

THE SHARPNESS OF CONDITION FOR SOLVING THE JUMP PROBLEM

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(Communicated by Palle Jorgensen)

Abstract

Let γ be a non-rectifiable closed Jordan curve in \mathbb{C} , which is merely assumed to be d -summable ($1 < d < 2$) in the sense of Harrison and Norton [7]. We are interested in the so-called jump problem over γ , which is that of finding an analytic function in \mathbb{C} having a prescribed jump across the curve.

The goal of this note is to show that the sufficient solvability condition of the jump problem given by $\nu > \frac{d}{2}$, being the jump function defined in γ and satisfying a Hölder condition with exponent ν , $0 < \nu \leq 1$, cannot be weakened on the whole class of d -summable curves.

AMS Subject Classification: 30E25, 28A80.

Keywords: Non-rectifiable curve, Fractional dimension, jump problem, Cauchy integral, analytic functions.

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1 Introduction

Let γ be a closed Jordan curve in \mathbb{C} and suppose that there is given a continuous function f on γ . The jump problem determined by the pair (γ, f) consists in finding a function $\Phi(z)$ with the following properties:

- $\Phi(z)$ is analytic for $z \in \overline{\mathbb{C}} \setminus \gamma$,
- The usual continuous limit values $\Phi^+(t)$ and $\Phi^-(t)$ of the desired function Φ from the plus and minus side of γ are related for $t \in \gamma$ by the boundary condition:

$$\Phi^+(t) - \Phi^-(t) = f(t). \quad (1.1)$$

- $\Phi(z)$ tends to 0 as $z \rightarrow \infty$.

During the last decades, several attempts were made on the non-trivial question concerning the minimal restrictions on the pair (γ, f) for solving the jump problem (1.1). The interest in this problem is based on its theoretical importance and the implications for applications in many branches of contemporary mathematics and physics.

The solution of (1.1) over smooth curves, when f satisfies a Hölder condition with exponent $\nu \in (0, 1]$ is obviously related to the boundary properties of the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau. \quad (1.2)$$

The major point established is that this solution is uniquely determined precisely by (1.2), which is clear after using the Plemelj-Sokhotskii formula. For a deeper discussion of this area of complex analysis we refer the reader to the classical Gakhov's monograph [5].

Subsequent developments put the study of the above-mentioned close connection in the context of non-smooth rectifiable curves. This was essentially worked out in the 1979's by Dynkin [3] and Salimov [11]. Since then, (1.2) has boundary values from both side of γ if f satisfies a Hölder condition with exponent $\nu \in (\frac{1}{2}, 1]$ which establishes simultaneously the solvability of the jump problem. Moreover, the bound for the Hölder exponent cannot be improved on the whole class of rectifiable curves.

In [8] (see also [9]), Kats proved that the jump problem on non-rectifiable curves is solvable whenever f satisfies a Hölder condition with exponent ν such that

$$\nu > \frac{\dim \gamma}{2}, \quad (1.3)$$

where $\dim \gamma$ denotes the upper metric dimension of γ . Moreover, assumption (1.3) cannot be relaxed on the class of curves of fixed upper metric dimension.

In [1, 10] solvability conditions of the jump problem, but needing a new dimensional metric characteristic and some possible formulation of curvilinear Cauchy integration, was reexamined and extended. The main result here shows that the solvability condition with upper metric dimension replaced by the new one is weaker than (1.3). As an application, solvability conditions of the Riemann boundary value problem are derived under weaker restrictions on the boundary. Besides the complex case the consideration can be extended to the framework of hyperanalytic functions theory (see [6] for the standard work here).

Inspired by the results in [8], recently the jump problem has been studied in case of d -summable ($1 < d < 2$) curves (to be defined later), but regarding the solution to be a hyperanalytic function. In particular, it is proved by Theorem 1 in [2] that if

$$\nu > \frac{d}{2}, \quad (1.4)$$

then the jump problem is solvable for any d -summable curve and any f satisfying the Hölder condition with exponent ν .

What is still lacking is the sharpness of (1.4) on the class of d -summable curves and it is the question we shall be concerned in this work. For this to be proved, it is worth noting that the curve constructed by Kats [8] cannot be used, since it is not d -summable with d being its upper metric dimension.

