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HAMILTON, RODRIGUES, GAUSS, QUATERNIONS, AND ROTATIONS: A HISTORICAL REASSESSMENT

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Abstract

Quaternions, invented in 1843 by the famous mathematical physicist Hamilton, largely fell out of favor long ago, being replaced by vectors, which are direct descendants of quaternions. Quaternions have become popular recently because rotation operations are simpler with quaternions than with matrices. Therefore, quaternions are widely used in computer applications involving rotations. The relation between rotations and quaternions was established shortly after the invention of the latter, but recently it has been claimed that Hamilton misunderstood the relation, and that credit should be given to Rodrigues, who had published a paper on rotations in 1840. This claim has been repeated widely, but a review of original sources shows that it is based on an incomplete reading of Hamilton's work. The same review shows, however, that Gauss had derived results similar to those of Hamilton and Rodrigues in 1819, but they remained unpublished until 1900, and thus unknown to either of them. These matters are described in a self-contained way in this article.

AMS Subject Classification: 01A55.

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1 Introduction

William Hamilton was a nineteen-century mathematical physicist who became famous because of his fundamental contributions to optics and dynamics, and for his invention (in 1843) of quaternions. The latter are an extension to 4-D space of the ordinary complex numbers, and Hamilton expected that they would have great significance in physics and geometry, but these expectations never materialized. As a consequence, they largely fell out

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of favor a few decades after they were introduced, being replaced by vectors, which are direct descendants of quaternions (e.g., [11]). Interestingly, quaternions have had a revival in recent years because they can be used in place of, and have numerical advantages over, rotations matrices, which are essential in modern applications such as computer graphics and in the aerospace industry [30], [26]. The relation between rotations and quaternions was established shortly after the invention of the latter, but there has been recent criticism of Hamilton's understanding of the relation. According to Altmann [1], Hamilton essentially misunderstood the relation, which should be credited to a little-known mathematician, O. Rodrigues, who made very important contributions to the study of rotations. Altmann's claim has been widely accepted (e.g., [15], [31]). This is not the only controversy regarding quaternions; a much earlier one revolved around work done by Gauss that seemed to presage Hamilton's quaternions. Another famous mathematician, Euler, also makes an appearance because of the so-called Euler-Rodrigues parameters, which constitute an essential link between quaternions and rotations. These parameters were introduced by Rodrigues in 1840, and there is no evidence that they were known to Euler. These matters have been discussed in the literature, but to the best of the author's knowledge, the discussions have been fragmentary and in some aspects also incomplete. In fact, examination of the digitized versions of original sources, available via the internet, shows a more complex history. In summary, Hamilton had discovered the Euler-Rodrigues parameters in 1844 independently of Rodrigues, and had a correct understanding of the relation between quaternions and rotations. Moreover, Hamilton's derivation of the relation was much simpler than that of Rodrigues. Another interesting fact is that the Euler-Rodrigues parameters had been introduced by Gauss in 1819 in a series of unpublished notes that also anticipated the basic properties of quaternions and rotations. Gauss did not prove, or even motivate his results, which were published posthumously, and thus were unknown to both Rodrigues and Hamilton. Finally, a paper of Euler that discussed a purely algebraic problem unrelated to rotations included a result, given without proof, that has a connection with rotations and quaternions.

The examination of original sources has provided another example of the process of discovery and rediscovery that is not uncommon in mathematics, and to disentangle the contributions made by the different authors, the starting point is a consideration of rotations parameterized in terms of two set of parameters: (a) an axis and an angle, and (b) the Euler-Rodrigues parameters. This will be followed by an introduction to quaternions and its relation to rotations. The final section gives a historical account that makes the connections between some of the basic results presented in the earlier sections and the mathematicians that contributed to them. For the benefit of the reader, digitized versions of the relevant historical papers and book excerpts can be found at https://umdrive.memphis.edu/jpujol/public/Quaternion

2 Overview of rotations

Let us consider a finite rotation of angle α of a vector $\mathbf{r}$ about an axis defined by a unit vector $\mathbf{a} = (a_1, a_2, a_3)$. The geometry of the problem is shown in Fig. 1. The result of this rotation is a vector $\mathbf{r}'$ given by

$$\mathbf{r}' = \cos\alpha\ \mathbf{r} + (1-\cos\alpha)(\mathbf{r}\cdot\mathbf{a})\mathbf{a} + \sin\alpha\ \mathbf{a}\times\mathbf{r}, \tag{2.1}$$

where the dot and cross denote the scalar (or dot) and vector products [23], [29].

