Communications in Mathematical Analysis
Volume 12, Number 2, pp. 58–70 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

Moments of Complex B-Splines

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(Communicated by Palle Jorgensen)

Abstract

A relation between double Dirichlet averages and multivariate complex B-splines is presented. Based on this relationship, a formula for the computation of certain moments of multivariate complex B-splines is derived. In addition, an infinite-dimensional analogue of the Lauricella function F_B is defined and a relation to the moments of multivariate complex B-splines is presented.

AMS Subject Classification: 41A15, 26A33, 41A63, 33C65

Keywords: Complex B-spline, Dirichlet average, Weyl fractional derivative and integral operator, R- hypergeometric function, moments, Appell functions, Lauricella functions

1 Introduction

Recently, a generalization of Schoenberg's polynomial splines to complex orders z with $\operatorname{Re} z > 1$ was introduced in [7]. These so-called complex B-splines $B_z : \mathbb{R} \to \mathbb{C}$ are defined in the Fourier domain by

$$\mathcal{F}(B_z)(\omega) =: \widehat{B_z}(\omega) := \int_{\mathbb{R}} B_z(t) e^{-i\omega t}\, dt = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z, \tag{1.1}$$

for $\operatorname{Re} z > 1$. Here $\mathcal{F}$ denotes the Fourier-Plancherel transform. At the origin, there exists the continuous continuation $\widehat{B_z}(0) = 1$. Note that since $\left\{\frac{1-e^{-i\omega}}{i\omega} \;\middle|\; \omega \in \mathbb{R}\right\} \cap \{y \in \mathbb{R} \mid y < 0\} = \emptyset$, complex B-splines reside on the main branch of the complex logarithm and are thus well-defined.

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Complex B-splines possess several interesting basic properties, which are discussed in [7]. In the following, we summarize the most important ones for our purposes.

Fourier inversion of (1.1) shows that complex B-splines are piecewise polynomials of complex degree. More precisely, the following result holds. (See [7] for the proof.)

Proposition 1.1. *Complex B-splines have a time-domain representation of the form*

$$B_z(t) = \frac{1}{\Gamma(z)} \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} (t-k)_+^{z-1}, \tag{1.2}$$

where the above sum exists pointwise for all $t \in \mathbb{R}$ and in $L^2(\mathbb{R})$-norm. Here,

$$t_+^z = \begin{cases} t^z = e^{z \ln t}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases}$$

is the complex-valued truncated power function, and $\Gamma : \mathbb{C} \setminus \mathbb{Z}_0^- \to \mathbb{C}$ denotes the Euler Gamma function, where $\mathbb{Z}_0^- := \{n \in \mathbb{Z} \mid n \le 0\}$.

Remark 1.2. For real $z > 0$, the function $z \mapsto \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z$ and its time domain representation (1.2) were already investigated in [26] in connection with fractional powers of operators and later also in [24] in the context of extending Schoenberg's polynomial splines to real orders. In the former, a proof that this function is in $L^1(0,\infty)$ was given using arguments from summability theory (cf. Lemma 2 in [26]), and in the latter the same result was shown but with a different proof. In addition, it was proved in [24] that for real $z > 0$, $z \mapsto \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z$ is in $L^2(\mathbb{R})$ for $z > 1/2$ (using our notation). (Cf. Theorem 3.2 in [24].)

Equation (1.2) shows that B_z has, in general, non-compact support contained in $[0,\infty)$. It was also shown in [7] that complex B-splines are elements of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and, due to their decay in frequency domain induced by the polynomial ω^z in the denominator of (1.1), belong to the Sobolev spaces $W_2^r(\mathbb{R})$ (with respect to the L^2-Norm and with weight $(1+|x|^2)^r$) for $r < \operatorname{Re} z - \frac{1}{2}$. The smoothness of their Fourier transform yields a fast decay in time domain:

$$B_z(x) = O(x^{-m}), \quad \text{for } \mathbb{N} \ni m < \operatorname{Re} z + 1, \text{ as } x \to \infty. \tag{1.3}$$

Remark 1.3. Prior to [7], the asymptotic behavior (1.3) of the function $z \mapsto \left(\frac{1-e^{-i\omega}}{i\omega}\right)^z$ for real $z > 1$ was already shown in [2], (Proposition 3.1), to be of order $O(x^{-z-1})$, as $x \to \infty$. The same estimate was proven later in [24], (Theorem 3.1), for real $z > 0$. As we are more interested in the approximation-theoretic aspects of complex B-splines, we restrict our attention to the case $\operatorname{Re} z > 1$, which yields continuous functions.

