Communications in **M**athematical **A**nalysis
Volume 12, Number 2, pp. 1–10 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

Oscillation Criteria for Bounded Solutions for some Nonlinear Diffusion Equations via Picone-type formulae

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(Communicated by Irena Lasiecka)

Abstract

By means of Picone's type identities and inequalities some comparison results for problems related to the equation

$$\nabla.\{A(u)\nabla u\} + c(x)u + f(u) = g(x) \text{ in } \mathbb{R}^n$$

are established.Because of its principal part,this type of equation finds its applications in various physical phenomena like in nonlinear diffusion problems,flows through porous media, plasma physics,...etc. In this paper we show how versatile can the use of Picone-type formulae be for these type of quasilinear equations. Our main focus is to establish some oscillation criteria for classical non trivial and bounded solutions of some of these types of equations.We will display here some criteria conditions for some model equations.The ultimate aim is that by means of comparison methods and some Picone-type formulae,this could lead to getting the oscillation criteria of some more general equations.

AMS Subject Classification: 35J60, 35J70, 34C10

Keywords: Picone's formulae, nonlinear diffusion, oscillation of solutions, quasilinear elliptic equations

1 Introduction

In the nonlinear diffusion problems,as seen in [2],from the general form

$$-\triangle\phi(u) + f(u) = g(x),$$

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the governing equation (in steady state condition) takes the Fickian diffusion form

$$-\nabla(K(u)\nabla u) + f(u) = g(x)$$

where K,the diffusion coefficient,is any primitive of ϕ. For the slow diffusion problems like for flows through porous media, $\phi'(0) = 0$, $\phi'(t) > 0$ for $t \neq 0$, e.g. $\phi(t) = |t|^{m-1}t$; $m > 1$ and for fast diffusion problems like in plasma physics, $\phi'(0) = +\infty$; $\phi'(t) > 0$ for $t \neq 0$, e.g. $\phi(t) = |t|^{m-1}t$; $m \in (0,\ 1)$.

Our aim is to investigate some qualitative aspects of the classical solutions (and possibly their extension in the whole space) for the equation

$$\nabla\Big\{A(u)\nabla u\Big\} + c(x)u + f(u) = g(x); \tag{1.1}$$

$$\text{where} \quad A \in C^1(\mathbb{R}, (0,\ \infty));\ A'(t) > 0 \quad \forall t \neq 0; \tag{1.2}$$

$$f \in C(\mathbb{R}) \text{ is non decreasing and } \ tf(t) > 0 \quad \forall t \neq 0;\ c,\ g \in C(\mathbb{R}^n, \mathbb{R}). \tag{1.3}$$

Keeping in mind that K can be any primitive of ϕ,we assume that

$$A(t) := a_0 + a(t)\ ;\ a_0 > 0, \quad a \in C^1(\mathbb{R},\ (0,\ \infty);\ a'(t) > 0 \quad \forall t \neq 0. \tag{1.4}$$

In the sequel,for any regular function w, $\overline{D(w)}$ would denote any bounded connected component of support(w). Also,a solution of the equation (1.1),say, will be a non trivial u which satisfies (1.1) a.e. and for any $D(u)$

$$u \in C^1\Big(\overline{D(u)},\ \mathbb{R}\Big)\bigcap C^2\Big(D(u),\ \mathbb{R}\Big) \quad \text{and} \quad A(u)\nabla u \in C^1\Big(D(u),\ \mathbb{R}^n\Big).$$

The hypotheses on the equations,unless indicated otherwise remain

$$\textbf{(}\boldsymbol{H}\textbf{)} \qquad \text{(1.2) through (1.4) are satisfied and } \ c(x) > 0 \text{ for large } |x|\ .$$

As far as we know,the nonlinear cases in multidimensional space has been mainly about $p-$Laplacian type equations (see [3, 4, 5, 6, 7, 9, 10]).The comparison methods needed here to get most of the results rely on a positive lower bound (and sometimes upper bound) of the coefficient of the principal part, $A(u)$,say. Moreover,the sign of the coefficient c can be any one but $c \neq 0$ a.e. is important for the method we use. More reading about the physical applications and properties of the data can be found in [1, 2]. We hope that this work will contribute in more qualitative aspects of solutions of some forms of nonlinear diffusion problems.