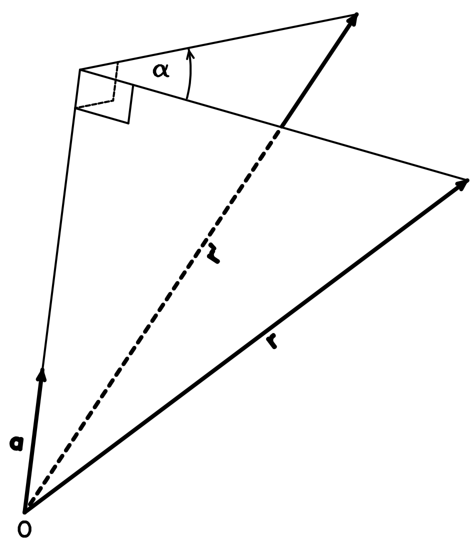

Figure 1. Rotation of angle α of a vector **r** about an axis identified by a unit vector **a**. The result is the vector **r′**.After [29].

Equation (2.1) can be written in matrix form as

$$\mathbf{r}' = \mathbf{R}\mathbf{r}, \tag{2.2}$$

where

$$\mathbf{R} = \begin{pmatrix} c\alpha + (1-c\alpha)a_1^2 & (1-c\alpha)a_1a_2 - s\alpha\ a_3 & (1-c\alpha)a_1a_3 + s\alpha\ a_2 \\ (1-c\alpha)a_2a_1 + s\alpha\ a_3 & c\alpha + (1-c\alpha)a_2^2 & (1-c\alpha)a_2a_3 - s\alpha\ a_1 \\ (1-c\alpha)a_3a_1 - s\alpha\ a_2 & (1-c\alpha)a_3a_2 + s\alpha\ a_1 & c\alpha + (1-c\alpha)a_3^2 \end{pmatrix} \tag{2.3}$$

and $c\alpha$ and $s\alpha$ stand for $\cos\alpha$ and $\sin\alpha$ respectively. To derive equation (2.3) write equation (2.1) in component form and expand the scalar and vector products.

Two important properties of rotations are as follows.

a) $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$, where the superscript T indicate matrix transposition and **I** represents the identity matrix [29].

b) From equation (2.1) it follows immediately that $\mathbf{Ra} = \mathbf{a}$. Therefore, **a** is an eigenvector of **R** with eigenvalue equal to one (regardless of the value of α). Moreover, **a** is the only real-valued eigenvector [24], [29].

In applications it may be needed to find α and **a** when **R** is given. The angle α is obtained from the trace of **R**, given by

$$\mathrm{tr}(\mathbf{R}) = 2\cos\alpha + 1, \tag{2.4}$$

The components of **a** are obtained by direct inspection of the off-diagonal elements in equation (2.3), which gives

$$\begin{aligned} a_1 &= (R_{32} - R_{23})/2\sin\alpha \\ a_2 &= (R_{13} - R_{31})/2\sin\alpha, \qquad\qquad \alpha \neq 0, \pi \\ a_3 &= (R_{21} - R_{12})/2\sin\alpha. \end{aligned} \tag{2.5}$$

When $\alpha = 0$, **R** is the identity matrix, and **a** is undefined. For the special case $\alpha = \pi$ see [3].

Rotations can also be expressed in terms of the Euler-Rodrigues parameters, which are the components of the vector **v** and the scalar s given by

$$\mathbf{v} = \mathbf{a}\sin\tfrac{1}{2}\alpha, \qquad\qquad s = \cos\tfrac{1}{2}\alpha, \tag{2.6}$$

where **a** and α are as before. Writing equation (2.1) in terms of $\alpha/2$ and then using these parameters gives

$$\begin{aligned} \mathbf{r}' &= (\cos^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\alpha)\,\mathbf{r} + 2\sin^2\tfrac{1}{2}\alpha\,(\mathbf{r}\cdot\mathbf{a})\mathbf{a} + 2\cos\tfrac{1}{2}\alpha\ \sin\tfrac{1}{2}\alpha\ \mathbf{a}\times\mathbf{r} = \\ &(s^2 - |\mathbf{v}|^2)\mathbf{r} + 2(\mathbf{r}\cdot\mathbf{v})\mathbf{v} + 2s\ \mathbf{v}\times\mathbf{r}. \end{aligned} \tag{2.7}$$

(after [32]). This is an important result that will be used in the next section.