If $\operatorname{Re} z, \operatorname{Re} z_1, \operatorname{Re} z_2 > 1$, then the convolution relation $B_{z_1} * B_{z_2} = B_{z_1+z_2}$ and the recursion relation

$$B_z(x) = \frac{x}{z-1} B_{z-1}(x) + \frac{z-x}{z-1} B_{z-1}(x-1)$$

hold. Complex B-splines are scaling functions and generate multiresolution analyses of $L^2(\mathbb{R})$ and wavelets. Furthermore, they relate difference and differential operators. For more details and proofs, we refer the interested reader to [7, 9, 8, 10, 16].

Unlike the classical cardinal B-splines, complex B-splines B_z possess an additional modulation and phase factor in the frequency domain:

$$\widehat{B_z}(\omega) = \widehat{B}_{\mathrm{Re}\,z}(\omega)\, e^{i\,\mathrm{Im}\,z \ln|\Omega(\omega)|}\, e^{-\mathrm{Im}\,z \arg \Omega(\omega)},$$

where $\Omega(\omega) := (1 - e^{-i\omega})/(i\omega)$. The existence of these two factors allows the extraction of additional information from sampled data and the manipulation of images. Phase information ($e^{i\,\mathrm{Im}\,z \ln|\Omega(\omega)|}$) and an adjustable smoothness parameter, namely $\mathrm{Re}\,z$, are already built into their definition. Thus, they define a *continuous* family, with respect to smoothness, of approximation spaces, and allow to incorporate phase information for single band frequency analysis [7, 10].

In [8] and [16], some further properties of complex B-splines were investigated. In particular, connections between complex derivatives of Riemann-Liouville or Weyl type and Dirichlet averages were exhibited. Whereas in [8] the emphasis was on univariate complex B-splines and their applications to statistical processes, multivariate complex B-splines were defined in [16] using a well-known geometric formula for classical multivariate B-splines [11, 17]. It was also shown that Dirichlet averages are especially well-suited to explore the properties of multivariate complex B-splines. Using Dirichlet averages, several classical multivariate B-spline identities were generalized to the complex setting. There also exist interesting relationships between complex B-splines, Dirichlet averages and difference operators, several of which are highlighted in [9].

In this paper, which is based on a short communiction [15], we present a generalization of some results found in [5, 19] to complex B-splines. For this purpose, the concept of double Dirichlet average [3] needs to be introduced and its definition extended via projective limits to an infinite-dimensional setting suitable for complex B-splines. Moments of complex B-splines are defined and a formula for their computation in terms of a special double Dirichlet average presented. Extending the representation of a Lauricella F_B function by Carlson's R-hypergeometric function [3] to the infinite-dimensional setting, we define an infinite-dimensional analogue F_B^∞ of F_B and present an identity relating F_B^∞ to the moments of multivariate complex B-splines.

2 Complex B-Splines

Let $n \in \mathbb{N}$ and let $\triangle^n$ denote the standard n-simplex in $\mathbb{R}^{n+1}$:

$$\triangle^n := \left\{ u := (u_0, \ldots, u_n) \in \mathbb{R}^{n+1} \;\middle|\; u_j \geq 0;\ j = 0, 1, \ldots, n;\ \sum_{j=0}^{n} u_j = 1 \right\}.$$

Note, that the set $\triangle_0^n := \{u \in \mathbb{R}^n \mid u_j \geq 0;\ j = 1, \ldots, n;\ \sum_{j=1}^n u_j \leq 1\}$, can be identified via the bijection

$$\triangle_0^n \to \triangle^n, \qquad (u_1, \ldots, u_n) \mapsto \left(1 - \sum_{i=1}^{n} u_i, u_1, \ldots, u_n\right),$$

with $\triangle^n$. When convenient, we will employ this identification.

The extension of $\triangle^n$ to infinite dimensions is done via projective limits. The resulting infinite-dimensional standard simplex is given by

$$\triangle^\infty := \left\{ u := (u_j)_{j\in(\mathbb{R}_0^+)^{\mathbb{N}_0}} \;\middle|\; \sum_{j=0}^{\infty} u_j = 1 \right\},$$

and endowed with the topology of pointwise convergence, i.e., the weak$*$-topology. We denote by $\mu_b = \varprojlim \mu_b^n$ the projective limit of *Dirichlet measures* μ_b^n on the n-dimensional standard simplex $\triangle^n$ with density

$$\frac{\Gamma(b_0)\cdots\Gamma(b_n)}{\Gamma(b_0+\cdots+b_n)} u_0^{b_0-1} u_1^{b_1-1}\cdots u_n^{b_n-1}, \tag{2.1}$$

where $b_0,\ldots,b_n \in \mathbb{C}$ with $\operatorname{Re} b_j > 0$, $j = 0,1,\ldots,n$. Note that by the Kolmogorov Extension Theorem (see, for instance, [23]), this measure μ_b exists.

