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Perturbations of Operator Functions in a Hilbert Space

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Abstract

Let A and $\tilde{A}$ be linear bounded operators in a separable Hilbert space, and f be a function analytic on the closed convex hull of the spectra of A and $\tilde{A}$. Let SN_2 and SN_1 be the ideals of Hilbert-Schmidt and nuclear operators, respectively. In the paper, a sharp estimate for the norm of $f(A)-f(\tilde{A})$ is established, provided A and $\tilde{A}$ have the so called Hilbert-Schmidt property. In particular, A has the Hilbert-Schmidt property, if one of the following conditions holds: $A-A^* \in SN_2$, or $AA^*-I \in SN_1$. Here A^* is adjoint to A, and I is the unit operator. Our results are new even in the finite dimensional case.

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1 Introduction and statement of the main result

Let H be a separable complex Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. All the considered operators are linear and bounded. For an operator A, $\sigma(A)$ denotes the spectrum, $R_z(A) = (A-zI)^{-1}$ $(z \notin \sigma(A))$ is the resolvent of A; $r_s(A)$ denotes the spectral radius of A, $\lambda_k(A)$ are the eigenvalues of A taken with their multiplicities, A^* is the adjoint to A and $A_I = (A-A^*)/2i$. Let f be a scalar-valued function, which is analytic on a neighborhood of $\sigma(A)$. Then we put

$$f(A) = -\frac{1}{2\pi i}\int_C f(\lambda)R_\lambda(A)d\lambda, \tag{1.1}$$

where C is a closed smooth contour surrounding $\sigma(A)$. Let SN_p be the ideal Shatten-von Neumann operators K with the finite norm $N_p(K) := [Trace\ (KK^*)^{p/2}]^{1/p}$ $(1 \le p < \infty)$. So SN_2 is the ideal of Hilbert-Schmidt operators, and SN_1 is the ideal of nuclear operators.

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This paper is devoted to perturbations of analytic operator valued functions of operators acting in a Hilbert space. The perturbation theory of operator functions in a Hilbert space is rather rich. I would like to mention the fundamental papers on double operator integrals by M. Birman and M. Solomyak, which are reflected in the survey [2], and also the papers by T. Ando [1] and V. Peller [11] which contributed to the topic most substantially. The remarkable Birman-Solomyak results allow us to establish bounds for the norm of $f(A) - f(\tilde{A})$ in the case when A and $\tilde{A}$ are selfadjoint and $A - \tilde{A}$ belongs to some nice ideal. Besides, A and $\tilde{A}$ may be unbounded. At the same time below we do not assume that A and $\tilde{A}$ are selfadjoint. The paper [12] also should be mentioned; it deals with a trace class perturbation of a normal operator with the spectrum on a smooth curve. The results of that paper can be applied to perturbation theory, scattering theory, functional models, and others. The interesting inequality

$$\|f(A) - f(\tilde{A})\| \leq const\, f(\|A - \tilde{A}\|)$$

was derived in [3]. Here f is a holomorphic function admitting certain integral representation.

In contrast to the Hilbert space, perturbations of functions of operators in a Banach space still do not attract much attention of mathematicians although it is very important for various applications. Mainly perturbations of concrete functions are considered, such as the exponential function (semigroup) [4], sine and cosine operator functions [10]. In the paper [7], perturbations of entire functions of some classes of certain infinite matrices are investigated. Of course we cannot survey the whole subject here and refer the reader to the above pointed papers and references given therein.

Recall that a linear operator V is called quasinilpotent if $\sigma(V) = \{0\}$. A compact quasinilpotent operator will be called *a Volterra operator*.

Let $E(t)$ be an orthogonal resolution of the identity in H, defined on a real segment $[a,b]$. E is called a maximal resolution of the identity (m.r.i.), if its every gap $E(t_0 + 0) - E(t_0 - 0)$ (if it exists) is one-dimensional, cf. [6]. We will say that a bounded linear operator A is *a E-triangular operator*, if there is a m.r.i. $E(t)$, such that $E(t)AE(t) = AE(t)$ for all $t \in [a,b]$ and

$$A = D + V, \tag{1.2}$$

where D is a normal operator and V is a Volterra one, satisfying the equalities

$$E(t)VE(t) = VE(t) \text{ and } DE(t) = E(t)D \quad (t \in [a,b]). \tag{1.3}$$

A E-triangular operator A has the property

$$\sigma(A) = \sigma(D), \tag{1.4}$$

cf. [6, Lemma 7.5.1]. Each compact operator is E-triangular and each operator having the Schatten-von Neumann Hermitian component is E-triangular; for more details see [6, Chapter 7]. We will call D and V *the diagonal part and nilpotent part of* A, respectively. We will say that A has *the Hilbert-Schmidt property, if A is a E-triangular operator and its nilpotent part is a Hilbert-Schmidt operator*: $N_2(V) < \infty$.

