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Radial Toeplitz Operators on the Unit Ball and Slowly Oscillating Sequences

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Abstract

In the paper we deal with Toeplitz operators acting on the Bergman space $\mathcal{A}^2(\mathbb{B}^n)$ of square integrable analytic functions on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. A bounded linear operator acting on the space $\mathcal{A}^2(\mathbb{B}^n)$ is called *radial* if it commutes with unitary changes of variables. Zhou, Chen, and Dong [9] showed that every radial operator S is diagonal with respect to the standard orthonormal monomial basis $(e_\alpha)_{\alpha\in\mathbb{N}^n}$. Extending their result we prove that the corresponding eigenvalues depend only on the length of multi-index α, i.e. there exists a bounded sequence $(\lambda_k)_{k=0}^\infty$ of complex numbers such that $S e_\alpha = \lambda_{|\alpha|} e_\alpha$.

Toeplitz operator is known to be radial if and only if its generating symbol g is a radial function, i.e., there exists a function a, defined on $[0,1]$, such that $g(z) = a(|z|)$ for almost all $z \in \mathbb{B}^n$. In this case $T_g e_\alpha = \gamma_{n,a}(|\alpha|)e_\alpha$, where the eigenvalue sequence $(\gamma_{n,a}(k))_{k=0}^\infty$ is given by

$$\gamma_{n,a}(k) = 2(k+n)\int_0^1 a(r)\, r^{2k+2n-1}\, dr = (k+n)\int_0^1 a(\sqrt{r})\, r^{k+n-1}\, dr.$$

Denote by Γ_n the set $\{\gamma_{n,a} \colon a \in L^\infty([0,1])\}$. By a result of Suárez [8], the C^*-algebra generated by Γ_1 coincides with the closure of Γ_1 in ℓ^∞ and is equal to the closure of

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d_1 in ℓ^∞, where d_1 consists of all bounded sequences $x = (x_k)_{k=0}^\infty$ such that

$$\sup_{k\geq 0}\Big((k+1)\,|x_{k+1} - x_k|\Big) < +\infty.$$

We show that the C^*-algebra generated by Γ_n does not actually depend on n, and coincides with the set of all bounded sequences $(x_k)_{k=0}^\infty$ that are *slowly oscillating* in the following sense: $|x_j - x_k|$ tends to 0 uniformly as $\frac{j+1}{k+1} \to 1$ or, in other words, the function $x\colon \{0,1,2,\ldots\} \to \mathbb{C}$ is uniformly continuous with respect to the distance $\rho(j,k) = |\ln(j+1) - \ln(k+1)|$. At the same time we give an example of a complex-valued function $a \in L^1([0,1], r\,dr)$ such that its eigenvalue sequence $\gamma_{n,a}$ is bounded but is not slowly oscillating in the indicated sense. This, in particular, implies that a bounded Toeplitz operator having *unbounded* defining symbol does not necessarily belong to the C^*-algebra generated by Toeplitz operators with *bounded* defining symbols.

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1 Introduction and Main Results

1.1 Bergman space on the unit ball

We shall use some notation and well-known facts from Rudin [3] and Zhu [10]. Denote by $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{C}^n$: $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w_j}$. Let $|\cdot|$ be the Euclidean norm in $\mathbb{C}^n$ induced by this inner product, and let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$. Denote by dv the Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$ normalized so that $v(\mathbb{B}^n) = 1$, and denote by $d\sigma$ the surface measure on the unit sphere $\mathbb{S}^{2n-1} = \partial\mathbb{B}^n$ normalized so that $\sigma(\mathbb{S}^{2n-1}) = 1$. Let $\mathbb{N} = \{0,1,2,\ldots\}$. Given a multi-index $\alpha \in \mathbb{N}^n$ and a vector $z \in \mathbb{C}^n$, we understand the symbols $|\alpha|$, $\alpha!$ and z^α in the usual sense:

$$|\alpha| = \sum_{j=1}^n \alpha_j, \qquad \alpha! = \prod_{j=1}^n \alpha_j!, \qquad z^\alpha = \prod_{j=1}^n z_j^{\alpha_j}.$$

Consider the *Bergman space* $\mathcal{A}^2 = \mathcal{A}^2(\mathbb{B}^n, v)$ of all square integrable analytic functions on $\mathbb{B}^n$. Denote by $(e_\alpha)_{\alpha\in\mathbb{N}^n}$ the standard orthonormal monomial basis in $\mathcal{A}^2$:

$$e_\alpha(z) = \sqrt{\frac{(n+|\alpha|!)}{n!\,\alpha!}}\,z^\alpha.$$

The *reproducing kernel* K_z of the space $\mathcal{A}^2$ at a point $z \in \mathbb{B}^n$ satisfies $\langle f, K_z\rangle = f(z)$ for all $f \in \mathcal{A}^2$, and is given by the following formula:

$$K_z(w) = \sum_{\alpha\in\mathbb{N}^n} \overline{e_\alpha(z)}\,e_\alpha = \frac{1}{(1-\langle w, z\rangle)^{n+1}}.$$

The *Berezin transform* of a bounded linear operator $S : \mathcal{A}^2 \to \mathcal{A}^2$ is a function $\mathbb{B}^n \to \mathbb{C}$ defined by

$$(\mathcal{B}(S))(z) = \frac{\langle S K_z, K_z\rangle}{\|K_z\|^2} = (1-|z|^2)^{n+1}\langle S K_z, K_z\rangle.$$

It is well known that the Berezin transform $\mathcal{B}$ is injective: if $\mathcal{B}(S)$ is identically zero, then $S = 0$. A proof of this fact for the one-dimensional case is given by Stroethoff [7].

Given a function $g \in L^1(\mathbb{B}^n)$, the *Toeplitz operator* T_g is defined on a dense subset of $\mathcal{A}^2$ by

$$(T_g(f))(z) := \int_{\mathbb{B}^n} \overline{K_z}\, g\, f\, dv.$$

If $g \in L^\infty(\mathbb{B}^n)$, then T_g is bounded and $\|T_g\| \le \|g\|_\infty$.

1.2 Radial operators on the unit ball

Following Zhou, Chen and Dong [9] we recall the concept of a *radial function* on $\mathbb{B}^n$ and of a *radial operator* acting on $\mathcal{A}^2$. The *radialization* of a measurable function $f : \mathbb{B}^n \to \mathbb{C}$ is given by

$$\mathrm{rad}(f)(z) := \int_{\mathcal{U}_n} f(Uz) dH(U),$$

where dH is the normalized Haar measure on the compact group $\mathcal{U}_n$ consisting of the unitary matrices of order n.

A function $f : \mathbb{B}^n \to \mathbb{C}$ is called *radial* if $\mathrm{rad}(f)$ coincides with f almost everywhere. For a continuous function f this means that $f(z) = f(|z|)$ for all $z \in \mathbb{B}^n$.

Given a unitary matrix $U \in \mathcal{U}_n$, denote by Ψ_U the corresponding "change of a variable operator" acting on $\mathcal{A}^2$:

$$(\Psi_U f)(z) := f(U^* z).$$

Here U^* is the conjugated transpose of U. Note that Ψ_U is a unitary operator on the space $\mathcal{A}^2$, its inverse is Ψ_{U^*}, and the formula $\Psi_{U_1 U_2} = \Psi_{U_1} \Psi_{U_2}$ holds for all $U_1, U_2 \in \mathcal{U}_n$.