Below, we will use the following notation: $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Definition 2.1 ([8])**.** Given a weight vector $b \in (\mathbb{C}^+)^{\mathbb{N}_0}$ and an increasing knot sequence $\tau := \{\tau_k\}_k \in \mathbb{R}^{\mathbb{N}_0}$ with the property that $\lim_{k\to\infty} \sqrt[k]{\tau_k} \leq \varrho$, for some $\varrho \in [0,e)$, a complex B-spline $B_z(\bullet \mid b;\tau)$ of order z, $\operatorname{Re} z > 1$, with weight vector b and knot sequence τ is a function satisfying

$$\int_{\mathbb{R}} B_z(t \mid b;\tau) g^{(z)}(t)\, dt = \int_{\triangle^\infty} g^{(z)}(\tau\cdot u)\, d\mu_b(u) \tag{2.2}$$

for all $g \in \mathcal{S}(\mathbb{R})$.

Remark 2.2. We may assume, without loss of generality, that the knot sequence τ is such that $\tau_0 = 0$.

Here, $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions on $\mathbb{R}$, and

$$\tau\cdot u = \sum_{k\in\mathbb{N}_0} \tau_k u_k, \quad \text{for } u = \{u_k\}_{k\in\mathbb{N}_0} \in \triangle^\infty.$$

In addition, we use the Weyl or Riemann-Liouville fractional derivative [13, 18, 22] of complex order z, $\operatorname{Re} z > 0$, $W^z : \mathcal{S}(\mathbb{R}_0^+) \to \mathcal{S}(\mathbb{R}_0^+)$, defined by

$$(W^z f)(x) := \frac{(-1)^n}{\Gamma(\nu)} \frac{d^n}{dx^n} \int_0^\infty (t-x)_+^{\nu-1} f(t)\, dt,$$

with $n = \lceil \operatorname{Re} z \rceil$, and $\nu = n - z$. Here, $\mathcal{S}(\mathbb{R}_0^+)$ denotes the space of Schwartz functions restricted to $\mathbb{R}_0^+$, and $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$, $x \mapsto \min\{n \in \mathbb{Z} \mid n \geq x\}$, the *ceiling function*.

The inverse operator of W^z, is the Weyl integral of complex order z, given by

$$W^{-z} f = \frac{1}{\Gamma(z)} \int_\bullet^\infty (t-\bullet)_+^{z-1} f(t)\, dt.$$

To simplify notation, we write $f^{(z)}$ for $W^z f$ and $f^{(-z)}$ for $W^{-z} f$.

Remark 2.3. Note that both W^z and W^{-z} are linear operators mapping $\mathcal{S}(\mathbb{R}_0^+)$ into itself [18, 22]. As the space $C^\omega(\mathbb{R}_0^+)$ of real-analytic functions on $\mathbb{R}_0^+$ is dense in $\mathcal{D}(\mathbb{R}_0^+)$, the space of compactly supported C^∞-functions on $\mathbb{R}_0^+$, (see, for instance, [20], p. 780) , (2.2) holds for all $g \in \mathcal{S}(\mathbb{R}_0^+)$ since $\mathcal{D}(\mathbb{R}_0^+)$ is dense in $\mathcal{S}(\mathbb{R}_0^+)$. Moreover, since $\mathcal{S}(\mathbb{R}_0^+)$ is dense in $L^2(\mathbb{R}_0^+)$, we deduce that $B_z(\bullet \mid b, \tau) \in L^2(\mathbb{R}_0^+)$.

Remark 2.4. For finite $\tau = \{\tau_0, \tau_1, \ldots, \tau_n\} \in (\mathbb{R}_0^+)^{n+1}$ and finite $b = \{b_0, b_1, \ldots, b_n\} \in (\mathbb{R}^+)^n$, $n \in \mathbb{N}$, and $z := n \in \mathbb{N}$, Eq. (2.2) defines also *Dirichlet splines*. (Cf. [6], where these splines were first introduced.) Recall that a Dirichlet spline $D_n(\bullet | b; \tau)$ of order n is that function for which the equality

$$\int_{\mathbb{R}} g^{(n)}(t)\, D_n(t|b;\tau)\, dt = \int_{\triangle^n} g^{(n)}(\tau \cdot u)\, d\mu_b(u), \tag{2.3}$$

holds for all $g \in C^n(\mathbb{R})$. Hence, (2.3) also holds for $g \in \mathcal{S}(\mathbb{R})$.