To write equation (2.7) in matrix form, namely,

$$\mathbf{r}' = \mathbf{M}\mathbf{r}. \tag{2.8}$$

let $\mathbf{v} = (l, m, n)$ and use

$$l^2 + m^2 + n^2 + s^2 = 1, \tag{2.9}$$

which follows form equation (2.6). Then,

$$\mathbf{M} = \begin{pmatrix} s^2 + l^2 - m^2 - n^2 & 2(lm - ns) & 2(nl + ms) \\ 2(lm + ns) & s^2 - l^2 + m^2 - n^2 & 2(mn - ls) \\ 2(nl - ms) & 2(mn + ls) & s^2 - l^2 - m^2 + n^2 \end{pmatrix}. \tag{2.10}$$

Clearly, **R** and **M** represent the same matrix; the two symbols are introduced to identify the differences in formulation. If α is replaced by $-\alpha$, **M** becomes $\mathbf{M}^T$.

Given **M**, α is obtained from tr(**M**) and the components of **v** are obtained using expressions similar to equations (2.5) [3], but because they will not be used here, they are not given. On the other hand, the squares of the components of **v** will be of interest in the last section. To obtain them, eliminate s^2 from the diagonal elements of **M** using equation (2.9) and then equate these elements to M_{11}, M_{22}, and M_{33} (preserving the appropriate ordering). This results in a system of three equations in l^2, m^2 and n^2, with solution given by

$$\begin{aligned} l^2 &= \tfrac{1}{4}(1 + M_{11} - M_{22} - M_{33}) \\ m^2 &= \tfrac{1}{4}(1 + M_{22} - M_{11} - M_{33}) \\ n^2 &= \tfrac{1}{4}(1 + M_{33} - M_{11} - M_{22}). \end{aligned} \tag{2.11}$$

Combining these equations with equation (2.9) gives

$$s^2 = \tfrac{1}{4}(1 + M_{11} + M_{22} + M_{33}). \tag{2.12}$$

3 Quaternions and their relation to rotations

Quaternions were invented by Hamilton in 1843 [20]. They are entities of the form

$$q = r + a\mathsf{i} + b\mathsf{j} + c\mathsf{k}, \tag{3.1}$$

where r, a, b, and c are real numbers and i, j, k satisfy

$$\begin{gathered} \mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1 \\ \mathsf{ij} = -\mathsf{ji} = \mathsf{k}, \qquad \mathsf{jk} = -\mathsf{kj} = \mathsf{i}, \qquad \mathsf{ki} = -\mathsf{ik} = \mathsf{j}. \end{gathered} \tag{3.2}$$

An excellent introduction to the genesis of quaternions is provided in [36], while [4] provides a good overview. Here only those aspects of quaternions related to rotations will be considered. The i, j, k can be interpreted as either extensions to 4-D space of the usual imaginary unit of 2-D space or unit vectors in 3-D space. In the latter case q can be written as

$$q = (r + \mathbf{u}), \qquad \mathbf{u} = (a, b, c), \tag{3.3}$$

where r and $\mathbf{u}$ are known as the scalar and vector of the quaternion. This notation is due to Hamilton [21], who also introduced the scalar and vector products, although not the symbols used today. Also, today's scalar product is the negative of Hamilton's. When $r = 0$, q is known as a pure quaternion. On the other hand, we will consider a vector to be a pure quaternion.

Given two quaternions q_1 and q_2, written as in equation (3.1), their product q_1q_2 is obtained by multiplying them as if their were polynomials, keeping the order of the units $\mathsf{i}, \mathsf{j}, \mathsf{k}$, and then using the relations given in equations (3.2). If the two quaternions are written as in equation (3.3), i.e., $q_1 = (r + \mathbf{u})$, $q_2 = (t + \mathbf{w})$, their product can be written as

$$q_1q_2 = (rt - \mathbf{u}\cdot\mathbf{w} + r\mathbf{w} + t\mathbf{u} + \mathbf{u}\times\mathbf{w}), \tag{3.4}$$

where the dot and cross represent the usual scalar and vector products, respectively (after [4]). This definition shows that, in general, $q_1q_2 \neq q_2q_1$.

We will need some additional definitions. Given a quaternion $q = (r + \mathbf{u})$, its conjugate (q^*), norm ($||q||$), and reciprocal (q^{-1}) are defined by

$$q^* = (r - \mathbf{u}), \qquad ||q|| = qq^*, \qquad q^{-1} = \frac{q^*}{||q||}. \tag{3.5}$$

The symbols used here for the conjugate and norm differ from those introduced by Hamilton (see, e.g., [4]). Also note that some authors define the norm as the square root of $||q||$. Using equation (3.4) we see that

$$||q|| = (r + \mathbf{u})(r - \mathbf{u}) = r^2 + |\mathbf{u}|^2 = r^2 + a^2 + b^2 + c^2. \tag{3.6}$$

Quaternions with norm equal to one are known as unit (or normalized) quaternions. If q is a unit quaternion, it can be written as

$$q = \cos\gamma + \sin\gamma\,\mathbf{e}, \qquad |\mathbf{e}| = 1, \tag{3.7}$$

where $\cos\gamma = r$, $\sin\gamma = |\mathbf{u}|$, and $\mathbf{e} = \mathbf{u}/|\mathbf{u}|$. Using equation (3.6) we see that $||q|| = 1$.