Before turning our attention to showing an example which leads the sharpness of condition (1.4), we would like to briefly recapitulate the minimal technical information will be needed.

2 Upper metric dimension and d -summable sets in \mathbb{R}^2

The upper metric dimension of a compact set $\mathbf{E} \subset \mathbb{R}^2$ is equal to

$$\dim \mathbf{E} := \limsup_{\varepsilon \rightarrow 0} \frac{\log N_{\mathbf{E}}(\varepsilon)}{-\log \varepsilon}, \quad (2.1)$$

where $N_{\mathbf{E}}(\varepsilon)$ stands for the least number of ε -balls needed to cover \mathbf{E} . This can be found in [4]. Note that the limit in (2.1) remains unchanged if $N_{\mathbf{E}}(\varepsilon)$ is replaced by the number $m_k(\mathbf{E})$ of k -cubes, with $2^{-k} \leq \varepsilon < 2^{-k+1}$, intersecting \mathbf{E} . For completeness we recall that a cube $Q \subset \mathbb{R}^2$ is called a k -cube if it is of the form

$$[l_1 2^{-k}, (l_1 + 1) 2^{-k}] \times [l_2 2^{-k}, (l_2 + 1) 2^{-k}]$$

where k and l_1, l_2 are integers.

One can therefore also write $\dim \mathbf{E}$ as

$$\dim \mathbf{E} := \limsup_{k \rightarrow \infty} \frac{\log m_k(\mathbf{E})}{k \log 2}.$$

The set \mathbf{E} is said to be d -summable if the improper integral

$$\int_0^\delta N_{\mathbf{E}}(t) t^{d-1} dt$$

converges, for some $\delta > 0$.

As we have mentioned, this geometric notion was introduced in [7], who showed that any d -summable set \mathbf{E} has upper metric dimension $\dim \mathbf{E} \leq d$. Moreover, the assumption $\dim \mathbf{E} < d$ implies the d -summability of \mathbf{E} .

3 Example

Fix $\beta > 1$ and set $M_n = 2^{\theta(n\beta)}$, where $\theta(p)$ denotes the integer part of the number p . The sequence of points $\{a_n^j\}_{j=0}^{M_n}$ with $a_n^0 = 2^{-n+1}$, $a_n^1 = 2^{-n+1} - \frac{2^{-n}}{M_n}$, ..., yields the division of the interval $I_n = [2^{-n}, 2^{-n+1}]$ into M_n equal parts.

Let us consider

$$T(n) = ((\ln 2(\beta - 1)n - 1) \log_2 n - \frac{2}{\ln 2}) \frac{1}{n^2 \log_2^3 n}, \quad n \geq \theta\left(\frac{6\beta}{\beta - 1}(\beta + 1)^2\right) + 1$$

and put

$$T_*(n) = 2^{\theta(\log_2 T(n))}.$$

Define the set \mathbf{E}_*^n to consist of all the vertical intervals

$$[a_n^j, a_n^j + i2^{-n}T_*(n)], \quad j = 1, \dots, M_n - 1,$$

the intervals of the real axis given by

$$[a_n^{2^j}, a_n^{2^{j-1}}], \quad j = 1, \dots, \frac{M_n}{2},$$

as well as all the horizontal intervals

$$[a_n^{2^{j+1}} + i2^{-n}T_*(n), a_n^{2^j} + i2^{-n}T_*(n)], \quad j = 1, \dots, \frac{M_n}{2} - 1.$$

Define $\mathbf{E}_* = \bigcup_{n=n_0}^{\infty} \mathbf{E}_*^n$, where $n_0 = \theta\left(\frac{6\beta}{\beta - 1}(\beta + 1)^2\right) + 1$.

Proposition 3.1. *The upper metric dimension of \mathbf{E}_* equals $\frac{2\beta}{1+\beta}$. Moreover, \mathbf{E}_* is $\frac{2\beta}{1+\beta}$ -summable.*

Sketch of the Proof.