To define a multivariate analogue of univariate complex B-splines, we proceed as follows. Let $\lambda \in \mathbb{R}^s \setminus \{0\}$, $s \in \mathbb{N}$, be a direction, and let $g : \mathbb{R} \to \mathbb{C}$ be a function. The *ridge function* g_λ corresponding to g is defined as the function $\mathbb{R}^s \to \mathbb{C}$ with

$$g_\lambda(x) := g(\langle \lambda, x \rangle), \quad \text{for all } x \in \mathbb{R}^s.$$

We denote the canonical inner product in $\mathbb{R}^s$ by $\langle \bullet, \bullet \rangle$ and the norm induced by it by $\|\bullet\|$.

Definition 2.5 ([16])**.** Let $\tau = \{\tau^n\}_{n\in\mathbb{N}_0} \in (\mathbb{R}^s)^{\mathbb{N}_0}$ be a sequence of knots in $\mathbb{R}^s$ with the property that

$$\exists \varrho \in [0, e) : \limsup_{n\to\infty} \sqrt[n]{\|\tau^n\|} \le \varrho. \tag{2.4}$$

The multivariate complex B-spline $\boldsymbol{B}_z(\bullet \mid b, \tau) : \mathbb{R}^s \to \mathbb{C}$ of order z, $\operatorname{Re} z > 1$, with weight vector $b \in (\mathbb{C}^+)^{\mathbb{N}_0}$ and knot sequence τ is defined by means of the identity

$$\int_{\mathbb{R}^s} g(\langle \lambda, x \rangle) \boldsymbol{B}_z(x \mid b, \tau)\, dx = \int_{\mathbb{R}} g(t) B_z(t \mid b, \lambda\tau)\, dt, \tag{2.5}$$

where $g \in \mathcal{S}(\mathbb{R})$, and where $\lambda \in \mathbb{R}^s \setminus \{0\}$ such that $\lambda\tau := \{\langle \lambda, \tau^n \rangle\}_{n\in\mathbb{N}_0}$ is separated, i.e., there exists a $\delta > 0$, so that $\inf\{|\langle \lambda, \tau^n \rangle - \langle \lambda, \tau^m \rangle| \mid m, n \in \mathbb{N}_0\} \ge \delta$.

Remark 2.6. Since ridge functions are dense in $L^2(\mathbb{R}^s)$ (see, for instance, [21]), we conclude that $\boldsymbol{B}_z(\bullet \mid b, \tau) \in L^2((\mathbb{R}_0^+)^s)$. Moreover, it follows from the Hermite-Genocchi formula for the univariate complex B-splines $B_z(\bullet | b, \lambda\tau)$ and (2.5), that

$$\boldsymbol{B}_z(x | b, \tau) = 0, \quad \text{when } x \notin [\tau],$$

where $[\tau]$ denotes the convex hull of τ.

3 Dirichlet Averages

Let Ω be a non-empty open convex set in $\mathbb{C}^s$, $s \in \mathbb{N}$, and let $b \in (\mathbb{C}^+)^{\mathbb{N}_0}$. Let $f \in \mathcal{S}(\Omega) := \mathcal{S}(\Omega, \mathbb{C})$, the Schwartz space of complex-valued functions on Ω, be a measurable function.

For $\tau \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ and $u \in \triangle^\infty$, define $\tau \cdot u$ to be the bilinear mapping $(\tau, u) \mapsto \sum_{i=1}^{\infty} u_i \tau^i$. The infinite sum exists if there exists a $\varrho \in [0, e)$ so that

$$\limsup_{n\to\infty} \sqrt[n]{\|\tau^n\|} \leq \varrho. \tag{3.1}$$

Here, $\|\bullet\|$ now denotes the canonical Euclidean norm on $\mathbb{C}^s$. (See also [8].)

Definition 3.1. Let $f : \Omega \subset \mathbb{C}^s \to \mathbb{C}$ be a measurable function. The Dirichlet average $F : (\mathbb{C}^+)^{\mathbb{N}_0} \times \Omega^{\mathbb{N}_0} \to \mathbb{C}$ over $\triangle^\infty$ is defined by

$$F(b;\tau) := \int_{\triangle^\infty} f(\tau \cdot u)\, d\mu_b(u),$$

where $\mu_b = \varprojlim \mu_b^n$ is the projective limit of Dirichlet measures on the n-dimensional standard simplex $\triangle^n$.

We remark that the Dirichlet average is holomorphic in $b \in (\mathbb{C}^+)^{\mathbb{N}_0}$ when $f \in C(\Omega, \mathbb{C})$ for every fixed $\tau \in \Omega^{\mathbb{N}_0}$. (See [4] for the finite-dimensional case and [16] for the infinite-dimensional setting.)

