon}$. The two following properties are equivalent.*

(a) For all p and q with $pq < 0$ and for all ε small enough, the roots of the characteristic polynomial of (7.1) are of modulus 1.

(b) For some $k \in \mathbb{R}$, we have $\square = \square^{[\frac{1}{2},\frac{1}{2}]} + ik\square^{[1,-1]}$.

Proof. The characteristic polynomial $D(\lambda)$ of the recurrence (7.1) is symmetric, that is to say $D(\lambda) = \lambda^4 D(\frac{1}{\lambda})$. We set $\mu = \lambda + \frac{1}{\lambda}$ and we define the quadratic $E(\mu) = \frac{\varepsilon^2}{p\lambda^2}D(\lambda)$. We easily get

$$E(\mu) = \gamma_1\gamma_{-1}\mu^2 + \gamma_0(\gamma_1 + \gamma_{-1})\mu + (\gamma_0^2 + \gamma_1^2 + \gamma_{-1}^2 - 2\gamma_1\gamma_{-1} + \frac{q}{p}\varepsilon^2).$$

$(a) \Rightarrow (b)$. We look for the parameters $\gamma_{-1}, \gamma_0, \gamma_1$ for which $D(\lambda)$ has roots on the unit circle, for ε small enough and for all p, q, $pq < 0$. This amounts to say that $E(\mu)$ has only real roots in $[-2,2]$ for all p,q, $pq < 0$, provided ε is small enough. Now, given three complex numbers α,β,γ with $\alpha \neq 0$, the two following properties are equivalent. First, the solutions of the quadratic $\alpha y^2+\beta y+\gamma+\eta=0$ are in $[-2,2]\subset\mathbb{R}$, for all η small enough. Second, we have $\alpha,\beta,\gamma\in\mathbb{R}$ and

$$|\beta|\leq 4|\alpha|,\ \ |\gamma|\leq 4|\alpha|,\ \ 4\alpha\gamma\leq\beta^2,\ \ 2|\beta|\leq 4|\alpha|+\gamma sgn(\alpha).$$

The proof of this equivalence is straightforward and relies on explicit computations using the usual solution of a quadratic. Let us apply this to $E(\mu)$. Since $\Box\in\tilde{O}_{1,\varepsilon}$, we have $\Box 1=\gamma_{-1}+\gamma_0+\gamma_1=0$ and $\Box t=\gamma_1-\gamma_{-1}=1$ inside $[a+\varepsilon,b-\varepsilon]$ if and only if $\gamma_1=r$, $\gamma_{-1}=r-1$ and $\gamma_0=-2r+1$ for some $r\in\mathbb{C}$. So, $\alpha=r(r-1)$, $\beta=-(2r-1)^2$ and $\gamma=4r^2-4r+2$ are real. Therefore we obtain $\Re(r)=1/2$ and easy computations lead to (b).
$(b)\Rightarrow(a)$. If (b) holds then, by setting $r=\frac{1}{2}+ik$, we get $\gamma_1=r$, $\gamma_{-1}=r-1$ and $\gamma_0=-2r+1$. Algebraic computations show that the solutions of

$$E(\mu)=r(r-1)\mu^2-(2r-1)^2\mu+(4r^2-4r+2)-\omega^2\varepsilon^2, \tag{7.2}$$

where $\omega^2=-\frac{q}{p}$, are in $[-2,2]\subset\mathbb{R}$ for all $\varepsilon\leq\frac{1}{|\omega|\sqrt{1+4k^2}}$. This implies (a). □

Remark 7.2. We recover $\Box^{[\frac{1}{2},\frac{1}{2}]}$ and $\Box^{[\frac{1-i}{2},\frac{1+i}{2}]}$ by setting $k=0$ and $k=-\frac{1}{2}$ in (b) respectively.

Remark 7.3. If $\Box$ lies in $O_{1,\varepsilon}$ but not necessarily in $\tilde{O}_{1,\varepsilon}$, the property (a) is equivalent to the following inequalities for $\gamma_{-1},\gamma_0,\gamma_1\in\mathbb{C}$

$$\begin{gathered}4|\gamma_1\gamma_{-1}|-|\gamma_0(\gamma_1+\gamma_{-1})|\geq 0,\ \ 4|\gamma_1\gamma_{-1}|-|\gamma_0^2+\gamma_1^2+\gamma_{-1}^2-2\gamma_1\gamma_{-1}|\geq 0,\\ \gamma_0^2(\gamma_1+\gamma_{-1})^2-4\gamma_1\gamma_{-1}(\gamma_0^2+\gamma_1^2+\gamma_{-1}^2-2\gamma_1\gamma_{-1})\geq 0 \text{ and}\\ 16|\gamma_1\gamma_{-1}|+4\gamma_0^2+4\gamma_1^2+4\gamma_{-1}^2-8\gamma_1\gamma_{-1}-8|\gamma_0(\gamma_1+\gamma_{-1})|\geq 0.\end{gathered}$$

There exist infinitely many solutions $(\gamma_{-1},\gamma_0,\gamma_1)$ of these inequalities which are not of the shape $(r-1,-2r+1,r)$ nor even $(-s,s-r,r)$ with $r,s\in\mathbb{C}$. So there exist operators $\Box\notin\tilde{O}_{1,\varepsilon}$ satisfying (a) but not (b).