The procedure is to give a lower and upper estimate for $m_k(\mathbf{E}_*)$. For this purpose we set $\rho_n = \frac{2^{-n}}{M_n}$ (the distance between two consecutive vertical intervals of \mathbf{E}_*). By construction we have

$$\begin{aligned} \sum_{2^{-k} < \rho_n, k > n} 2^{k-n+\theta(\log_2 T(n))+\theta(n\beta)} &\leq m_k(\mathbf{E}_*) \leq \sum_{2^{-k} < \rho_n, k > n} 2^{k-n+\theta(\log_2 T(n))+\theta(n\beta)} + \\ &+ 2 \sum_{2^{-k} < \rho_n, k > n} 2^{k-n} + \sum_{2^{-k} < \rho_n, k > n} 2^{2k-2n+\theta(\log_2 T(n))} + 2\theta(\beta - 1) + 2. \end{aligned}$$

Let

$$S_1 = \sum_{2^{-k} < \rho_n, k > n} 2^{k-n+\theta(\log_2 T(n))+\theta(n\beta)}.$$

Then we have

$$S_1 \leq 2^k \sum_{2^{-k} > \rho_n, k > n} 2^{(\beta-1)n} T(n) = 2^k \sum_{n=n_0}^{P_k} 2^{(\beta-1)n} T_1(n),$$

where P_k is the integer positive number defined by the condition

$$\frac{k}{\beta+1} - 1 < P_k \leq \frac{k}{\beta+1}$$

Consequently, a short computation shows that

$$S_1 \leq c_1 \frac{2^{(\beta-1)\frac{k}{\beta+1}} 2^k}{\frac{k}{(\beta+1)} \log_2^2 \frac{k}{\beta+1}} \leq c_2 \frac{2^{\frac{2\beta}{\beta+1}k}}{k \log_2^2 k}$$

Here and subsequently, c_1, c_2, \dots denote generic positive constants.

On the other hand

$$\begin{aligned} S_1 &\geq \sum_{2^{-k} < \rho_n, k > n} 2^{k-n+\log_2 T(n)-1+n\beta-1} = 2^{k-2} \sum_{n=n_0}^{P_k} 2^{(\beta-1)n} T(n) \geq \\ &\geq c_3 2^k \frac{2^{(\beta-1)P_k}}{P_k \log_2^2 P_k} \geq c_4 \frac{2^{(\frac{2\beta}{\beta+1})k}}{k \log_2^2 k}. \end{aligned}$$

Next

$$\begin{aligned} \sum_{2^{-k} < \rho_n, k > n} 2^{k-n} &\leq c_5 \cdot 2^k \sum_{n=n_0}^{P_k} 2^{(\beta-1)n} T(n) \leq c_6 \frac{2^{\frac{2\beta}{\beta+1}k}}{k \log_2^2 k}, \\ \sum_{2^{-k} > \rho_n, k > n} 2^{2k-2n+\theta(\log_2 T(n))} &\leq 2^{2k} \sum_{2^{-k} > \rho_n, k > n} 2^{-2n} T(n) = \\ = 2^{2k} \sum_{n=N_k+1}^k 2^{-2n} T_1(n) &\leq c_7 2^{2k} \sum_{n=N_k+1}^k 2^{-2n} \cdot \frac{1}{(P_k+1)^2 \log_2^2 (P_k+1)} \leq \\ &\leq c_8 \cdot \frac{2^{(\frac{2\beta}{\beta+1})k}}{k \log_2^2 k}, \end{aligned}$$

and finally

$$2\theta(\beta-1) + 2 < 2\beta = \frac{2^{(\frac{2\beta}{\beta+1})n_0}}{n_0 \log_2^2 n_0} \cdot \frac{n_0 \log_2^2 n_0}{2^{\frac{2\beta n_0}{\beta+1}}} \cdot 2\beta \leq c_8 \cdot \frac{2^{\frac{2\beta k}{\beta+1}}}{k \log_2^2 k}.$$

In view of the foregoing considerations we have

$$c_9 \frac{2^{\frac{2\beta k}{\beta+1}}}{k \log_2^2 k} \leq m_k(\mathbf{E}_*) \leq c_{10} \frac{2^{\frac{2\beta k}{\beta+1}}}{k \log_2^2 k}$$

or equivalently

$$c_{11} \frac{t^{-\frac{2\beta}{1+\beta}}}{\ln \frac{1}{t} \ln^2 \ln \frac{1}{t}} \leq N_{\mathbf{E}_*}(t) \leq c_{12} \frac{t^{-\frac{2\beta}{1+\beta}}}{\ln \frac{1}{t} \ln^2 \ln \frac{1}{t}}. \quad (3.1)$$