Given two quaternions q_1 and q_2, the conjugate, reciprocal, and norm of their product satisfy

$$(q_1q_2)^* = q_2^*q_1^*, \qquad (q_1q_2)^{-1} = q_2^{-1}q_1^{-1}, \qquad ||q_1q_2|| = ||q_1||\,||q_2||. \tag{3.8}$$

Now we are ready to consider the relation between quaternions and rotation. First note that, in general, the product of a vector (i.e., a pure quaternion) by a quaternion is not a pure quaternion (i.e., it is not a vector). Therefore, quaternions, in general, cannot be used to represent vector rotations. The only exception is for quaternions of the form

$$q = (\cos\theta + \sin\theta\,\mathbf{e}) = \mathbf{b}\mathbf{a}^{-1}, \tag{3.9}$$

where $|\mathbf{e}| = 1$, $\mathbf{a}$ and $\mathbf{b}$ are any two vectors having the same length, the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, $\mathbf{e}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{e}$ form a right-handed set [4]. This result can be verified by direct quaternionic multiplication:

$$\mathbf{b}\mathbf{a}^{-1} = (0+\mathbf{b})(0-\mathbf{a}/||\mathbf{a}||) = \frac{1}{||\mathbf{a}||}(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b}) = q. \tag{3.10}$$

The equality $q = \mathbf{b}\mathbf{a}^{-1}$ was another way Hamilton used to define quaternions [22], (arts. 106, 107), but its derivation was not simple. To understand the geometric meaning of equation (3.9) multiply by $\mathbf{a}$ on the right and expand the resulting quaternionic product, which gives

$$\mathbf{b} = q\mathbf{a} = q(0+\mathbf{a}) = (0 + \cos\theta\,\mathbf{a} + \sin\theta\,\mathbf{e}\times\mathbf{a}). \tag{3.11}$$

Comparison of this result with equation (2.1) after a change in symbols ($\mathbf{r}\to\mathbf{a}; \mathbf{a}\to\mathbf{e}$) shows that $q\mathbf{a}$ corresponds to a rotation of angle θ of $\mathbf{a}$ about $\mathbf{e}$ in a plane perpendicular to $\mathbf{e}$, which is a particular case of rotation.

In order to use quaternions to represent general rotations two things are needed. First, for a rotation of angle α about a unit vector $\mathbf{a}$, q must be of the form

$$q = (\cos\tfrac{1}{2}\alpha + \mathbf{a}\sin\tfrac{1}{2}\alpha). \tag{3.12}$$

Note that q is a unit quaternion. Second, the operation to be performed on a vector $\mathbf{r}$ to produce a rotated vector $\mathbf{r}'$ is

$$\mathbf{r}' = q\mathbf{r}q^{-1} = (\cos\tfrac{1}{2}\alpha + \mathbf{a}\sin\tfrac{1}{2}\alpha)\;\mathbf{r}\;(\cos\tfrac{1}{2}\alpha - \mathbf{a}\sin\tfrac{1}{2}\alpha). \tag{3.13}$$

It is easy to verify that this operation produces a rotation by writing q in terms of the s and $\mathbf{v}$ introduced in equations (2.6), expressing $\mathbf{r}$ as a pure quaternion, and then performing the products indicated using the definition given by equation (3.4):

$$\begin{aligned}\mathbf{r}' &= (s+\mathbf{v})(0+\mathbf{r})(s-\mathbf{v}) = (s+\mathbf{v})(\mathbf{r}\cdot\mathbf{v} + s\mathbf{r} - \mathbf{r}\times\mathbf{v}) = \\ &= \Big(0 + (s^2 - |\mathbf{v}|^2)\mathbf{r} + 2(\mathbf{r}\cdot\mathbf{v})\mathbf{v} + 2s\;\mathbf{v}\times\mathbf{r}\Big).\end{aligned} \tag{3.14}$$

This result shows that $\mathbf{r}'$ is indeed a vector (the scalar part is equal to zero), equal to the vector $\mathbf{r}'$ given by equation (2.7), which we know is the result of rotating $\mathbf{r}$ using the parameters in equation (3.12) [32].