8 A second example: the discretization of a null-lagrangian

We address here the minimization of the discrete action $\mathcal{A}_d$ defined as

$$\mathcal{A}_d(x)=\frac{1}{\varepsilon}\int_a^b x(t)\left(\gamma_1 x(t+\varepsilon)\chi(t+\varepsilon)+\gamma_0 x(t)\chi(t)+\gamma_{-1}x(t-\varepsilon)\chi(t-\varepsilon)\right)dt. \tag{8.1}$$

Here, $d=1$ and $N=1$. The action (8.1) is associated to the stationary lagrangian $\mathcal{L}(t,x,\Box x)=x\Box x$ and is constrained by the boundary conditions $x(a)=\alpha$ and $x(b)=\beta$.

The continuous version of this problem deals with $\mathcal{L}(x)=x\dot{x}$ which is a null-lagrangian. The C.E.L. is equivalent to $0=0$, the action $\mathcal{A}_c$ defined as in (4.1) is constant in $C^1([a,b],\mathbb{C})$ and is equal to $\frac{1}{2}(\beta^2-\alpha^2)$. In constrast, the discrete case is quite different.

Proposition 8.1. *Let $\gamma_{-1},\gamma_0,\gamma_1\in\mathbb{C}$ and let $\mathcal{A}_d$ defined by (8.1).*

- *$\mathcal{A}_d$ is constant in $C^0([a,b],\mathbb{C})$ if and only if $\mathcal{A}_d\equiv 0$.*
- *If $\gamma_{-1},\gamma_0,\gamma_1\in\mathbb{R}$ and $|\gamma_{-1}+\gamma_1|\leq|\gamma_0|$ then $\mathcal{A}_d(x)$ has a constant sign in $C^0([a,b],\mathbb{R})$.*

Proof. Indeed, the discrete action (8.1) may be written as a combination of potential energy and autocorrelation function since we have

$$\mathcal{A}_d(x) = \frac{\gamma_0}{\varepsilon}\int_a^b x(t)^2 dt + \frac{\gamma_1+\gamma_{-1}}{\varepsilon}\int_a^{b-\varepsilon} x(t)x(t+\varepsilon)dt. \tag{8.2}$$

Both integrals are linearly independent functions w.r.t. x in $C^0([a,b],\mathbb{C})$. So, $x\Box x$ is a null-lagrangian if and only if $\mathcal{A}_d$ is constant in $C^0([a,b],\mathbb{C})$. This amounts to $\gamma_1+\gamma_{-1}=\gamma_0=0$ and hence, $\mathcal{A}_d\equiv 0$.
The second statement follows from using Cauchy-Schwarz inequality in (8.2) and shows that $\frac{1}{\gamma_0}\mathcal{A}_d(x)$ is strictly positive for all $x\in C^0([a,b],\mathbb{R})$. □

From now on, for sake of conciseness, we assume that $\gamma_{-1},\gamma_1\in\mathbb{R}$, $\gamma_0\in\mathbb{R}^\star$, $\gamma=\frac{\gamma_1+\gamma_{-1}}{2\gamma_0}\neq 0$ and $M=\frac{b-a}{\varepsilon}\in\mathbb{N}$. We shall examine the minimization of $\mathcal{A}_d(x)$ in two cases. The first one deals with piecewise constant maps while the second one uses the explicit solutions of D.E.L..

Proposition 8.2. *If $|\gamma|>\frac{1}{2}$ and ε is small enough, the range of the action $\mathcal{A}_d$, defined by (8.1) on the space of piecewise constant maps, is $\mathbb{R}$.*