From the above it follows that

$$\dim_{\mathbf{E}_*} = \frac{2\beta}{1+\beta} - \limsup_{t \rightarrow 0} \frac{\ln \ln \frac{1}{t} \ln^2 \ln \frac{1}{t}}{\ln \frac{1}{t}} = \frac{2\beta}{1+\beta}.$$

On the other hand

$$\int_0^{\frac{1}{3}} N_{\mathbf{E}_*}(t) t^{\frac{2\beta}{1+\beta}-1} dt \leq C_2 \int_0^{\frac{1}{3}} \frac{dt}{t \ln \frac{1}{t} \ln^2 \ln \frac{1}{t}} < +\infty,$$

which implies the $\frac{2\beta}{1+\beta}$ -summability of \mathbf{E}_* .

4 Sharpness of the solvability condition (1.4)

Theorem 4.1. *For any pair of numbers ν and d subjected to the condition $0 < \nu \leq \frac{d}{2} < 1$ it is possible to construct a d -summable curve γ_* and a function f_* , which satisfies a Hölder condition with exponent ν such that the jump problem is not solvable.*

Proof:

Let $\beta = \frac{d}{2-d}$ and consider the d -summable closed Jordan curve $\gamma_* = \mathbf{E}_* \cup [0, n_0 - i] \cup [n_0 - i, n_0]$, the interior of which is denoted by Ω_* .

We follow Kats [8] in constructing the function f_* . To do this, all the points a_n^j , $j = 0, \dots, M_n$, $n = 1, 2, \dots$ (from that of Section 3) are numbered in decreasing order as $\delta_0 = a_1^0, \delta_1 = a_1^1, \dots$.

Denote by $\Delta_k = \delta_k - \delta_{k+1}$. Hence, $\Delta_0 = \dots = \Delta_{M_1-1} = \rho_1$, $\Delta_{M_1} = \dots = \Delta_{M_2-1} = \rho_2, \dots$, etc.

The series

$$\sum_{j=k}^{\infty} (-1)^j \Delta_j^\nu$$

converges because of the decreasing character of the sequence $\{\Delta_k\}_{k=0}^{\infty}$.

Define the function φ at the points δ_k as

$$\varphi(\delta_k) = \sum_{j=k}^{\infty} (-1)^j \Delta_j^\nu$$

and extend it to the whole interval $[0, 2^{-n_0+1}]$ requiring to be linear on every intervals $[\delta_{k+1}, \delta_k]$ and $\varphi(0) = 0$.

Further we set $\varphi(x+iy) = \varphi(x)$ if $z = x+iy$ lies in Ω_* , and $\varphi(z) = 0$ for $z \in \mathbb{C} \setminus (\Omega_* \cup \gamma_*)$.

It can be directly proved that φ satisfies on $[0, 2^{-n_0+1}]$ the Hölder condition with exponent ν and so on $\Omega_* \cup \gamma_*$.

Denote $\mu = (1+\beta)(1-\nu) = \frac{2(1-\nu)}{2-d}$ and $m = \theta(\mu) - 1$. Now we define the desired function to be $f_*(z) = x^m \varphi(z)$, $z = x+iy$.

An easy computation shows that f_* has the following properties:

a) $\partial_{\bar{z}} f_*$ is a bounded function away from zero and summable in the p -th power with $p > 1$.

b) There exist constants $\alpha < 2$ and $c > 0$ such that

$$\left| \int_{\Omega_*} \frac{\partial_{\bar{\xi}} f_*(\xi)}{\xi - z} d\xi d\bar{\xi} \right| \leq c|z|^{-\alpha}$$

c) There exist constants a and $b > 0$, such that

$$\Re \left(\int_{\Omega_*} \frac{\partial_{\bar{\xi}} f_*(\xi)}{\xi + x} d\xi d\bar{\xi} \right) \geq b \ln \frac{1}{x} + a, \quad 0 < x \leq 1.$$

Having disposed of this preliminary step, we can now follow in exactly similar way the arguments apply in [8] to prove that there is not solution of the jump problem determined by the pair (γ_*, f_*) . The details are left to the reader.

Acknowledgments

The final version of the paper was written when the first two authors were visiting the IMPA, Rio de Janeiro in the spring of 2011; the financial support and kind hospitality are gratefully acknowledged.

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