Given a bounded linear operator $S : \mathcal{A}^2 \to \mathcal{A}^2$, its *radialization* $\mathrm{Rad}(S)$ is defined by

$$\mathrm{Rad}(S) := \int_{\mathcal{U}_n} \Psi_U S \Psi_{U^*}\, dH(U),$$

where the integration is understood in the weak sense.

A bounded linear operator S is called *radial* if $S\Psi_U = \Psi_U S$ for all $U \in \mathcal{U}_n$ or, equivalently, if $\mathrm{Rad}(S) = S$.

Zhou, Chen, and Dong [9] proved that the Berezin transform "commutes with the radialization" in the following sense: for every bounded linear operator S acting in $\mathcal{A}^2$

$$\mathcal{B}(\mathrm{Rad}(S)) = \mathrm{rad}\, B(S).$$

It follows that S is radial if and only if $\mathcal{B}(S)$ is radial. In the one-dimensional case (i.e., for $n = 1$) these facts were proved by Zorboska [11].

For each $\alpha \in \mathbb{N}^n$ denote by P_α the orthogonal projection onto the one-dimensional space generated by e_α:

$$P_\alpha(x) := \langle x, e_\alpha \rangle e_\alpha.$$

Given a bounded sequence $\lambda = (\lambda_m)_{m=0}^{\infty}$ of complex numbers, denote by R_λ the following operator (*radial operator with eigenvalue sequence* λ):

$$R_\lambda := \sum_{\alpha\in\mathbb{N}^n} \lambda_{|\alpha|} P_\alpha,$$

where the convergence of the series is understood in the strong operator topology. The Berezin transform of R_λ was computed in [1, 9]:

$$(\mathcal{B}(R_\lambda))(z) = (1-|z|^2)^{n+1} \sum_{m=0}^{\infty} \frac{2(m+n)!}{m!\,(n-1)!}\,\lambda_m\,|z|^{2m}. \tag{1.1}$$

Since the function $\mathcal{B}(R_\lambda)$ is radial, the operator R_λ is radial.

Theorem 1.1. *Let S be a bounded linear radial operator in $\mathcal{A}^2$. Then there exists a bounded complex sequence λ such that $S = R_\lambda$.*

Zhou, Chen, and Dong [9] proved one part of this theorem, namely, that S is diagonal with respect to the monomial basis. In Section 2 we prove the remaining part: the eigenvalues of S depend only on the length of the multi-index.

1.3 Radial Toeplitz operators on the unit ball

Zhou, Chen, and Dong [9] proved that a Toeplitz operator T_g is radial if and only if its generating symbol g is radial, i.e., if there exists a function a defined on $[0,1]$ such that $g(z) = a(|z|)$ for almost all $z \in \mathbb{B}^n$. Then T_g is diagonal with respect to the orthonormal monomial basis, and the corresponding eigenvalues depend only on the length of multi-indices. Denote the eigenvalue sequence of such operator by $\gamma_{n,a}$:

$$T_g e_\alpha = \gamma_{n,a}(|\alpha|)e_\alpha.$$

An explicit expression of the eigenvalues $\gamma_{n,a}(m)$ in terms of a was found by Grudsky, Karapetyants and Vasilevski [1] (see also [9]):

$$\gamma_{n,a}(m) = (m+n)\int_0^1 a(\sqrt{r})\,r^{m+n-1}\,dr, \tag{1.2}$$

or, changing a variable,

$$\gamma_{n,a}(k) = 2(m+n)\int_0^1 a(r)\,r^{2m+2n-1}\,dr. \tag{1.3}$$

Denote by $\Gamma_n(L^\infty([0,1]))$, or Γ_n in short, the set of all these eigenvalue sequences, which are generated by the radial Toeplitz operators with bounded generating functions:

$$\Gamma_n := \Gamma_n(L^\infty([0,1])) = \{\gamma_{n,a} :\ a \in L^\infty([0,1])\}. \tag{1.4}$$

Define $\gamma_{1,a}$ and Γ_1 by (1.3) and (1.4) with $n = 1$:

$$\gamma_{1,a}(k) = 2(k+1)\int_0^1 a(r)\,r^{2k+1}\,dr, \tag{1.5}$$

$$\Gamma_1 := \Gamma_1(L^\infty([0,1])) = \{\gamma_{1,a} : \ a \in L^\infty([0,1])\}. \tag{1.6}$$

Denote by $d_1(\mathbb{N})$ the set of all bounded sequences $x = (x_j)_{j\in\mathbb{N}}$ satisfying the condition

$$\sup_{k\in\mathbb{N}} \Big((k+1)(\Delta x)_k\Big) < +\infty,$$

where $(\Delta x)_k = x_{k+1} - x_k$.

Then the C^*-algebra generated by radial Toeplitz operators with bounded generated symbols is isometrically isomorphic to the C^*-algebra generated by Γ_n.

Theorem 1.2 (Suárez [8])**.** *The C^*-algebra generated by Γ_1 coincides with the topological closure of Γ_1 in $\ell^\infty(\mathbb{N})$, being the topological closure of $d_1(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$.*

1.4 Slowly oscillating sequences

Denote by $\mathrm{SO}(\mathbb{N})$ the set of all bounded sequences that *slowly oscillate* in the sense of Schmidt [5] (see also Landau [2] and Stanojević and Stanojević [6]):

$$\mathrm{SO}(\mathbb{N}) := \Big\{x \in \ell^\infty : \ \lim_{\frac{j+1}{k+1}\to 1} |x_j - x_k| = 0\Big\}.$$

In other words, $\mathrm{SO}(\mathbb{N})$ consists of all bounded functions $\mathbb{N} \to \mathbb{C}$ that are uniformly continuous with respect to the “logarithmic metric” $\rho(j,k) := |\ln(j+1) - \ln(k+1)|$. In Section 3 we give some properties and equivalent definitions of the C^*-algebra $\mathrm{SO}(\mathbb{N})$.

In Section 4 we prove that the C^*-algebra generated by Γ_n does not actually depend on n. Applying Theorem 1.2 and some standard approximation techniques (de la Vallée-Poussin means) we obtain the main result of the paper.

Theorem 1.3. *For each n the C^*-algebra generated by Γ_n coincides with the topological closure of Γ_n in ℓ^∞ and is equal to* $\mathrm{SO}(\mathbb{N})$.

As shown by Grudsky, Karapetyants and Vasilevski [1], if $a \in L^1([0,1], r^{2n-1}dr)$ and the sequence $\gamma_{n,a}$ is bounded, then $\gamma_{n,a}(m+1) - \gamma_{n,a}(m) \to 0$. At the same time, in this situation $\gamma_{n,a}$ does not necessarily belong to $\mathrm{SO}(\mathbb{N})$. The next result is proved in Section 5.