Definition 3.2. [3] Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be continuous. Let $b \in (\mathbb{C}^+)^{k+1}$ and $\beta \in (\mathbb{C}^+)^{\varkappa+1}$. Suppose that for fixed $k, \varkappa \in \mathbb{N}$, $X \in \mathbb{C}^{(k+1)\times(\varkappa+1)}$, and that the convex hull $[X]$ of X is contained in Ω. Then the double Dirichlet average of f is defined by

$$\mathscr{F}(b; X; \beta) := \int_{\triangle^k} \int_{\triangle^\varkappa} f(u \cdot Xv) d\mu_b^k(u) d\nu_\beta^\varkappa(v),$$

where $u \cdot Xv := \sum_{i=0}^{k} \sum_{j=0}^{\varkappa} u_i X_{ij} v_j$ and $\sum_{i=0}^{k} u_i = 1 = \sum_{j=0}^{\varkappa} v_j$.

We remark that $\mathscr{F}(b; X; \beta)$ is holomorphic on Ω in the elements of b, β, and X ([3]).

We again use projective limits to extend the notion of double Dirichlet average to an infinite-dimenional setting. To this end, let $u, v \in \triangle^\infty$ and let $\mu_b = \varprojlim \mu_b^n$ and $\nu_\beta = \varprojlim \nu_\beta^n$ be the projective limits of Dirichlet measures μ_b^n and ν_β^n of the form (2.1) on the n-dimensional standard simplex, where $b, \beta \in (\mathbb{C}^+)^{\mathbb{N}_0}$.

Now suppose that $X \in \mathbb{C}^{\mathbb{N}_0 \times \mathbb{N}_0}$ is a infinite matrix with the property that $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |X_{ij}|$ converges. Let

$$u \cdot Xv := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_i X_{ij} v_j.$$

Here, we have $\sum_{i=0}^{\infty} u_i = 1 = \sum_{j=0}^{\infty} v_j$.

Suppose that $\Omega \subset \mathbb{C}$ contains the convex hull $[X]$ of X and that $f : \Omega \to \mathbb{C}$ is continuous. The double Dirichlet average of f over $\triangle^\infty$ is then given by

$$\mathscr{F}(b; X; \beta) := \int_{\triangle^\infty} \int_{\triangle^\infty} f(u \cdot Xv) d\mu_b(u) d\nu_\beta(v). \tag{3.2}$$

(In order to ease notation, we use the same symbol for the (double) Dirichlet average over $\triangle^\infty$ and its finite-dimensional projections $\triangle^n$.) It is easy to show that

$$\mathscr{F}(b;X;\beta) = \int_{\triangle^\infty} F(\beta;uX)d\mu_b(u), \tag{3.3}$$

where $uX := \{\langle u, X_j\rangle\}_{j\in\mathbb{N}_0}$, with X_j denoting the j-column of X. We note that $\mathscr{F}(b;X;\beta)$ is holomorphic in the elements of b, β, and X over $\triangle^\infty$.

For $z \in \mathbb{C}^+$, we define

$$\mathscr{F}^{(z)}(b;X;\beta) := \int_{\triangle^\infty}\int_{\triangle^\infty} f^{(z)}(u\cdot Xv)d\mu_b(u)d\nu_\beta(v).$$

(See also [16] for the case of a single Dirichlet average.)

4 Double Dirichlet Averages and Complex B-Splines

Assume now that the matrix X is real-valued and of the form $X_{ij} = 0$, for $i \geq s$ and all $j \in \mathbb{N}_0$, some $s \in \mathbb{N}$. In other words, $X \in \mathbb{R}^{s\times\mathbb{N}_0}$.

Theorem 4.1. *Suppose that* $\beta \in (\mathbb{R}^+)^\infty$ *and that* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 1$. *Let* $b := (b_0, b_1, \ldots, b_{s-1}) \in \mathbb{R}^s$ *be such that* $\sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. *Assume that* $f \in \mathcal{S}(\mathbb{R}_0^+)$. *Further assume that uX is separated for all* $u \in \triangle^{s-1}$. *Then*

$$\mathscr{F}^{(z)}(b;X;\beta) = \int_{\mathbb{R}^s} \boldsymbol{B}_z(x\,|\,\beta, uX)\, F^{(z)}(b;x)dx.$$