Finally, let us consider the composition of two rotations about axes with the same origin. Let q be the quaternion defined by equation (3.12) and let p be a similar quaternion corresponding to some other rotation. Now we will rotate the $\mathbf{r}'$ given by the first equality in equation (3.13) using p, which results in the vector

$$\mathbf{r}'' = p\mathbf{r}'p^{-1} = p(q\mathbf{r}q^{-1})p^{-1} = (pq)\mathbf{r}(pq)^{-1} \tag{3.15}$$

because the quaternionic product is associative (e.g., [4]). Therefore, the composition of rotations is equivalent to the product of quaternions.

4 Historical notes

As in many other branches of applied mathematics, Euler made fundamental contributions to the study of rotations, and three of his papers include results presented here, as follows. The elements of $\mathbf{R}$, the relation $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, and equations (2.4) and (2.5) were derived by Euler [13] (§38, §23, §49), although he did not express his results in matrix form (matrices were introduced about 80 years later). The equation $\mathbf{Ra} = \mathbf{a}$ appears in Euler [14] (§21) in the form of three separate equations. The same paper includes a theorem (§24) that states that in whatever way a sphere is turned about its center, it is always possible to assign a diameter whose direction in the translated state agrees with that of the initial state. This diameter corresponds to the axis of rotation. Euler [12] also presented results similar to those expressed by equation (2.10) in a paper that discussed a purely algebraic problem, unrelated to rotations. Expressed in modern terminology, Euler wanted to find the components of a 3×3 matrix whose rows and columns formed two sets of orthonormal vectors, which is equivalent to $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$, and although Euler did not use matrices, he placed the nine unknowns in a square array, which makes the association with rotation matrices obvious. A particular solution, introduced in §XXXIII, had the nine parameters expressed in terms of four numbers p, q, r, s such that the sum of their squares was equal to some other number u. When $u = 1$, p, q, r, s satisfy a relation similar to equation (2.9) and the array corresponds to $\mathbf{M}^T$. Euler did not provide a proof for his result but made reference to the Diophantine method. It is worth noting that the well-known Euler angles, which afford another parameterization of rotations, were introduced in the same paper. Interestingly, a theorem of Euler on the product of two sums of four squares, which results in another sum of four squares, follows without effort from the norm of the product of two quaternions, given by the third of equations (3.8) [4].

As equations (3.13) and (3.14) show, rotations and quaternions are linked through the Euler-Rodrigues parameters, which were introduced by Rodrigues in 1840 ([18], and reference therein). Some of the results derived by Rodrigues can be found in [5]. As Altmann [1] noted, these parameters were not known to Euler, but because his extensive work on rotations it must have been easy to associate his name to the parameters. However, they had been discovered by Gauss in 1819, while Hamilton discovered them independently in 1844, but before addressing the question of the priority of the Euler-Rodrigues parameters, we will consider the origin of equation (3.13), which was introduced in 1845 by Cayley [6] in a slightly different form. Cayley started with $q^{-1}\mathbf{r}q$ and $q = 1 + \lambda \mathsf{i} + \mu \mathsf{j} + \nu \mathsf{k}$. After performing the product Cayley ended up with a system of three equations corresponding to the transformation of one set of rectangular coordinates into another (with a common origin), with the matrix of the transformation similar to $\mathbf{M}^T$. Then he noted that the transformation would correspond to a rotation of angle θ about an axis with angles f, g, and h with respect to the x, y, and z axes if

$$\lambda = \tan\tfrac{1}{2}\theta\cos f, \qquad \mu = \tan\tfrac{1}{2}\theta\cos g, \qquad \nu = \tan\tfrac{1}{2}\theta\cos h, \tag{4.1}$$

which were introduced by Rodrigues, although Cayley was not able to account, *à priori*, for the appearance of these coefficients. Note the similarity between them and the parameters in equation (2.6). Equation (3.15) was introduced by Cayley in 1848 [7].