Theorem 1.4. *There exists a function $a \in L^1([0,1], r\,dr)$ such that $\gamma_{n,a} \in \ell^\infty(\mathbb{N}) \setminus \mathrm{SO}(\mathbb{N})$.*

That is, a bounded Toeplitz operator having *unbounded* defining symbol does not necessarily belong to the C^*-algebra generated by Toeplitz operators with *bounded* defining symbols.

2 Diagonalization of Radial Operators in the Monomial Basis

Lemma 2.1 (Zhou, Chen, and Dong [9])**.** *Let $S : \mathcal{A}^2 \to \mathcal{A}^2$ be a bounded radial operator and α be a multi-index. Then e_α is an eigenfunction of S, i.e., $\langle S e_\alpha, e_\beta\rangle = 0$ for every multi-index β different from α.*

Proof. For a reader convenience we give here a proof, slightly different from [9]. Choose an index $j \in \{1,\dots,n\}$ such that $\alpha_j \neq \beta_j$ and a complex number t such that $|t| = 1$ and $t^{\alpha_j} \neq t^{\beta_j}$. For example, put

$$t = e^{i\phi} \qquad \text{where} \qquad \phi = \frac{\pi}{|\alpha_j - \beta_j|}.$$

Denote by U the diagonal matrix with (j, j)st entry equal to t^{-1} and all other diagonal entries equal to 1:

$$U = \operatorname{diag}(1,\dots,1, \underbrace{t^{-1}}_{j\text{st position}}, 1,\dots,1).$$

Then U is a unitary matrix, $\Psi_U e_\alpha = t^{\alpha_j} e_\alpha$, and

$$t^{\alpha_j}\langle S e_\alpha, e_\beta\rangle = \langle S\Psi_U e_\alpha, e_\beta\rangle = \langle \Psi_U S e_\alpha, e_\beta\rangle = \langle S e_\alpha, \Psi_{U^*} e_\beta\rangle = t^{\beta_j}\langle S e_\alpha, e_\beta\rangle.$$

Since $t^{\alpha_j} \neq t^{\beta_j}$, it follows that $\langle S e_\alpha, e_\beta\rangle = 0$. □

Lemma 2.2 (Berezin transform of basic projections)**.** *Let $\alpha \in \mathbb{N}^n$ and $z \in B$. Then*

$$\mathcal{B}(P_\alpha)(z) = (1 - |z|^2)^{n+1} q_\alpha(z),$$

where $q_\alpha \colon B \to \mathbb{C}$ is the square of the absolute value of e_α:

$$q_\alpha(z) = |e_\alpha(z)|^2 = \frac{(n+|\alpha|)!}{n!\,\alpha!}\,|z^\alpha|^2.$$

Proof. We calculate $P_\alpha K_z$ for an arbitrary $z \in B$:

$$P_\alpha K_z = P_\alpha\left(\sum_{\beta\in\mathbb{N}^n} \overline{e_\beta(z)}\, e_\beta\right) = \overline{e_\alpha(z)}\, e_\alpha.$$

The reproducing property of K_z implies that $\langle e_\alpha, K_z\rangle = e_\alpha(z)$. Therefore

$$\mathcal{B}(P_\alpha)(z) = \frac{1}{K_z(z)}\langle P_\alpha K_z, K_z\rangle = (1-|z|^2)^{n+1}\langle \overline{e_\alpha(z)}\, e_\alpha, K_z\rangle = (1-|z|^2)^{n+1}|e_\alpha(z)|^2. \qquad \square$$

Lemma 2.3. *For each $m \in \mathbb{N}$, the function $z \mapsto |z|^{2m}$ is $\frac{n}{m+n}$ times the arithmetic mean of the functions q_α with $|\alpha| = m$:*

$$|z|^{2m} = \frac{m!\,n!}{(m+n)!}\sum_{|\alpha|=m} q_\alpha(z) = \frac{n}{m+n}\,\frac{m!\,(n-1)!}{(m+n-1)!}\sum_{|\alpha|=m} q_\alpha(z).$$

Proof. Apply the multinomial theorem and the definition of q_α:

$$|z|^{2m} = \left(\sum_{j=1}^{n} |z_j|^2\right)^m = \sum_{|\alpha|=m}\frac{m!}{\alpha!}\prod_{j=1}^{n}|z_j|^{2\alpha_j} = \sum_{|\alpha|=m}\frac{m!}{\alpha!}|z^\alpha|^2 = \frac{m!\,n!}{(m+n)!}\sum_{|\alpha|=m} q_\alpha(z). \qquad \square$$

Lemma 2.4. *Let $\alpha \in \mathbb{N}^n$. Then for all $z \in B$,*

$$\operatorname{rad}(q_\alpha)(z) = \frac{n+|\alpha|}{n}\,|z|^{2|\alpha|}.$$

Proof. Express the integration over $\mathcal{U}_n$ through the integration over $\mathbb{S}^{2n-1}$:

$$\operatorname{rad}(q_\alpha)(z) = \int_{\mathcal{U}_n} \frac{n+|\alpha|!}{n!\,\alpha!} |(Uz)^\alpha|^2 \, dH(U) = \frac{n+|\alpha|!}{n!\,\alpha!} |z|^{2|\alpha|} \int_{\mathbb{S}^{2n-1}} |\zeta^\alpha|^2 \, d\sigma(\zeta).$$

The value of the latter integral is well known (e.g., see [3, Proposition 1.4.9]):

$$\int_{\mathbb{S}^{2n-1}} |\zeta^\alpha|^2 \, d\sigma(\zeta) = \frac{(n-1)!\,\alpha!}{(n-1+|\alpha|)!}. \qquad \square$$

Lemma 2.5 (radialization of basic projections)**.** *Let $\alpha \in \mathbb{N}^n$. Then the radialization of P_α is the arithmetic mean of all P_β with $|\beta| = |\alpha|$:*

$$\operatorname{Rad}(P_\alpha) = \frac{(n-1)!\,|\alpha|!}{(n-1+|\alpha|)!} \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=|\alpha|}} P_\beta. \tag{2.1}$$

Proof. We shall prove that both sides of (2.1) have the same Berezin transform, then (2.1) will follow from the injectivity of the Berezin transform. We use the fact the Berezin transform "commutes with the radialization" [9], and apply then Lemmas 2.2 and 2.4:

$$\mathcal{B}(\operatorname{Rad}(P_\alpha))(z) = \operatorname{rad}(\mathcal{B}(P_\alpha))(z) = (1-|z|^2)^{n+1} \operatorname{rad}(q_\alpha)(z) = \frac{n+|\alpha|}{n} |z|^{2|\alpha|}(1-|z|^2)^{n+1}.$$

On the other hand, by Lemmas 2.4 and 2.3,

$$\frac{(n-1)!\,|\alpha|!}{(n-1+|\alpha|)!} \sum_{|\beta|=|\alpha|} \mathcal{B}(P_\beta)(z) = (1-|z|^2)^{n+1} \frac{(n-1)!\,|\alpha|!}{(n-1+|\alpha|)!} \sum_{|\beta|=|\alpha|} q_\beta(z) = \frac{n+|\alpha|}{n} |z|^{2|\alpha|} (1-|z|^2)^{n+1}.$$