Proof. We shall consider real piecewise constant maps $x:[a,b]\to\mathbb{R}$. Having in mind that $x(a)=\alpha$ and $x(b)=\beta$, we introduce an arbitrary vector $Y=(y_1,\dots,y_{M-2})\in\mathbb{R}^{M-2}$. We define next $x(t)=y_k$ if $a+k\varepsilon\le t<a+(k+1)\varepsilon$ for all $k=1,\dots,M-2$, $x(t)=\alpha$ if $t\in[a,a+\varepsilon[$ and lastly, $x(t)=\beta$ if $t\in[b-\varepsilon,b]$. Let V be the $(M-2)$-dimensional affine space generated by those functions. If $x\in V$, we easily get

$$\frac{1}{\gamma_0}\mathcal{A}_d(x) = \sum_{k=0}^{M-1} x_k^2 + 2\gamma\sum_{k=0}^{M-2} x_k x_{k+1} = {}^{\mathrm{T}}XS_MX \tag{8.3}$$

where X and the symmetric tridiagonal matrix S_M are respectively defined by

$$X=\begin{pmatrix}\alpha\\ Y\\ \beta\end{pmatrix}\in\mathbb{R}^M,\qquad S_M=\begin{pmatrix}1&\gamma&&\\ \gamma&1&\ddots&\\ &\ddots&\ddots&\gamma\\ &&\gamma&1\end{pmatrix}\in\mathbb{R}^{M\times M}. \tag{8.4}$$

Then we have

$$\frac{1}{\gamma_0}\mathcal{A}_d(x) = {}^{\mathrm{T}}YS_{M-2}Y + 2\gamma(\alpha y_1+\beta y_{M-2})+\alpha^2+\beta^2. \tag{8.5}$$

The eigenvalues of the matrix S_{M-2} are the $M-2$ real numbers $\lambda_k=1+2\gamma\cos\left(\frac{\pi k}{M-1}\right)$ with $k\in\{1,\dots,M-2\}$. For ε small enough, i.e. M large enough, the product of the eigenvalues of S_{M-2} associated to $k=1$ and $k=M-2$ is negative since $|\gamma|>\frac{1}{2}$. Hence, S_{M-2} has eigenvalues of opposite sign. Let $k\in\{1,\dots,M-2\}$ and Y be an eigenvector of S_{M-2} associated to λ_k and $X=(\alpha,Y,\beta)\in\mathbb{R}^M$. Then, the value of the action of the function $x\in V$ associated to the previous datas is equal to

$$\mathcal{A}_d(x)=\gamma_0\left(\lambda_k\|Y\|^2+2\gamma(\alpha y_1+\beta y_{M-2})+\alpha^2+\beta^2\right). \tag{8.6}$$

Since $\lambda_1\lambda_{M-2} < 0$, if Y lies in the union of the two eigenspaces associated to λ_1 or λ_{M-2} and $\|Y\| \to \infty$, we see that $\mathcal{A}_d$ is surjective help to the intermediate value theorem. As a consequence the range of $\mathcal{A}_d$ on the larger set V is also $\mathbb{R}$. $\square$

Now, let us solve the discrete Euler-Lagrange equation associated to (8.1). By using (6.4), D.E.L. may be written as $\square x + \square^\star x = 0$, that is to say

$$(\gamma_1+\gamma_{-1})(\chi(t+\varepsilon)x(t+\varepsilon)+\chi(t-\varepsilon)x(t-\varepsilon))+2\gamma_0 x(t)\chi(t) = 0. \tag{8.7}$$

Proposition 8.3. *We suppose that* $|\gamma| = \left|\frac{\gamma_1+\gamma_{-1}}{2\gamma_0}\right| \neq \frac{1}{2}$.

(a) Each solution $x \in C^0([a,b],\mathbb{C})$ of D.E.L. (8.7) is uniquely determined by its restriction φ to $[a,a+\varepsilon[$.

(b) D.E.L. admits a solution if and only if

$$\beta = \alpha\exp(\pm i(M+2)\theta) \text{ and } \gamma = \frac{-1}{2\cos\theta} \tag{8.8}$$

where $\theta = \frac{k\pi}{M+2}$ for some integer k.

(c) If (8.8) holds, the restriction of $\mathcal{A}_d$ to the space of solutions of D.E.L. has range $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}$ or $\{0\}$.

Proof. We define r_1 and r_2 as the roots of the polynomial $r^2+\frac{r}{\gamma}+1$. Since $|\gamma| \neq \frac{1}{2}$, we have $r_1 \neq \pm 1$, $r_2 \neq \pm 1$ and $r_1 \neq r_2$.

(a) Let us consider a solution $x \in C^0([a,b],\mathbb{C})$ of (8.7) and its restriction φ to $[a,a+\varepsilon[$. Let us express $x(t)$ as simple function of k and $\varphi(\tau)$ where $t = \tau + k\varepsilon$ with $\tau \in [a,a+\varepsilon[$ and $k \in \{0,\dots,M\}$. We find it convenient to denote $u_k = x(\tau+k\varepsilon)$. When $k \in \{1,\dots,M-1\}$, D.E.L. (8.7) is a linear recurrence relationship with three terms

$$\gamma u_{k+1} + u_k + \gamma u_{k-1} = 0. \tag{8.9}$$

In contrast, when $k=0$ and $k=M-1$, we get respectively $\gamma u_1 + u_0 = 0$ and $\gamma u_{M-1} + u_M = 0$. Solving the difference equation (8.9) gives

$$u_k = (d_1 r_1^k + d_2 r_2^k)\varphi(\tau) \text{ for all } k \in \{1,\dots,M-1\}.$$

By using this formula with $k=1$ and $k=2$, we obtain the two explicit constants d_1 and d_2 and we may write

$$u_k = \frac{1}{r_1-r_2}\varphi(\tau)[(1-\gamma^{-2})(r_2^{k-1}-r_1^{k-1})-\gamma^{-1}(r_1 r_2^{k-1}-r_2 r_1^{k-1})]. \tag{8.10}$$

Thus, u_k depends exclusively on the index k and the function $u_0 = \varphi(\tau)$.