Proof. We prove the formula first for $b \in (\mathbb{R}^+)^s$. To this end, we identify $u = (u_0, u_1, \ldots, u_{s-1}, 0, 0, \ldots) \in \triangle^\infty$ with $(u_0, u_1, \ldots, u_{s-1}) \in \triangle^{s-1}$. By the Hermite-Genocchi formula for complex B-splines (see [8] and to some extend [16]), we have that

$$F^{(z)}(\beta;uX) = \int_{\triangle^\infty} f^{(z)}(u'\cdot uX)\, d\mu_\beta(u') = \int_{\mathbb{R}} f^{(z)}(t) B_z(t\,|\,\beta, uX)dt.$$

Substituting this expression into (3.3) and using (2.5) yields

$$\mathscr{F}^{(z)}(b;X;\beta) = \int_{\triangle^\infty}\int_{\mathbb{R}^s} f^{(z)}(\langle u, x\rangle)\boldsymbol{B}_z(x\,|\,\beta, uX)\, dx\, d\mu_b(u).$$

Interchanging the order of integration, which is justified by the Lebesgue Dominated Convergence Theorem, proves the statement for $b \in (\mathbb{R}^+)^s$. To obtain the general case $b \in \mathbb{R}^s$, we note that by Theorem 6.3-7 in [4], the Dirichlet average F can be holomorphically continued in the b-parameters provided that $\sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. □

Remark 4.2. Theorem 4.1 extends Theorem 6.1 in [19] to complex B-splines and the $\triangle^\infty$-setting.

5 Moments of Complex B-Splines

Following [4], we define the R-hypergeometric function $R_a(b;\tau) : (\mathbb{R}^+)^s \times \Omega^s \to \mathbb{C}$ by

$$R_a(b;\tau) := \int_{\triangle^{s-1}} (\tau \cdot u)^a d\mu_b^{s-1}(u), \tag{5.1}$$

where $\Omega := H$, H a half-plane in $\mathbb{C}\setminus\{0\}$, if $a \in \mathbb{C}\setminus\mathbb{N}$, and $\Omega := \mathbb{C}$, if $a \in \mathbb{N}$. It can be shown (see [4]) that R_{-a}, $a \in \mathbb{C}^+$, has a holomorphic continuation in τ to $\mathbb{C}_0$, where $\mathbb{C}_0 := \{\zeta \in \mathbb{C} \mid -\pi < \arg\zeta < \pi\}$.

Taking in the definition of the double Dirichlet average (3.2) for f the real-valued function $t \mapsto t^{-c}$, where $c := \sum_{i=0}^{s-1} b_i$, the resulting double Dirichlet average is denoted by $\mathscr{R}_{-c}(b;X;\beta)$ and generalizes power functions. The corresponding single Dirichlet average $R_{-c}(b;x)$, where $x = (x_0,\ldots,x_{s-1})$, is given by

$$R_{-c}(b;x) = \prod_{i=0}^{s-1} x_i^{-b_i}, \quad x \notin [X]. \tag{5.2}$$

(See [4], (6.6-5).)

Definition 5.1. Let $p = (p_0, p_1, \ldots, p_{s-1}) \in \mathbb{R}^s$, $s \in \mathbb{N}$, be a multi-index with the property that $p_i < -\frac{1}{2}$, for all $i = 1,\ldots,s$. The moment $\mathsf{M}_p(\beta;X;z) := \mathsf{M}_p((\boldsymbol{B}_z(\bullet \mid \beta, X))$ of order $p := \sum_{i=1}^{s} p_i$ of the complex B-spline $\boldsymbol{B}_z(\bullet \mid \beta, X)$ is defined by

$$\mathsf{M}_p(\beta;X;z) := \int_{\mathbb{R}^s} x^p \boldsymbol{B}_z(x \mid \beta, X)\, dx. \tag{5.3}$$

Note that since $\boldsymbol{B}_z(\bullet \mid \beta, X) \in L^2((\mathbb{R}^+)^s)$ and $\boldsymbol{B}_z(\bullet \mid \beta, X) = 0$, for $x \notin [X]$, an easy application of the Cauchy-Schwartz inequality shows that the above integral exists provided the multi-index p satisfies the afore-mentioned condition on its components.

Using a result from [13], namely Property 2.5 (b), and requiring that $\operatorname{Re} z < \operatorname{Re} c$, we substitute the function $f := \frac{\Gamma(c-z)}{\Gamma(c)} (\bullet)^{-(c-z)}$ into (5.1) to obtain

$$R^{(z)}_{-(c-z)}(b;x) = R_{-c}(b;x) = \prod_{i=0}^{s-1} x_i^{b_i}.$$

The above considerations together with Theorem 4.1 immediately yield the next result.