$\square$

Lemma 2.6 (radialization of a diagonal operator)**.** *Let $(c_\alpha)_{\alpha\in\mathbb{N}^n}$ be a bounded family of complex numbers. Consider the operator $S : \mathcal{A}^2 \to \mathcal{A}^2$ given by*

$$S = \sum_{\alpha\in\mathbb{N}^n} c_\alpha P_\alpha.$$

Then

$$\operatorname{Rad}(S) = \sum_{m=0}^{\infty} \left(\frac{m!\,(n-1)!}{(m+n-1)!} \sum_{|\beta|=m} c_\beta \right) \left(\sum_{|\alpha|=m} P_\alpha \right).$$

Proof. Follows from Lemma 2.5 and the fact that the sum of a converging serie of mutually orthogonal vectors does not depend on the order of summands. $\square$

Proof of Theorem 1.1. Let S be a bounded linear radial operator in $\mathcal{A}^2$. By Lemma 2.1,

$$S = \sum_{\alpha\in\mathbb{N}^n} c_\alpha P_\alpha.$$

Since $\operatorname{Rad}(S) = S$, it follows from Lemma 2.6 that the coefficients c_α depend only on $|\alpha|$. Defining λ_m equal to c_α for some α with $|\alpha| = m$, we obtain

$$S = \sum_{m=0}^{\infty} \lambda_m \left(\sum_{|\alpha|=m} P_\alpha \right) = R_\lambda. \qquad \square$$

3 Slowly Oscillating Sequences

Definition 3.1 (logarithmic metric on $\mathbb{N}$)**.** Define $\rho\colon \mathbb{N}\times\mathbb{N}\to[0,+\infty)$ by

$$\rho(j,k) := \big|\ln(j+1)-\ln(k+1)\big|.$$

The function ρ is a metric on $\mathbb{N}$ because it is obtained from the usual metric $d\colon \mathbb{R}\times\mathbb{R}\to[0,+\infty)$, $d(t,u):=|t-u|$, via the injective function $\mathbb{N}\to\mathbb{R}$, $j\mapsto\ln(j+1)$.

Definition 3.2 (modulus of continuity of a sequence with respect to the logarithmic metric)**.** Given a complex sequence $x=(x_j)_{j\in\mathbb{N}}$, define $\omega_{\rho,x}\colon[0,+\infty)\to[0,+\infty]$ by

$$\omega_{\rho,x}(\delta) := \sup\{|x_j-x_k|:\ \ j,k\in\mathbb{N},\ \rho(j,k)\le\delta\}.$$

Definition 3.3 (slowly oscillating sequences)**.** Denote by $\mathrm{SO}(\mathbb{N})$ the set of the bounded sequences that are uniformly continuous with respect to the logarithmic metric:

$$\mathrm{SO}(\mathbb{N}) = \{\lambda\in\ell^\infty(\mathbb{N}):\ \lim_{\delta\to0^+}\omega_{\rho,\lambda}(\delta)=0\}.$$

Note that the class $\mathrm{SO}(\mathbb{N})$ plays an important role in Tauberian theory, see Landau [2], Schmidt [5, § 9], Stanojević and Stanojević [6].

For every sequence x the function $\omega_{\rho,x}\colon[0,+\infty)\to[0,+\infty]$ is increasing (in the non-strict sense). Therefore the condition $\lim_{\delta\to0^+}\omega_{\rho,x}(\delta)=0$ is equivalent to the following one: for all $\varepsilon>0$ there exists a $\delta>0$ such that $\omega_{\rho,x}(\delta)<\varepsilon$.

The same class $\mathrm{SO}(\mathbb{N})$ can be defined using another special metric ρ_1 on $\mathbb{N}$:

Definition 3.4. Define $\rho_1\colon\mathbb{N}\times\mathbb{N}\to[0,+\infty)$ by

$$\rho_1(j,k) = \frac{|j-k|}{\max(j+1,k+1)} = 1-\frac{\min(j+1,k+1)}{\max(j+1,k+1)}.$$

Proposition 3.5. *ρ_1 is a metric on $\mathbb{N}$.*

Proof. Clearly ρ_1 is non-negative, symmetric, and $\rho_1(j,k)=0$ only if $j=k$. We have to prove that for all $j,k,p\in\mathbb{N}$

$$\rho_1(j,p)+\rho_1(p,k)-\rho_1(j,k)\ge0. \tag{3.1}$$

Denote the left-hand side of (3.1) by $\Lambda(j,k,p)$. Since $\Lambda(j,k,p)$ is symmetric with respect to j and k, assume without loss of generality that $j\le k$. If $j\le p\le k$, then

$$\Lambda(j,k,p)=\left(1-\frac{j+1}{p+1}\right)+\left(1-\frac{p+1}{k+1}\right)-\left(1-\frac{j+1}{k+1}\right)=\frac{p-j}{p+1}-\frac{p-j}{k+1}=\frac{(p-j)(k-p)}{(k+1)(p+1)}\ge0.$$

If $j\le k<p$, then $\Lambda(j,k,p)=\frac{(p-k)(j+k+2)}{(k+1)(p+1)}\ge0$. If $p<j\le k$, then $\Lambda(j,k,p)=\frac{(j-p)(j+k+2)}{(j+1)(k+1)}\ge0$. ◻

Proposition 3.6 (relations between ρ and ρ_1)**.**

1. *For all $j,k\in\mathbb{N}$,*

$$\rho_1(j,k)\le\rho(j,k). \tag{3.2}$$

2. *For all* $j,k \in \mathbb{N}$ *satisfying* $\rho_1(j,k) \le \frac{1}{2}$,

$$\rho(j,k) \le 2\ln(2)\rho_1(j,k). \tag{3.3}$$

Proof. Since the functions ρ and ρ_1 are symmetric and vanish on the diagonal ($\rho(j,j) = \rho_1(j,j) = 0$), consider only the case $j < k$. Denote $\frac{k+1}{j+1} - 1$ by t, then

$$\rho(j,k) = \ln(1+t), \qquad \rho_1(j,k) = 1 - \frac{1}{1+t} = \frac{t}{1+t}.$$

Define $f\colon (0,+\infty) \to (0,+\infty)$ by

$$f(t) := \frac{\ln(1+t)}{1 - \frac{1}{1+t}}.$$

Then

$$f'(t) = \frac{t - \ln(1+t)}{t^2} > 0,$$

and thus f is strictly increasing on $(0,+\infty)$. Since $\lim_{t\to 0^+} f(t) = 1$ and $f(1) = 2\ln(2)$, we see that $f(t) > 1$ for all $t > 0$ and $f(t) \le 2\ln(2)$ for all $t \in (0,1]$. Substituting t by $\frac{k+1}{j+1} - 1$ we obtain (3.2) and (3.3). □

Corollary 3.7. *The set* $\mathrm{SO}(\mathbb{N})$ *can be defined using the metric* ρ_1 *instead of* ρ*:*

$$\mathrm{SO}(\mathbb{N}) = \left\{\lambda \in \ell^\infty(\mathbb{N}) \colon \quad \lim_{\delta\to 0^+} \sup_{\rho_1(j,k)\le\delta} |\lambda_j - \lambda_k| = 0\right\}.$$

Let us mention some simple properties of $\mathrm{SO}(\mathbb{N})$.