(b) Since the numbers r_1 and r_2 satisfy $r_1+r_2 = -\frac{1}{\gamma}$ and $r_1 r_2 = 1$, the formula (8.10) holds also for $k=0$. The values $\tau = a$ and $k = M$, that is $\tau = b$, yield two relationships

$$(\gamma_1+\gamma_{-1})(u_M+u_{M-2})+2\gamma_0 u_{M-1}=0,$$
$$(\gamma_1+\gamma_{-1})(u_{M-1})+2\gamma_0 u_M=0,$$

from which we deduce $\beta=\alpha(d_1 r_1^M+d_2 r_2^M)$ and $d_1 r_1^{M+1}+d_2 r_2^{M+1}=0$. Since $r_1 r_2=1$, we obtain

$$\frac{\beta}{\alpha}=r_1^M d_1\frac{(r_2-r_1)}{r_2} \text{ and } r_2^{2M}=-\frac{d_1 r_1}{d_2 r_2}.$$

By using the explicit formulas for r_1,r_2,d_1,d_2, straightforward computations yield the result.

(c) Let us define ω_i and ω_s as the infimum and supremum of $\mathcal{A}_d(x)$ when x lies in the subspace of piecewise continuous solutions of D.E.L.. If x is a solution of D.E.L., plugging (8.10) into (8.2) and using Chasles formula gives

$$\mathcal{A}_d(x)=c\int_a^{a+\varepsilon}\varphi(t)(\Box\varphi)(t)dt=c(\gamma_0-\gamma^{-1}\gamma_1)\int_a^{a+\varepsilon}\varphi(t)^2dt, \tag{8.11}$$

where $c=\sum_{k=0}^{M-1}(d_1 r_1^k+d_2 r_2^k)^2$. Let us suppose that $c(\gamma_0-\gamma^{-1}\gamma_1)>0$. Then, for all $\eta>0$, we may find φ such that $\varphi(a)=\alpha$ and $\varphi(a+\varepsilon)=-\gamma^{-1}\alpha$ and lastly, $\mathcal{A}_d(x)\leq\eta$. As a consequence, $\omega_i=0$. A similar argument yields $\omega_s=+\infty$. Otherwise, if $c(\gamma_0-\gamma^{-1}\gamma_1)<0$, $\omega_s=0$ and $\omega_i=-\infty$. Lastly, if $c=0$ which is equivalent to $c_1=c_{-1}$, $\mathcal{A}_d$ vanishes for every solution of D.E.L..

□

Let us give some explanations about Proposition 8.3. The first equality in (8.8) expresses the fact that (8.7) behaves as a first order difference equation and not as a general second order one as (6.4) since the matrix P is absent from the lagrangian. So β is determined from α. The second equality in (8.8) expresses the non-resonance property enabling the linear combination of $\exp(ikt)$ and $\exp(-ikt)$ to fit the Dirichlet conditions. Lastly, we notice that the restriction $\varphi(\tau)$ of a solution of D.E.L. is arbitrary except for the boundary values $\varphi(a)$ and $\varphi(a+\varepsilon)$.

Table 1 shortens some of the previous results:

	Continuous case	Discrete case
Gauge	yes	depends on $\Box$ (see (8.2))
Values of action	$\frac{1}{2}(\beta^2-\alpha^2)$	$\mathbb{R}$ or $\mathbb{R}^+$ or $\mathbb{R}^-$ (depends on γ)
Eqn. of motion	$0=0$	$\Box x+\Box^\star x=0$
Sol. of eqn. of motion	all functions	depends on β/α and M

Table 1. Properties of actions with lagrangians $x\dot{x}$ and $x\Box x$

The examples presented in the last two sections show that the discretization process using generalized scale derivatives leads to intricate conclusions compared to the analogous continuous setting. Given a specific property of D.E.L. or $\mathcal{A}_d$ or $\mathcal{L}_d$, the description of the discretization operators $\square \in \mathcal{O}_{N,\varepsilon}$ ensuring this property gives rise to various and nontrivial algebraic or analytic problems. Among these problems, convergence of schemes and/or solutions, shape of solutions of dynamic equations, oscillatory, resonance and spectral properties are of a special interest.

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