A has the Hilbert-Schmidt property, for instance, if one of the following conditions holds: $A_I \in SN_2$ or $AA^* - I \in SN_1$.

Indeed, due to Lemma 7.7.2 from [6] we have

$$N_2(V) = u(A), \text{ where } u(A) := \left[2N_2^2(A_I) - 2\sum_{k=1}^{\infty} |Im\ \lambda_k(A)|^2 \right]^{1/2}, \text{ provided } A_I \in SN_2. \quad (1.5)$$

Due to Lemmas 7.15.2 from [6],

$$N_2(V) = \theta(A), \text{ where } \theta(A) := \left[N_2^2(AA^* - I) - \sum_{k=1}^{\infty} (|\lambda_k(A)|^2 - 1) \right]^{1/2}, \text{ provided } AA^* - I \in SN_1. \quad (1.6)$$

Moreover, due to Lemmas 6.3.6 and 2.3.2 we can write

$$N_2(V) = g(A)\ , \text{ where } g(A) := \left[N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 \right]^{1/2}, \text{ if } A \in SN_2. \quad (1.7)$$

Obviously, $u(A) \le \sqrt{2}N_2(A_I)$ and $\theta(A) \le \sqrt{2}N_2(AA^* - I)$. In addition, $g^2(A) \le N_2^2(A) - |Trace\ A^2|$.

Denote by $co(A, \tilde{A})$ the closed convex hull of $\sigma(A) \cup \sigma(\tilde{A})$. Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let operators A and $\tilde{A}$ have the Hilbert - Schmidt property, and $N_2(A - \tilde{A}) < \infty$. In addition, let $f(\lambda)$ be holomorphic on a neighborhood of $co(A, \tilde{A})$. Then with the notation*

$$\psi_{j,k} := \sup_{z \in co\ (A, \tilde{A})} \frac{|f^{(k+j+1)}(z)|}{\sqrt{k!\,j!}(k+j+1)!} \quad (j, k = 0, 1, 2, ...),$$

the inequality

$$N_2(f(A) - f(\tilde{A})) \le N_2(A - \tilde{A}) \sum_{j,k=0}^{\infty} \psi_{j,k} N_2^j(V) N_2^k(\tilde{V})$$

is valid, where V and $\tilde{V}$ are the nilpotent parts of A and $\tilde{A}$, respectively.

The proof of this theorem is presented in the next section.

In the paper [8, Theorem 5.1], a sharp bound for $N_2(f(A) - f(\tilde{A}))$ was derived under the conditions $A_I \in SN_2$ and $A - \tilde{A} \in SN_2$. So Theorem 1.1 is an essential generalization of the the mentioned result from [8].

Let A and $\tilde{A}$ be normal operators and $f(\lambda)$ be holomorphic on a neighborhood of $co(A, \tilde{A})$. Then Theorem 1.1 implies the inequality.

$$N_2(f(A) - f(\tilde{A})) \le N_2(A - \tilde{A}) \sup_{z \in co\ (A, \tilde{A})} |f'(z)|.$$

Furthermore, let U be a unitary operator, commuting with $E(t)$. Then it is simple to check that UA is a E-triangular operator. Due to Lemma 7.3.3 [6] we can assert that UV is the nilpotent part of UA, and UD is its diagonal part UA. Assume that

$$(UA)_I := (UA - (UA)^*)/2i \in SN_2. \quad (1.8)$$

Since $N_2(V) = N_2(U^{-1}UV) \le N_2(UV)$ and $N_2(UV) \le N_2(V)$, according to (1.5) we obtain

$$N_2(V) = u(UA) \le \sqrt{2}N_2((UA)_I). \tag{1.9}$$

Here

$$u^2(UA) := 2\left[N_2^2((UA)_I) - \sum_{k=1}^{\infty}(Im\,(\lambda_k(UA))^2\right].$$

So under condition (1.8), in Theorem 1.1, one can apply inequality (1.9).

Note that one can take the operator U defined by the multiplication by e^{it} for a real t. Then condition (1.8) takes the form

$$(e^{it}A)_I := (e^{it}A - e^{-it}A^*)/2i \in SN_2.$$

Besides,

$$u^2(UA) = u^2(e^{it}A)) = 2\left[N_2^2((e^{it}A)_I) - \sum_{k=1}^{\infty}(Im\,(e^{it}\lambda_k(A)))^2\right].$$

2 Proof of Theorem 1.1

Lemma 2.1. *Let A and $\tilde{A}$ have n-dimensional ranges ($n < \infty$) and $f(\lambda)$ be holomorphic on a neighborhood of $co(A,\tilde{A})$. Then*

$$N_2(f(A) - f(\tilde{A})) \le N_2(A - \tilde{A})\sum_{j,k=0}^{n-1}\psi_{j,k}N_2^j(V)N_2^k(\tilde{V}).$$