Proposition 3.8. $\mathrm{SO}(\mathbb{N})$ *is a closed subalgebra of the* C^**-algebra* $\ell^\infty(\mathbb{N})$.

Proof. It is a general fact that the set of the uniformly continuous functions on some metric space M is a closed subalgebra of the C^*-algebra of the bounded continuous functions on M. In our case $M = (\mathbb{N},\rho)$. Since

$$\omega_{\rho,f+g} \le \omega_{\rho,f} + \omega_{\rho,g}, \quad \omega_{\rho,\lambda f} = |\lambda|\omega_{\rho,f}, \quad \omega_{\rho,fg} \le \omega_{\rho,f}\|g\|_\infty + \omega_{\rho,g}\|f\|_\infty, \quad \omega_{\rho,\overline{f}} = \omega_{\rho,f},$$

the set $\mathrm{SO}(\mathbb{N})$ is closed with respect to the algebraic operations. The topological closeness of $\mathrm{SO}(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$ follows from the inequality

$$\omega_{\rho,f}(\delta) \le 2\|f-g\|_\infty + \omega_{\rho,g}(\delta). \qquad \square$$

Proposition 3.9 (comparison of $\mathrm{SO}(\mathbb{N})$ to $c(\mathbb{N})$)**.** *The set of the converging sequences* $c(\mathbb{N})$ *is a proper subset of* $\mathrm{SO}(\mathbb{N})$.

Proof. 1. Denote by $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ the one-point compactification (Alexandroff compactification) of $\mathbb{N}$. The topology on $\overline{\mathbb{N}}$ can be induced by the metric

$$d_{\overline{\mathbb{N}}}(j,k) := \left|\frac{j}{j+1} - \frac{k}{k+1}\right|.$$

If $\sigma \in c(\overline{\mathbb{N}})$, then σ is uniformly continuous with respect to the metric $d_{\overline{\mathbb{N}}}$, but $d_{\overline{\mathbb{N}}}$ is less or equal than ρ:

$$d_{\overline{\mathbb{N}}}(j,k) = \frac{|j-k|}{(j+1)(k+1)} \le \frac{|j-k|}{\max(j+1,k+1)} = \rho_1(j,k) \le \rho(j,k).$$

2. The sequence $x = (x_j)_{j\in\mathbb{N}}$ with $x_j = \cos(\ln(j+1))$ does not converge but belongs to $\mathrm{SO}(\mathbb{N})$ since

$$\left|x_j - x_k\right| = \left|\cos(\ln(j+1)) - \cos(\ln(k+1))\right| \le \left|\ln(j+1) - \ln(k+1)\right| = \rho(j,k). \qquad \square$$

We define now the *left and right shifts* of a sequence. Given a complex sequence $x = (x_j)_{j\in\mathbb{N}}$, define the sequences $\tau_L(x)$ and $\tau_R(x)$ as follows:

$$\tau_L(x) := (x_1, x_2, x_3, \ldots), \qquad \tau_R(x) := (0, x_0, x_1, \ldots).$$

More formally,

$$\tau_L(x)_j := x_{j+1}; \qquad \tau_R(x)_j := \begin{cases} 0, & j = 0; \\ x_{j-1}, & j \in \{1,2,3,\ldots\}. \end{cases}$$

Note that $\tau_L(\tau_R(x)) = x$ for every sequence x.

Both τ_L and τ_R are bounded linear operators on $\ell^\infty(\mathbb{N})$. In the following two propositions we show that $\mathrm{SO}(\mathbb{N})$ is an invariant subspace of each one of these operators.

Proposition 3.10. *For every* $x \in \mathrm{SO}(\mathbb{N})$, $\tau_L(x) \in \mathrm{SO}(\mathbb{N})$.

Proof. The image of $\tau_L(x)$ is a subset of the image of x, therefore $\|\tau_L(x)\| \le \|x\|$. If $\delta > 0$, $j,k \in \mathbb{N}$, $j < k$ and $\rho(j,k) \le \delta$, then

$$\rho(j+1,k+1) = \ln\frac{k+2}{j+2} = \ln\frac{k+1}{j+1} + \ln\left(1 + \frac{1}{k+1}\right) - \ln\left(1 + \frac{1}{j+1}\right) < \ln\frac{k+1}{j+1} = \rho(j,k) \le \delta.$$

It follows that $\omega_{\rho,\tau_L(x)}(\delta) \le \omega_{\rho,x}(\delta)$ and $\lim\limits_{\delta\to 0^+} \omega_{\rho,\tau_L(x)}(\delta) = 0$. $\square$

Proposition 3.11. *For every* $x \in \mathrm{SO}(\mathbb{N})$, $\tau_R(x) \in \mathrm{SO}(\mathbb{N})$.

Proof. The sequences x and $\tau_R(x)$ have the same image up to one element zero:

$$\{\tau_R(x)_j : \ j \in \mathbb{N}\} = \{x_j : \ j \in \mathbb{N}\} \cup \{0\}.$$

Therefore $\|\tau_R(x)\|_\infty = \|x\|_\infty$.

2. Let $\delta \in \left(0, \frac{1}{3}\right)$, $j,k \in \mathbb{N}$, $j < k$ and $\rho(j,k) \le \delta$. Then $j \ge 1$, $k \ge 2$, and

$$\rho_1(j-1,k-1) = \frac{k-j}{k} = \frac{k+1}{k} \cdot \frac{(k+1)-(j+1)}{k+1} \le \frac{3}{2}\rho_1(j,k).$$

Applying Proposition 3.6 we see that

$$\rho_1(j-1,k-1) \le \frac{3}{2}\rho(j,k) = \frac{3}{2}\delta \le \frac{1}{2}$$

and

$$\rho(j-1,k-1) \le 2\ln(2)\rho_1(j-1,k-1) \le 2\ln(2)\,\frac{3}{2}\,\delta = 3\ln(2)\delta.$$

Thus for every $\delta \in \left(0,\frac{1}{3}\right)$,

$$\omega_{\rho,\tau_R(x)}(\delta) \le \omega_{\rho,x}(3\ln(2)\delta).$$

Therefore $\lim\limits_{\delta\to 0^+} \omega_{\rho,\tau_R(x)}(\delta) = 0.$ □

4 Γ_n is a Dense Subset of SO($\mathbb{N}$)

First we prove that Γ_n is contained in $\mathrm{SO}(\mathbb{N})$.