Corollary 5.2. *Suppose that* $\beta \in (\mathbb{R}^+)^\infty$ *and that* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 1$. *Let* $b := (b_0, b_1, \ldots, b_{s-1}) \in (-\infty, -\frac{1}{2})^s$ *be such that* $c := \sum_{i=0}^{s-1} b_i \notin -\mathbb{N}_0$. *Moreover, suppose that* $\operatorname{Re} z < \operatorname{Re} c$. *Then*

$$\mathsf{M}_{-c}(\beta;X;z) = \mathscr{R}^{(z)}_{-(c-z)}(b;X;\beta). \tag{5.4}$$

Remark 5.3. Corollary 5.2 extends Corollary 6.2 in [19] to the infinite dimensional case and complex order setting.

6 Complex B-splines and Lauricella Functions

We briefly review some properties of the Lauricella function F_B, which are important for the purposes of this section and the relationship to complex B-splines and Dirichlet averages.

The Lauricella function $F_B : \mathbb{R}^n \to \mathbb{C}$ (cf. [1, 14]) is defined by the infinite series

$$F_B(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\gamma;x_1,\ldots,x_n) := \sum_{m_1,\ldots,m_n\in\mathbb{N}_0} \frac{(\alpha_1)_{m_1}\cdots(\alpha_n)_{m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma+m_1+\cdots+m_n)m_1!\cdots m_n!}\, x_1^{m_1}\cdots x_n^{m_n},$$

where the parameters $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots\beta_n$ and γ are elements of $\mathbb{C}$, and $(z)_n$ is the Pochhammer symbol, given by

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(n)}, \quad n\in\mathbb{N},\ z\in\mathbb{C}\setminus\mathbb{Z}_0.$$

The region of convergence for F_B is the interior of the n-cube $W^n := [-1,+1]^n \subset \mathbb{R}^n$, $n\in\mathbb{N}$.

Remark 6.1. For $n := 2$, the Lauricella function F_B becomes the Appell function F_2, and for $n := 1$ Gauß's hypergeometric ${}_2F_1$ function.

Remark 6.2. There are three other Lauricella functions, F_A, F_C, and F_D, defined in a similar fashion and with different regions of convergence. For our intentions, however, in particular in light of Euler-type integral representations, we will deal exclusively with F_B in this article.

Remark 6.3. For a connection between Dirichlet averages, the Lauricella function F_D, and the generalized Mittag-Leffler function $E^\gamma_{\alpha,\delta}$, defined by

$$E^\gamma_{\alpha,\delta}(z) := \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\alpha k+\delta)\,k!}\, z^k,$$

we refer the interested reader to [12].

Using multi-index notation with $\alpha := (\alpha_1,\ldots,\alpha_n)$, $\beta := (\beta_1,\ldots,\beta_n)$, $v := (v_1,\ldots,v_n)$, and $x := (x_1,\ldots,x_n)$, we can express the Euler-type integration representation of the Lauricella function F_B on the simplex $\triangle^n_0$ found in [14] in the following form:

$$\begin{aligned} F_B(\alpha,\beta,\gamma;x) &:= \frac{1}{B(\alpha,\gamma-|\alpha|)}\int_{\triangle^n_0} v^{\alpha-1}(1-|v|)^{\gamma-|\alpha|}(1-vx)^{-\beta}dv \\ &= \int_{\triangle^n_0}(1-vx)^{-\beta}d\mu^n_{(\alpha,\gamma-|\alpha|)}(v). \end{aligned} \tag{6.1}$$

Here, we set $vx := (v_1x_1,\ldots,v_nx_n)$ and denoted by B the $n+1$-dimensional Beta function:

$$B(\alpha,\gamma-|\alpha|) := \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\Gamma(\gamma-\alpha_1-\cdots\alpha_n)}{\Gamma(\gamma)}.$$

As usual, $|\alpha|$ denotes the length of a multi-index α.

Note that, following [3], but using a different matrix Z, which is more amenable to a generalization to infinite dimensions, we may write (6.1) in the form

$$\int_{\triangle_0^n}(1-vx)^{-\beta}d\mu^n_{(\alpha,\gamma-|\alpha|)}(v) = \mathscr{R}_{-\gamma}(\gamma-|\beta|,\beta;Z;\gamma-|\alpha|,\alpha),$$

with

$$Z := Z^{n+1} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-x_1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1-x_{n-1} & 1 \\ 1 & 1 & 1 & 1 & 1-x_n \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{6.2}$$