Next we will consider Hamilton's work on rotations, which can be divided into two stages. In the first stage Hamilton derived, independently, an expression identical (except for the symbols used) to the second equality in equation (3.13). Hamilton presented this result to the Royal Irish Academy in 1844 but was published in 1847 [21] (p. 13). The derivation is very simple, and is given next, but to simplify the comparison with preceding equations, Hamilton's symbols have been modified as follows: $\beta \to \mathbf{r}, \beta' \to \mathbf{r}'$, $a \to \alpha$, $\alpha \to \mathbf{a}$. Hamilton's starting result was

$$\mathbf{r}' = \tfrac{1}{2}(\mathbf{r} - \mathbf{ara}) + \tfrac{1}{2}\cos\alpha\,(\mathbf{r} + \mathbf{ara}) + \tfrac{1}{2}\sin\alpha\,(\mathbf{ar} - \mathbf{ra}). \tag{4.2}$$

Interpreting the vectors as pure quaternions and operating shows that this $\mathbf{r}'$ is equal to that in equation (2.1). Equation (4.2) was rewritten by Hamilton as

$$\begin{aligned}\mathbf{r}' = \left(\cos\tfrac{1}{2}\alpha\right)^2\,\mathbf{r} - \left(\sin\tfrac{1}{2}\alpha\right)^2\,\mathbf{ara} + \cos\tfrac{1}{2}\alpha\,\sin\tfrac{1}{2}\alpha\,(\mathbf{ar} - \mathbf{ra}) = \\ \left(\cos\tfrac{1}{2}\alpha + \mathbf{a}\,\sin\tfrac{1}{2}\alpha\right)\mathbf{r}\left(\cos\tfrac{1}{2}\alpha - \mathbf{a}\sin\tfrac{1}{2}\alpha\right),\end{aligned} \tag{4.3}$$

which is equation (3.13). This derivation is the simplest available and half-angles arise naturally. In contrast, Rodrigues' derivation (given by Altmann [1]) is more complicated and requires spherical trigonometry. In a footnote to his paper Hamilton also showed that his results were equivalent to those published by Cayley in 1845 [6], who accepted Hamilton's priority when his collected papers were published [8], although Cayley noted that he had been led to his results by Rodrigues' formulas.

In the 1847 paper Hamilton also considered the composition of two or more rotations, and showed that it was equivalent to a single rotation, which could be represented by the product of the quaternions corresponding to the rotations involved, in agreement with equation (3.15), although Hamilton did not use that notation. To relate that work to Rodrigues' results consider the rotation of $\mathbf{r}'$ about $\mathbf{a}'$ through an angle α'. Following Hamilton [21] and using equation (4.3), the resulting vector can be written as

$$\mathbf{r}'' = \left(\cos\tfrac{1}{2}\beta + \mathbf{b}\,\sin\tfrac{1}{2}\beta\right)\mathbf{r}\left(\cos\tfrac{1}{2}\beta - \mathbf{b}\sin\tfrac{1}{2}\beta\right), \tag{4.4}$$

where β and $\mathbf{b}$ are determined as follows. Let q be the quaternion in equation (3.12) and p the quaternion corresponding to $\mathbf{a}'$ and α'. Then, the quaternion in parentheses in equation (3.15) is given by

$$\begin{aligned} t = pq = \Big(&\cos\tfrac{1}{2}\alpha' \ \cos\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}\alpha' \ \sin\tfrac{1}{2}\alpha \ (\mathbf{a}'\cdot\mathbf{a}) + \sin\tfrac{1}{2}\alpha' \cos\tfrac{1}{2}\alpha \ \mathbf{a}' + \\ &\cos\tfrac{1}{2}\alpha' \sin\tfrac{1}{2}\alpha \ \mathbf{a} + \sin\tfrac{1}{2}\alpha' \ \sin\tfrac{1}{2}\alpha \ (\mathbf{a}'\times\mathbf{a})\Big) \equiv (\cos\tfrac{1}{2}\beta + \mathbf{b} \ \sin\tfrac{1}{2}\beta) \end{aligned} \tag{4.5}$$

As t is a unit quaternion, it can be written as in equation (3.7). This expression for t is a special case of a general equation derived by Hamilton [21], although he did not give explicit expressions for β and $\mathbf{b}$.

Rodrigues's work for the composition of two rotations resulted in four equations. Aside from symbol changes, the first equation is obtained by equating the scalar parts of the two rightmost quaternions in equation (4.5). As vectors had not been introduced yet, $\mathbf{a}'\cdot\mathbf{a}$ should be replaced by its component form. The other three equations are obtained by equating the vector parts of the two quaternions, also written in component form. The original Rodrigues' equations can be found in [18]. Altmann [1] (p. 302-303), however, expressed them using vector notation, and combined them in an expression equivalent to the quaternion on the right-hand side of equation (4.5). This reworking of the Rodrigues' equations has the benefit of hindsight, but as Gray [19] noted, Rodrigues did not offer an interpretation in terms of the product of unit quaternions.

The importance of the 1847 Hamilton's paper in the context of rotations does not seem to have been widely recognized. The present author learned about it from [3] (p. 201). A more recent reference to this paper is [19], but the chronology of the results of Hamilton and Cayley presented there (p. 93) is incorrect. Also, the results of Cayley described in pp. 96-97 come from [6], not [5].