Proof. By (1.1),

$$f(A) - f(\tilde{A}) = -\frac{1}{2\pi i}\int_L f(\lambda)(R_\lambda(A) - R_\lambda(\tilde{A}))d\lambda = \tag{2.1}$$

$$\frac{1}{2\pi i}\int_L f(\lambda)R_\lambda(\tilde{A})ER_\lambda(A)d\lambda,$$

where $E = \tilde{A} - A$ and L surrounds $\sigma(A) \cup \sigma(\tilde{A})$. By the triangular (Schur) representation

$$A = D + V\ (\sigma(A) = \sigma(D)), \tag{2.2}$$

where D is a normal and V is a nilpotent operators having the same invariant subspaces. Similarly,

$$\tilde{A} = \tilde{D} + \tilde{V}\ (\sigma(\tilde{A}) = \sigma(\tilde{D})), \tag{2.3}$$

where $\tilde{D}$ is a normal and $\tilde{V}$ is a nilpotent operators having the same invariant subspaces. But

$$R_\lambda(A) = (D + V - I\lambda)^{-1} = (I + R_\lambda(D)V)R_\lambda(D).$$

Consequently,

$$R_\lambda(A) = \sum_{k=0}^{n-1}(-1)^k(R_\lambda(D)V)^kR_\lambda(D).$$

Similarly,

$$R_\lambda(\tilde{A}) = \sum_{k=0}^{n-1}(-1)^k(R_\lambda(\tilde{D})\tilde{V})^k R_\lambda(\tilde{D}).$$

So we have

$$f(A) - f(\tilde{A}) = \sum_{m,k=0}^{n-1} C_{mk} \tag{2.4}$$

where

$$C_{mk} = (-1)^{k+m}\frac{1}{2\pi i}\int_L f(\lambda)(R_\lambda(D)V)^m R_\lambda(D)E(R_\lambda(\tilde{D})\tilde{V})^k R_\lambda(\tilde{D})d\lambda.$$

Since D is a diagonal matrix in the orthonormal basis of the triangular representations of A (the Schur basis) $\{e_k\}$, and $\tilde{D}$ is a diagonal matrix in the Schur basis $\{\tilde{e}_k\}$ of $\tilde{A}$, we can write out

$$R_\lambda(D) = \sum_{j=1}^n \frac{Q_j}{\lambda_j - \lambda}, R_\lambda(\tilde{D}) = \sum_{j=1}^n \frac{\tilde{Q}_j}{\tilde{\lambda}_j - \lambda},$$

where $\lambda_j = \lambda_j(A), \tilde{\lambda}_j = \lambda_j(\tilde{A})$, $Q_k = (., e_k)e_k, \tilde{Q}_k = (., \tilde{e}_k)\tilde{e}_k$. Consequently,

$$C_{mk} = \sum_{i_1=1}^n Q_{i_1} V \sum_{i_2=1}^n Q_{i_2} V \dots V \sum_{i_{m+1}=1}^n Q_{i_{m+1}} E \sum_{j_1=1}^n \tilde{Q}_{j_1}\tilde{V}\sum_{j_2=1}^n \tilde{Q}_{j_2}\tilde{V}\dots$$

$$\tilde{V}\sum_{j_{k+1}=1}^n \tilde{Q}_{j_{k+1}} J_{i_1,i_2,\dots,i_{m+1},j_1 j_2 \dots j_{k+1}}. \tag{2.5}$$

Here

$$J_{i_1,i_2,\dots,i_{m+1},j_1 j_2\dots j_{k+1}} =$$

$$\frac{(-1)^{k+m}}{2\pi i}\int_L \frac{f(\lambda)d\lambda}{(\lambda_{i_1}-\lambda)\dots(\lambda_{i_{m+1}}-\lambda)(\tilde{\lambda}_{j_1}-\lambda)\dots(\tilde{\lambda}_{j_{k+1}}-\lambda)}.$$

Below the symbol $|V|$ means the operator whose entries are absolute values of V in the basis $\{e_k\}$ and $|\tilde{V}|$ means the operator whose entries are absolute values of $\tilde{V}$ in the basis $\{\tilde{e}_k\}$. Furthermore, denote $E_{kj} = (E\tilde{e}_j, e_k)$ and $c_{kj}^{(ml)} = (C_{ml}\tilde{e}_j, e_k)$. Then

$$E = \sum_{j,k=1}^n E_{kj}(., \tilde{e}_j)e_k \text{ and } C_{ml} = \sum_{j,k=1}^n c_{kj}^{(ml)}(., \tilde{e}_j)e_k.$$