Proposition 4.1. *Let $a \in L^\infty([0,1])$. Then $\gamma_{n,a} \in \mathrm{SO}(\mathbb{N})$. More precisely,*

$$\|\gamma_{n,a}\|_\infty \le \|a\|_\infty, \tag{4.1}$$

and for all $j,k\in\mathbb{N}$,

$$\left|\gamma_{n,a}(j) - \gamma_{n,a}(k)\right| \le 2\|a\|_\infty \rho(j,k). \tag{4.2}$$

Proof. The inequality (4.1) follows directly from (1.3):

$$|\gamma_{n,a}(j)| \le 2(n+j)\int_0^1 r^{2n+2j-1}\|a\|_\infty\,dr = \|a\|_\infty.$$

The proof of (4.2) is based on an idea communicated to us by K. M. Esmeral García. Since both sides of (4.2) are symmetric with respect to the indices j and k, without loss of generality we consider the case $j<k$. First factorize $a(r)$ and bound it by $\|a\|_\infty$:

$$\left|\gamma_{n,a}(j) - \gamma_{n,a}(k)\right| = \left|\int_0^1 ((n+j)r^{2n+2j-1} - (n+k)r^{2n+2k-1})a(r)\,dr\right| \tag{4.3}$$

$$\le \|a\|_\infty \int_0^1 \left|(n+j)r^{2n+2j-1} - (n+k)r^{2n+2k-1}\right| dr. \tag{4.4}$$

Denote by r_0 the unique solution of the equation $(n+j)r^{2n+2j-1} - (n+k)r^{2n+2k-1} = 0$ on the interval $(0,1)$:

$$r_0 = \left(\frac{n+j}{n+k}\right)^{\frac{1}{2(k-j)}}.$$

The function $r \mapsto (n+j)r^{2n+2j-1} - (n+k)r^{2n+2k-1}$ takes positive values on the interval $(0,r_0)$ and negative values on the interval $(r_0,1)$. Dividing the integral (4.4) on two parts by the point r_0, we obtain:

$$\left|\gamma_{n,a}(j) - \gamma_{n,a}(k)\right| \le 2\|a\|_\infty (r_0^{2n+2j} - r_0^{2n+2k}) = 2\|a\|_\infty r_0^{2n+2j}\rho_1(j,k).$$

Since $r_0 < 1$ and $\rho_1(j,k) \le \rho(j,k)$, the inequality (4.2) follows. □

Definition 4.2. Denote by $d_1(\mathbb{N})$ the set of the bounded sequences x such that

$$\sup_{j\in\mathbb{N}}\Big((j+1)|x_{j+1} - x_j|\Big) < +\infty.$$

Proposition 4.3. *$d_1(\mathbb{N})$ is a proper subset of* $\mathrm{SO}(\mathbb{N})$.

Proof. 1. Let $x \in d_1(\mathbb{N})$ and

$$M = \sup_{j\in\mathbb{N}}\Big((j+1)|x_{j+1}-x_j|\Big).$$

Then for all $j,k \in \mathbb{N}$ with $j<k$ we have

$$|x_k - x_j| \le \sum_{q=j}^{k-1} |x_{q+1}-x_q| \le M\sum_{q=j}^{k-1}\frac{1}{q+1} \le M\sum_{q=j}^{k-1}\ln\frac{q+1}{q} = M\ln\frac{k}{j} < M\ln\frac{k+1}{j+1} = M\rho(j,k).$$

Therefore $d_1(\mathbb{N})$ is contained in $\mathrm{SO}(\mathbb{N})$.

2. Consider the sequence

$$x_j := \sin\frac{\pi\lfloor\log_2(j+2)\rfloor}{\sqrt{\log_2(j+2)}}.$$

For every j and k with $k>j$,

$$\begin{aligned}
\left|x_k - x_j\right| &\le \frac{\pi\lfloor\log_2(k+2)\rfloor}{\sqrt{\log_2(k+2)}} - \frac{\pi\lfloor\log_2(j+2)\rfloor}{\sqrt{\log_2(j+2)}}\\
&\le \frac{\pi\log_2(k+2)}{\sqrt{\log_2(k+2)}} - \frac{\pi(\log_2(j+2)-1)}{\sqrt{\log_2(j+2)}}\\
&= \pi\Big(\sqrt{\log_2(k+2)} - \sqrt{\log_2(j+2)}\Big) + \frac{\pi}{\sqrt{\log_2(j+2)}}\\
&= \frac{\pi\log_2\frac{k+2}{j+2}}{\sqrt{\log_2(k+2)}+\sqrt{\log_2(j+2)}} + \frac{\pi}{\sqrt{\log_2(j+2)}}.
\end{aligned}$$

Thus $x \in \mathrm{SO}(\mathbb{N})$. On the other hand, if $j = 2^{k^2}-3$, then

$$|x_{j+1}-x_j| = |x_j| = \left|\sin\left(\frac{\pi(k^2-1)}{\sqrt{\log_2(2^{k^2}-1)}}\right)\right| = \left|\sin\left(k\pi - \frac{\pi(k^2-1)}{\sqrt{\log_2(2^{k^2}-1)}}\right)\right|.$$

Appying the inequality $|\sin(t)| \ge \frac{2|t|}{\pi}$, which holds for all t with $|t| \le \frac{\pi}{2}$, we obtain:

$$|x_{j+1}-x_j| \ge 2\left(k - \frac{(k^2-1)}{\sqrt{\log_2(2^{k^2}-1)}}\right) \ge 2\Big(k-\sqrt{k^2-1}\Big) \ge \frac{1}{k} = \frac{1}{\sqrt{\log_2(j+3)}}.$$

Therefore $x \notin d_1(\mathbb{N})$. □

Lemma 4.4. *Let $x \in \ell^\infty(\mathbb{N})$ and $\delta \in (0,1)$. Denote by y the sequence of the de la Vallée-Poussin means of x:*

$$y_j = \frac{1}{1+\lfloor j\delta\rfloor}\sum_{k=j}^{j+\lfloor j\delta\rfloor} x_k. \tag{4.5}$$

Then $y \in d_1(\mathbb{N})$ and

$$\|y-x\|_\infty \le \omega_{\rho,x}(\delta). \tag{4.6}$$

Proof. Note that for all $j \in \mathbb{N}$, the sum in the right-hand side of (4.5) contains $1+\lfloor j\delta \rfloor$ terms. Therefore

$$|y_j| \le \frac{1}{1+\lfloor j\delta\rfloor} \sum_{k=j}^{j+\lfloor j\delta\rfloor} \|x\|_\infty = \|x\|_\infty.$$

For $j \in \mathbb{N}$, let us estimate the difference $|y_{j+1}-y_j|$:

$$\begin{aligned}
|y_{j+1}-y_j| &= \left| \frac{1}{1+\lfloor (j+1)\delta\rfloor} \sum_{k=j}^{j+\lfloor (j+1)\delta\rfloor} x_k - \frac{1}{1+\lfloor j\delta\rfloor} \sum_{k=j}^{j+\lfloor j\delta\rfloor} x_k \right| \\
&\le \frac{\lfloor (j+1)\delta\rfloor - \lfloor j\delta\rfloor}{(1+\lfloor (j+1)\delta)(1+\lfloor j\delta\rfloor)} \sum_{k=j}^{j+\lfloor (j+1)\delta\rfloor} |x_k| + \frac{1}{1+\lfloor (j+1)\delta\rfloor} |x_{j+\lfloor (j+1)\delta\rfloor}| \\
&\le \frac{\|x\|_\infty(\lfloor j\delta\rfloor+1)}{(j+1)\delta(1+\lfloor j\delta\rfloor)} + \frac{\|x\|_\infty}{(j+1)\delta} \\
&= \frac{\|x\|_\infty}{(j+1)\delta}.
\end{aligned}$$

Thus $y \in d_1(\mathbb{N})$. Let us prove (4.6). If $j \le k \le j+\lfloor j\delta\rfloor$, then

$$\rho(j,k) = \ln\frac{k+1}{j+1} \le \ln\frac{k}{j} \le \ln(1+\delta) \le \delta.$$

Therefore

$$|y_j - x_j| \le \frac{1}{1+\lfloor j\delta\rfloor} \sum_{k=j}^{j+\lfloor j\delta\rfloor} |x_k - x_j| \le \omega_{\rho,x}(\delta). \qquad \square$$

Proposition 4.5. *$d_1(\mathbb{N})$ is a dense subset of* SO($\mathbb{N}$).