Hamilton's subsequent work on quaternions was based mostly on a coordinate-free approach, and his treatment of rotations in a book published in 1853 [22] was very different from that in [21]. In [22], Hamilton considered the operation rqr^{-1}, where r and q are quaternions, and showed that it represents a rotation of double the angle of r of the axis of q about the axis of r (art. 282). His proof was geometric and relied on reflections (see also [10]). Hamilton referred to this rotation as *conical*, which is consistent with Fig. 1.

Another result proved by Hamilton is that the difference between qrq^{-1} and $q^{-1}rq$ was in the direction of the rotation, not on the angle (art. 283). This is in agreement with our earlier observation that Cayley's results in [6] correspond to $\mathbf{M}^T$ (i.e., to a rotation of angle $-\alpha$). Another result of Hamilton is that given a vector $\mathbf{r}$, $q\mathbf{r}q^{-1}$ is another vector obtained by rotation of $\mathbf{r}$ about the axis of q through an angle equal to double the angle of q (art. 286), and noted that if one was interested in a rotation through the angle of the quaternion (instead of double its angle), q had to be replaced by its square root (art. 288). Taking the square root of a quaternion divides its angle by two [4]. A practical example is given in [33] (art. 121), who discussed the apparent diurnal rotation of stars, which was done using equation (3.12) with α an appropriate observed angle. Hamilton also considered the special case of a quaternion of the form $q = \mathbf{b}\mathbf{a}^{-1}$ and found that the effect of $(\mathbf{b}\mathbf{a}^{-1})^{1/2}B(\mathbf{a}\mathbf{b}^{-1})^{1/2}$, where B stood for body, is a rotation of B around an axis perpendicular to $\mathbf{a}$ and $\mathbf{b}$ that brings **a** into the direction of **b**. Therefore, this is a *rotation in one plane* or *plane rotation* (Hamilton's italics, art. 288). This result agrees with the interpretation of equation (3.11). The composition of two conical rotations with axes passing through a common point O was also discussed by Hamilton [22] (arts. 286, 341), who showed (again) that their composition is equivalent to a single rotation around an axis trough O, with the new rotation derived from the product of the quaternions corresponding to the original rotations.

This work of Hamilton on rotations was considered in detail here because some of its aspects seem to have been overlooked by Altmann [1], who presented a rather negative view of Hamilton's work on this subject while at the same time emphasizing Rodrigues contributions. At the core of Altmann's argument is a comparison of the expressions for q given by equations (3.7) and (3.12). The difference between the two expressions is the presence of a half-angle in the latter. Recall that equation (3.7) generates an arbitrary rotation of angle 2β when used in $q\mathbf{r}q^{-1}$ (with a simple modification when a rotation of angle β is desired), and that a plane rotation is a special case of rotation (which Altmann referred to as a rectangular transformation). Altmann [1] (p. 303) stated that "Hamilton committed a serious error of judgment in basing his parameterization on the special case of the rectangular transformation." In view of the general study of rotations carried out by Hamilton [21], [22], this statement seems unwarranted. Moreover, the statement in Altmann [1] (p. 300) to the effect that "Hamilton and his colleagues" searched for a quaternion transformation of a pure quaternion that would produce another pure quaternion is incorrect. As noted earlier, in his 1847 paper Hamilton [21] did not search for that transformation, corresponding to equation (3.13); he was led to it by the straightforward consideration of the most general case of rotation, represented by equation (4.2).

Altmann's statement "... the germ of the canker that eventually consumed the quaternion body." [1] (p. 297) in the context of Hamilton's work on rotations seems to imply that this was the cause of the quaternions demise, but the reasons for it lie elsewhere. In fact, Hamilton had expected that quaternions would play a fundamental role in applications to physics and geometry, but that was not the case. There was nothing wrong with the results derived using them. The problem was that there was little that quaternions could do that could not be done with more conventional methods or with vectors. In addition, quaternions had become coordinate-free entities, and using them to solve particular problems was not necessarily simpler than using methods based on coordinates. Some of the arguments in favor and against quaternions can be found in, e.g, [9], [11], [28], and [34]. It is ironic that the scalar and vector products, which were introduced by Hamilton, proved to be so useful by themselves that they ended up replacing quaternions. In this context, it is worth mentioning that this move from quaternions to vectors was championed by two famous mathematical physicists, W. Gibbs (American) and O. Heaviside (English). Gibbs [17] (p. 157), however, acknowledged that "the quaternion affords a convenient notation for rotations", although he also noted that an alternative representation for rotations was available. The quaternionic notation referred to corresponds to the first equality in equation (3.13). It is interesting that the quaternion operation singled out by Gibbs is precisely the same operation that has led to the current popularity of quaternions.