Put

$$|E| = \sum_{j,k=1}^n |E_{kj}|(., \tilde{e}_j)e_k \text{ and } |C_{ml}| = \sum_{j,k=1}^n |c_{kj}^{(ml)}|(., \tilde{e}_j)e_k.$$

By Lemma 1.5.1 [6],

$$|J_{i_1,i_2,\dots,i_{m+1},j_1 j_2\dots j_{k+1}}| \le \tilde{\psi}_{m,k} := \sup_{z\in co\,(A,\tilde{A})} \frac{|f^{(k+m+1)}(z)|}{(m+k+1)!}.$$

Now (2.5) implies

$$|C_{mk}| \le \tilde{\psi}_{m,k} \sum_{i_1=1}^{n} Q_{i_1}|V| \sum_{i_2=1}^{n} Q_{i_2}|V| \dots |V| \sum_{i_{m+1}=1}^{n} Q_{j_{m+1}}|E| \sum_{j_1=1}^{n} \tilde{Q}_{j_2}|\tilde{V}| \dots |\tilde{V}| \sum_{j_{k+1}=1}^{n} \tilde{Q}_{j_{k+1}}.$$

Thus

$$|C_{mk}| \le \tilde{\psi}_{m,k}|V|^m|E||\tilde{V}|^k. \tag{2.6}$$

Note that

$$N_2^2(|E|) = \sum_{k=1}^{n} \||E|\tilde{e}_k\|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} |E_{jk}|^2 = N_2^2(E).$$

Hence (2.6) yields the inequality $N_2(C_{mk})\| \le \tilde{\psi}_{m,k}\||V|^m\|N_2(E)\||\tilde{V}|^k\|$. By Theorem 2.5.1 from [6] we have $\| \, |V|^m\| \le \frac{N_2^m(V)}{\sqrt{m!}}$. So

$$N_2(C_{mk}) \le \tilde{\psi}_{m,k} N_2(E) \frac{N_2^m(V) N_2^k(\tilde{V})}{\sqrt{m!}\,\sqrt{k!}}.$$

Now (2.4) implies the required result. □

Lemma 2.2. *Let A be a E-triangular operator. Then there is a sequence of m-dimensional operators* B_m *(*$m = 1,2,...$*) strongly converging to A, such that* $\sigma(B_m) \subseteq \sigma(A)$*. Moreover, the nilpotent parts of* B_m *tend to the nilpotent part V of A in the operator norm.*

Proof. The diagonal part D of A commutes with $E(t)$; so by the classical von Neumann theorem, it is a function of $\int_a^b t dE(t)$. Therefore D is a strong limit the operators

$$D_n = \sum_{k=1}^{n} \phi_{nk}\Delta E_k \;\; (E_k = E(t_k), \Delta E_k = E_k - E_{k-1}; a = t_0 < t_1 < ... < t_n = b).$$

Here ϕ_{nk} $(k = 1,...,n)$ are numbers. Put

$$V_n = \sum_{k=1}^{n} E_{k-1} V \Delta E_k = V - \sum_{k=1}^{n} \Delta E_k V \Delta E_k.$$

Due to the well-known Lemma I.5.1 from [9] and equality (3.1) from Section I.3 of that book, we can assert that $V_n \to V$ in the operator norm. Furthermore, let $\{e_m^{(k)}\}_{m=1}^{\infty}$ be an orthogonal normal basis in $\Delta E_k H$. Put

$$Q_l^{(k)} = \sum_{m=1}^{l} (., e_m^{(k)}) e_m^{(k)} \;\; (k = 1,...,n;\; l = 1,2,....).$$

Clearly, $Q_l^{(k)}$ strongly converge to ΔE_k as $l \to \infty$. So the operators

$$D_{ln} = \sum_{k=1}^{n} \phi_{nk} Q_l^{(k)}$$

strongly tend to D_n as $l \to \infty$. Introduce the operators

$$W_{ln} = \sum_{k=1}^{n} \sum_{i=1}^{k-1} Q_l^{(i)} V Q_l^{(k)}.$$

Since projectors $Q_l^{(k)}$ strongly converge to ΔE_k as $l \to \infty$, and V_n is compact, operators W_{ln} converge to V_n in the operator norm. So the finite dimensional operators $T_{ln} = D_{ln} + W_{ln}$ strongly converge to A_n as $l \to \infty$. Therefore they strongly converge to A as $l, n \to \infty$. Moreover W_{ln} converge to V in the operator norm. But W_{ln} are nilpotent, and W_{ln} and D_{ln} have the same invariant subspaces. Consequently,

$$\sigma(D_{ln}) = \sigma(T_{ln}) \subseteq \sigma(A_n) = \{\phi_{nk}\} \subseteq \sigma(A).$$

Taking $B_m = T_{ln}$, we obtain the required result. □

Proof of Theorem 1.1: We get the required result due to the previous lemma by passing to the limit $n \to \infty$ in Lemma 2.1. □

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