Proof. Let $\varepsilon > 0$. Using the fact that $\omega_{\rho,x}(\delta) \to 0$ as $\delta \to 0$, choose a $\delta > 0$ such that $\omega_{\rho,x}(\delta) < \varepsilon$. Define y by (4.5). Then $y \in d_1(\mathbb{N})$ and $\|x-y\|_\infty < \varepsilon$ by Lemma 4.4. $\square$

Theorem 1.3 follows from Proposition 4.5 and Theorem 1.2:

Proposition 4.6. *Γ_1 is a dense subset of* SO($\mathbb{N}$).

Proof. Proposition 4.1 implies that Γ_1 is contained in SO($\mathbb{N}$). Let $x \in$ SO($\mathbb{N}$) and $\varepsilon > 0$. Applying Proposition 4.5 find a sequence $y \in d_1(\mathbb{N})$ such that

$$\|y-x\|_\infty < \frac{\varepsilon}{2}.$$

Using Theorem 1.2 we find a function $a \in L^\infty([0,1])$ such that $\|\gamma_{1,a} - y\|_\infty < \frac{\varepsilon}{2}$. Then

$$\|\gamma_{1,a} - x\|_\infty \le \|\gamma_{1,a} - y\|_\infty + \|y-x\|_\infty < \varepsilon. \qquad \square$$

Lemma 4.7. *Let $a \in L^\infty([0,1])$. Then $\gamma_{n,a} = \tau_L^{n-1}(\gamma_{1,a})$.*

Proof. Follows directly from the definitions of $\gamma_{n,a}$ and $\gamma_{1,a}$, see (1.3) and (1.5). $\square$

Proposition 4.8. *Γ_n is a dense subset of* SO($\mathbb{N}$).

Proof. By Proposition 4.1, Γ_n is a subset of $\mathrm{SO}(\mathbb{N})$.

Let $x \in \mathrm{SO}(\mathbb{N})$ and $\varepsilon > 0$. Denote $\tau_R^{n-1}(x)$ by y. By Proposition 3.11, $y \in \mathrm{SO}(\mathbb{N})$. Using Proposition 4.6 find a function $a \in L^\infty([0,1])$ such that $\|y-\gamma_{1,a}\|_\infty < \varepsilon$. Then apply Lemma 4.7:

$$\|x-\gamma_{n,a}\|_\infty = \|\tau_L^{n-1}(y)-\tau_L^{n-1}(\gamma_{1,a})\|_\infty = \|\tau_L^{n-1}(y-\gamma_{1,a})\|_\infty \le \|y-\gamma_{1,a}\|_\infty < \varepsilon. \qquad \square$$

We finish this section with an important observation. The results stated up to this moment do not take into account the multiplicities of the eigenvalues. In this connection we recall that for each bounded radial operator R_λ on $\mathcal{A}^2(\mathbb{B}^n)$ with the eigenvalue sequence $\lambda \in \ell^\infty(\mathbb{N})$, the equality

$$R_\lambda e_\alpha = \lambda_p e_\alpha$$

holds for all multi-indices $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| = p$, and there are $\binom{n+p-1}{n-1}$ such multi-indices.

As was mentioned, for each natural number n the C^*-algebra generated by Toeplitz operators on $\mathcal{A}^2(\mathbb{B}^n)$ with bounded radial symbols is isomorphic and isometric to the C^*-algebra of multiplication operators R_λ on $\ell^2(\mathbb{N})$ whose eigenvalue sequences belong to $\mathrm{SO}(\mathbb{N})$, and thus its C^* structure does not depend on n. At the same time these algebras, when n is varied, are quite different if we count multiplicities of eigenvalues, that is when the operators forming the algebra are considered by their action on the basis elements of the corresponding Hilbert space $\mathcal{A}^2(\mathbb{B}^n)$.

Let us consider in more detail sequences of eigenvalues with multiplicities. Formula for the rising sum of binomial coefficients states that

$$\sum_{m=0}^{p}\binom{n+m-1}{n-1} = \binom{n+p-1}{n}.$$

Now, for every $j \in \mathbb{N}$ there exists a unique p in $\mathbb{N}$ such that

$$\binom{n+p-1}{n} \le j < \binom{n+p}{n}.$$

Denote this p by $\pi_n(j)$, and say that the index j is located on the p-st "level".

Given a sequence $\lambda \in \ell^\infty$, define $\Phi_n(\lambda)$ as the sequence obtained from λ by repeating each λ_p according to its multiplicity. That is,

$$\Phi_n(\lambda) := \Big(\overbrace{\underbrace{\lambda_0}_{\binom{n-1}{n-1}\text{ times}}, \underbrace{\lambda_1}_{\binom{n}{n-1}\text{ times}}, \underbrace{\lambda_2}_{\binom{n+1}{n-1}\text{ times}}, \underbrace{\lambda_3}_{\binom{n+2}{n-1}\text{ times}}, \dots,}^{\binom{n+p-1}{n}\text{ elements}} \underbrace{\lambda_p}_{\binom{n+p-1}{n-1}\text{ times}}, \dots\Big).$$

Since the isometric homomorphism Φ_n of $\ell^\infty(\mathbb{N})$ is injective, the C^*-algebra generated by the set $\{\Phi_n(\gamma_{n,a})\colon\ a \in L^\infty[0,1]\}$ coincides with $\Phi_n(\mathrm{SO}(\mathbb{N}))$, that is, with the C^*-algebra obtained from $\mathrm{SO}(\mathbb{N})$ by applying the mapping Φ_n.

Note that for all p, q with $p < q$ the following estimates hold:

$$\ln\frac{q+1}{p+1} \le \ln\binom{n+q}{n} - \ln\binom{n+p}{n} \le n\ln\frac{q+1}{p+1},$$

which implies that $\Phi_n(\mathrm{SO}(\mathbb{N}))$ coincides with the C^*-algebra $\mathrm{SO}_{rep,n}(\mathbb{N})$, a subalgebra $\mathrm{SO}(\mathbb{N})$, which consists of all sequences having the same elements on each "level":

$$\mathrm{SO}_{rep,n}(\mathbb{N}) := \Big\{\mu \in \mathrm{SO}(\mathbb{N})\colon \quad \text{if} \quad \pi_n(j) = \pi_n(k), \quad \text{then} \quad \mu_j = \mu_k\Big\}.$$

That is, the described above eigenvalue repetitions do not change in essence a slowly oscillating behavior of sequences.