Altmann [1] also pointed out that the Euler-Rodrigues parameters were not known to Euler, which is probably correct, as there seems to be no evidence that Euler was aware of the importance of the half-angles. These parameters were, however, known to Gauss in 1819, although his results were published in 1900, well after his death [16]. Moreover, as noted above, they were derived independently by Hamilton [21], although Altmann [1] (p. 302) claimed that Hamilton had never considered half-angles. Therefore, what we see here is another example of the process of discovery and rediscovery that is not uncommon in mathematics. The transpose of the matrix **M** (see 2.8) is another case; it was introduced without a proof by Euler in 1771 [12] (§XXXIII), was rediscovered by Cayley in 1845 [6], and a similar matrix also appeared in Gauss' posthumously published notes. This latter work is extremely important because it essentially introduced quaternions in the context of transformations of coordinates involving magnification and rotation. This work appeared in three separate notes that were combined for publication. In the published version they are divided into two parts (I and II). Part I has eight subsections, with the first seven dated 1819 and the last one dated between 1822 and 1823. These dates were inferred by one of the editors of the notes. Part II was later and could not be dated. In barely three pages corresponding to Part I.2-6 Gauss did the following:

1) Introduced a quadruple of scalars $g = (a,b,c,d)$ with a, b, c and d proportional to the s, l, m, and n, respectively, that are obtained by taking the positive square roots in equations (2.12) and (2.11). These scalars satisfy $a^2+b^2+c^2+d^2 = 2$. Gauss referred to his quadruple as a scale.

2) Introduced a 3×3 scheme similar to $\frac{1}{2}\mathbf{M}^T$, although there are differences in the signs of some of the off-diagonal elements.

3) Gave the rule for the composition of two transformation of coordinates, which agrees with equation (3.4) except for the sign of the vector product. However, a transposition of the second and third components of Gauss' quadruples produces Hamilton's result [27] (p. 433). Gauss also noted that the composition was not commutative.

4) Gave expressions for a, b, c and d in terms of a stationary point and a rotation angle that coincide with the Euler-Rodrigues parameters introduced in equations (2.6) when the magnification is equal to 1.

5) Introduced the quadruple $g' = (a,-b,-c,-d)$ (which corresponds to the conjugation of a quaternion) and found that the composition of g and g' was equal to the quadruple $(a^2+b^2+c^2+d^2,0,0,0)$, which is equivalent to equation (3.6). However, his expression for ghg', where h is another quadruple, is not equivalent to the results presented here.

In Part I.8 Gauss referred to three rotation points P,P',P'' and angles $\lambda,\lambda',\lambda''$ and noted that the angles of the spherical triangle formed by the three points were $\lambda/2,\lambda'/2,\lambda''/2$. In Part II Gauss elaborated on the meaning of his scale (a,b,c,d). Using modern terminology, if (b,c,d) are the elements of a vector $\mathbf{u}$ and $\rho = |\mathbf{u}|$, then the components of $\mathbf{u}/\rho$ are the direction cosines of a line OP (O indicates the origin of the coordinate system). Moreover, if $k = \sqrt{a^2+\rho^2}$, then, $a = k\cos\theta$ and $\rho = k\sin\theta$ is a rotation of angle 2θ about OP with a magnification k.

All of these results were presented without proof or even motivation. Gauss did not introduce units similar to the i, j, k of Hamilton, but other than that, it is clear that Gauss had anticipated some of the basic properties of quaternions. Some German mathematicians knew these results before their posthumous publication, and claimed that Gauss had some priority on the discovery of the quaternions. This happened after Hamilton's death (in 1865), and two of his followers addressed these claims and other related matters [35], [25]. It is also clear that Gauss had described rotations in terms of the parameters introduced later by Rodrigues and that he was aware of the relation between rotations, spherical triangles, and half-angles before Rodrigues, although Altmann [1] (p. 302) claimed priority for the latter. Therefore, what constituted major achievements of two other mathematicians was three short unpublished notes of Gauss. Those familiar with Gauss work will not be surprised by that; it is just another example of his amazing foresight and his reluctance to publish anything that he did not consider to be ripe (e.g., [2]).

Finally, a few words regarding Rodrigues. He was a French mathematician, banker, and social reformer. His life was remarkable [18], [1], and although his number of mathematical contributions was not large, they were important. In fact, in addition to the rotation parameters named after him, he is also known for the formula that bears his name in the context of Legendre polynomials.

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