To obtain the above identity, we used that $\sum_{i=0}^{n} v_i = 1$,

$$\mathscr{R}_{-\gamma}(b;Z;\beta) = \int_{\triangle^n}\prod_{i=0}^{n}({}_iZv)^{-b_i}d\mu^n_\beta(v), \quad \sum_{i=0}^{n} b_i = \gamma = \sum_{i=0}^{n}\beta_i,$$

and introduced the factor $1^{\gamma-|\beta|}$ in front of $(1-vx)^{-\beta}$. We chose the (immaterial) exponent of 1 so that the multi-indices $(\gamma-|\beta|,\beta)$ and $(\gamma-|\alpha|,\alpha)$ have the same length, namely, γ, (See also [3].), and denoted by ${}_iZ$ the $(i+1)$-st row of the matrix Z, $i = 0,1,\ldots,n$. Thus, we have

$$F_B(\alpha,\beta,\gamma;x) = \mathscr{R}_{-\gamma}(\gamma-|\beta|,\beta;Z;\gamma-|\alpha|,\alpha),$$

where $Z \in \mathbb{R}^{(n+1)\times(n+1)}$ is given by (6.2).

The form of the matrix Z now lends itself to an extension of the above concepts to infinite dimensions. We define

$$Z^\infty = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots \\ 1 & 1-x_1 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \cdots \\ 1 & 1 & 1 & 1-x_{n-1} & 1 & \cdots \\ 1 & 1 & 1 & 1 & 1-x_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{R}^{\infty\times\infty}, \tag{6.3}$$

where $|x_n| < 1$, for all $n \in \mathbb{N}$, and note that the finite sections of Z^∞ are of the form (6.2), so that one may represent Z^∞ as a projective limit of the matrices $Z^{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}$, $n \in \mathbb{N}$ of the form (6.2). Similarly, one has $\mathbb{R}^{\infty\times\infty} = \varprojlim \mathbb{R}^{(n+1)\times(n+1)}$ in the sense of matrix rings.

As $|x_i| < 1$, for all $i \in \mathbb{N}$, and $\sum_{j=0}^{\infty} v_j = 1$, we obtain, using a computation in [25], the convergence of the infinite product $\prod_{i=0}^{\infty}({}_iZ^\infty v)^{-b_i}$ for $\operatorname{Re} b_i > 0$. Thus, $\mathscr{R}_{-\gamma}(b;Z;\beta)$ may be extended to the infinite-dimensional simplex $\triangle^\infty$ by a projective limit procedure. For the sake of notational simplicity, we denote this extension again by $\mathscr{R}_{-\gamma}(b;Z;\beta)$.

Note that this extension allows the definition of an infinite-dimensional Lauricelli function F_B^∞:

$$F_B^\infty(\alpha,\beta,\gamma;x) := \mathscr{R}_{-\gamma}(\gamma-|\beta|,\beta;Z^\infty;\gamma-|\alpha|,\alpha), \tag{6.4}$$

where $Z \in \mathbb{R}^{\infty\times\infty}$ is given by (6.3). Here the parameters α,β are elements of $\mathbb{C}^\infty$, the projective limit of $\mathbb{C}^n$, and $\operatorname{Re}\beta > 0$, in the sense of multi-indices. We remark, that F_B^∞ converges in the interior of the infinite-dimensional cube $W^\infty := \prod_{n=1}^{\infty}[-1,1]^n$, endowed with the weak*-topology, i.e., the topology of pointwise convergence.

Combining Eqns. (5.4) and (6.4), we obtain an identity between the moments of complex B-splines and the infinite-dimensional Lauricella function F_B^∞, namely,

$$(F_B^\infty)^{(z)}(\alpha,\beta,\gamma;x) = \mathsf{M}_{-\gamma}(\gamma-|\alpha|,\alpha;Z^\infty;\gamma-|\beta|,\beta), \tag{6.5}$$

where the z-th fractional derivative of F_B^∞ exists by the above identity (6.4).

Eqn. (6.5) is an extension of Corollary 6.4 in [19] to the infinite-dimensional setting involving multivariate complex B-splines of order z, $\operatorname{Re} z > 1$.

7 Summary

We employed the natural infinite-dimensional setting for multivariate complex B-splines to extend the concept of double Dirichlet averages. As a result of this extension, we obtained in the following results.

- The moments of multivariate complex B-splines were defined.
- A formula for the moments of multivariate complex B-splines in terms of double Dirichlet averages associated with the infinite-dimensional analogue of Carlson's hypergeometric R-function was derived.
- Employing an Euler-type integral representation, an infinite-dimensional analogue of Lauricella's F_B-function was obtained and related to the double Dirichlet average of Carlson's R-hypergeometric function on the infinite-dimensional simplex $\triangle^\infty$.
- An identity between the infinite-dimensional extension of Lauricella's F_B-function and the moments of multivariate complex B-splines was presented.

The results presented in this article generalize those given in [19] to infinite dimensions and splines of complex order.

Acknowledgments

The author thanks the referee for the careful reading of the manuscript and helpful comments.

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