5 Example

In this section we construct a *bounded* sequence $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ such that $\lambda = \gamma_{n,a}$ for a certain function $a \in L^1([0,1], r\,dr)$ but $\lambda \notin \mathrm{SO}(\mathbb{N})$. This implies that the corresponding radial Toeplitz operator is bounded, but it does not belong to the C^*-algebra generated by radial Toeplitz operators with bounded symbols.

Proposition 5.1. *Define* $f\colon \{z \in \mathbb{C}\colon\ \Re(z) \ge 0\} \to \mathbb{C}$ *by*

$$f(z) := \frac{1}{z+n} \exp\Big(\frac{i}{3\pi} \ln^2(z+n)\Big), \tag{5.1}$$

where $\ln$ *is the principal value of the natural logarithm (with imaginary part in* $(-\pi,\pi]$*). Then there exists a unique function* $A \in L^1(\mathbb{R}_+, \mathrm{e}^{-u}\,du)$ *such that* f *is the Laplace transform of* A*:*

$$f(z) = \int_0^{+\infty} A(u)\mathrm{e}^{-zu}\,dz.$$

Proof. For every $z \in \mathbb{C}$ with $\Re(z) \ge 0$ we can write $\ln(z+n)$ as $\ln|z+n| + i\arg(z+n)$ with $-\frac{\pi}{2} < \arg(z+n) < \frac{\pi}{2}$. Then

$$\begin{aligned} |f(z)| &= \frac{1}{|z+n|}\left|\exp\Big(\frac{i}{3\pi}(\ln|z+n| + i\arg(z+n))^2\Big)\right| \\ &= \frac{1}{|z+n|}\exp\Big(-\frac{2\arg(z+n)}{3\pi}\ln|z+n|\Big) \\ &= \frac{1}{|z+n|^{1+\frac{2\arg(z+n)}{3\pi}}}. \end{aligned}$$

Since $|z+n| \ge 1$ and $-\frac{1}{3} < -\frac{2\arg(z+n)}{3\pi} < \frac{1}{3}$,

$$|f(z)| \le \frac{1}{|z+n|^{2/3}}.$$

Therefore for every $x > 0$,

$$\int_{\mathbb{R}} |f(x+iy)|^2\,dy \le \int_{\mathbb{R}} \frac{dy}{((x+n)^2+y^2)^{2/3}} < \int_{\mathbb{R}} \frac{dy}{(1+y^2)^{2/3}} < +\infty,$$

and f belongs to the Hardy class H^2 on the half-plane $\{z \in \mathbb{C}\colon\ \Re(z) > 0\}$. By Paley–Wiener theorem (see, for example, Rudin [4, Theorem 19.2]), there exists a function $A \in L^2(0,+\infty)$ such that for all $x > 0$

$$f(x) = \int_0^{+\infty} A(u)\mathrm{e}^{-ux}\,du.$$

The uniqueness of A follows from the injective property of the Laplace transform. Applying Hölder's inequality we easily see that $A \in L^1(\mathbb{R}_+, \mathrm{e}^{-u}\,du)$:

$$\int_0^{+\infty} |A(u)|\,\mathrm{e}^{-u}\,du \le \|A\|_2 \left(\int_0^{+\infty} \mathrm{e}^{-2u}\,du\right)^{1/2} = \frac{\|A\|_2}{\sqrt{2}}. \qquad \square$$

Proposition 5.2. *The sequence* $\lambda = (\lambda_j)_{j\in\mathbb{N}}$, *where*

$$\lambda_j := \exp\left(\frac{i}{3\pi}\ln^2(j+n)\right), \tag{5.2}$$

belongs to $\ell^\infty(\mathbb{N}) \setminus \mathrm{SO}(\mathbb{N})$. *Moreover there exists a function* $a \in L^1([0,1], r\,dr)$ *such that* $\lambda = \gamma_{n,a}$.

Proof. Since $|\lambda_j| = 1$ for all $j \in \mathbb{N}$, the sequence λ is bounded. Let A be the function from Proposition 5.1. Define $a\colon [0,1] \to \mathbb{C}$ by

$$a(r) = A(-2\ln r).$$

Then

$$\int_0^1 |a(r)|\,r\,dr = \frac{1}{2}\int_0^1 |a(\sqrt{t})|\,dt = \frac{1}{2}\int_0^1 |A(-\ln(t))|\,dt = \frac{1}{2}\int_0^{+\infty} |A(u)|\,\mathrm{e}^{-u}\,du < +\infty,$$

and

$$\begin{aligned}\gamma_{n,a}(j) &= (j+n)\int_0^1 a(\sqrt{r})\,r^{j+n-1}\,dr = (j+n)\int_0^1 A(-\ln r)\,r^{j+n-1}\,dr \\ &= (j+n)\int_0^{+\infty} A(u)\,\mathrm{e}^{-(j+n)u}\,du = (j+n)f(j+n) = \lambda_j.\end{aligned}$$

Let us prove that $\lambda \notin \mathrm{SO}(\mathbb{N})$. For every $j,k \in \mathbb{N}$ we have

$$|\lambda_j - \lambda_k| = \left|\exp\left(\frac{i}{3\pi}\left(\ln^2(j+n) - \ln^2(k+n)\right)\right) - 1\right|.$$

Replace j by the following function of k:

$$j(k) := k + \left\lfloor \frac{k+n}{\ln^{1/2}(k+n)} \right\rfloor.$$

Then

$$\frac{j(k)-k}{k+n} = \frac{1}{\ln^{1/2}(k+n)} + O\left(\frac{1}{k+n}\right)$$

and

$$\begin{aligned}\ln(j(k)+n) &= \ln(k+n) + \ln\left(1 + \frac{j(k)-k}{k+n}\right) \\ &= \ln(k+n) + \frac{1}{\ln^{1/2}(k+n)} - \frac{1}{2\ln(k+n)} + O\left(\frac{1}{\ln^{3/2}(k+n)}\right).\end{aligned}$$

Denote $\ln^2(j(k)+n)-\ln^2(k+n)$ by L_k and consider the asymptotic behavior of L_k as $k \to \infty$:

$$L_k := \ln^2(j(k)+n)-\ln^2(k+n) = -1+2\ln^{1/2}(k+n)+O\left(\frac{1}{\ln(k+n)}\right).$$

Since L_k increases slowly for large k, for every $K>0$ there exists an integer $k \geq K$ such that L_k+1 is close enough to an integer multiple of $6\pi^2$, say to $6m\pi^2$:

$$L_k+1 \approx 6m\pi^2.$$

For such k,

$$|\lambda_{j(k)}-\lambda_k| = \left|\exp\left(\frac{i}{3\pi}(L_k+1-6m\pi^2)\right)\exp\left(-\frac{i}{3\pi}\right)-1\right| \approx \left|\exp\left(-\frac{i}{3\pi}\right)-1\right| \neq 0.$$

It means that $|\lambda_{j(k)}-\lambda_k|$ does not converge to 0 as k goes to infinity. On the other hand,

$$\rho(j(k),k) = \ln\frac{j(k)+1}{k+1} \leq \frac{(k+n)}{(k+1)\ln^{1/2}(k+n)} \to 0.$$

It follows that $\lambda \notin \mathrm{SO}(\mathbb{N})$. □

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