2 Half-linear equations

Consider for some $a_1 > 0$ the equations

$$\nabla\{A(u)\nabla u\} + c(x)u = 0 \quad \text{and} \tag{2.1}$$

$$\nabla\{a_1\nabla v\} + c(x)v = 0 \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Following some Picone-type formulae for presumed solutions,

$$\nabla\left\{uA(u)\nabla u - \frac{u^2}{v}a_1\nabla v\right\} = [A(u)-a_1]|\nabla u|^2 + a_1|\nabla u - \frac{u}{v}\nabla v|^2; \tag{2.3}$$

$$\nabla\left\{va_1\nabla v - \frac{v^2}{u}A(u)\nabla u\right\} = [a_1 - A(u)]|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2, \tag{2.4}$$

where whenever there is a denominator,it is supposed to be non zero where the formula applies. We then get the following

Lemma 2.1. *Let u be a solution of (2.1) , v that of (2.2),with $D(v)$ and $D(u)$ defined earlier,*

1) If $a_0 < A(u) < a_1 \quad \forall x \in D(v)$,then u has a zero in $D(v)$;

2) If v_0 is such a solution in (2.2) where a_0 replaces a_1 then v_0 has a zero in any $D(u)$;

3) In general, any two solutions u_1 and u_2 of (2.1) intersect in the sense that for any $i \neq j$ u_i has a zero in $D(u_j)$.

Proof. The integration over $D(v)$ of (2.4) gives

$$0 = \int_{D(v)} \left[[a_1 - A(u)]|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2\right]dx$$

and the integration over $D(u)$ of (2.3) where a_0 replaces a_1 gives

$$0 = \int_{D(u)} \left([A(u)-a_0]|\nabla u|^2 + a_0|\nabla u - \frac{u}{v_0}\nabla v_0|^2\right)dx.$$

Each case presents a contradiction since the right hand side of each is strictly positive. For the last part,the conclusion follows from (2.3) and (2.4) in which $v = u_2$, $a_1 = A(v)$ and $u = u_1$. □

Definition 2.2. 1) A function w will be said to be strongly oscillatory in $\mathbb{R}^n$ if for any $R > 0$,w has a nodal set in $\Omega_R^c = \Omega^c(R) := \{x \in \mathbb{R}^n \,;\, |x| > R\}$. A nodal set of w being any connected component of supp.(w). An equation will be said to be strongly oscillatory if any of its bounded solutions is strongly oscillatory.

2) A function will be said to be weakly oscillatory in $\mathbb{R}^n$ if it has a zero in any $\Omega^c(R)$ and similarly as above would be defined an equation to be weakly oscillatory in $\mathbb{R}^n$.

Finally a non trivial $w \in C(\mathbb{R}^n)$ will be said to be non oscillatory if either

$$\liminf_{|x|\nearrow\infty} |w(x)| = 0$$

or there exist $\tau > 0$ and $R > 0$ such that $|w(x)| > \tau$ for all $x \in \Omega_R^c$.

In the sequel we will just say oscillatory for strongly oscillatory. For $n \geq 3$,we have the following result from Lemma 3.1 and Theorem 3.1 of [3]:

Lemma 2.3. *Because $a_1 > 0$ is constant, if*

$$\int_0^{\infty} r^{n-1}\,[\min_{|x|=r} c(x)]dr = \infty \tag{2.5}$$

then the equation (2.2) is oscillatory in $\mathbb{R}^n$

Remark 2.4. R1) For the equation $\nabla\{A(u)\nabla u\} + c(x)u + f(u) = 0$ where $tf(t) > 0$ and $A(t) > a_0 > 0\ \forall t \neq 0$,if $c(x) > 0$ *a.e.*,the term $c(x)u$ ($c(x)|u|^{p-2}u$ for $p-$Laplacian equations) insures that the solution u say, would not be of compact support (i.e G :=support(u) is compact and $u = |\nabla u| = 0$ outside G).In fact it ensures that for $\Psi(x,t) := \int_0^t [c(x)s + f(s)]ds$,

$$\int_{0^+} dt/\sqrt{\Psi(x,t)} = +\infty \quad \text{whenever } c(x) \neq 0\ . \tag{2.6}$$

Moreover the fact that $t \mapsto t\{c(x)t + f(t)\}$ keeps the sign of $c(x)t^2 + |tf(t)| \quad \forall t \neq 0$ makes $-u$ to be also solution whenever u is;this in a way helps the oscillatory property for bounded solutions.

R2) Also the fact that $c \in C^{\theta}(\mathbb{R}^n)$; $\theta > 0$ implies that weakly oscillatory solutions are oscillatory.(see e.g. [4]).

3 Main results

Theorem 3.1. *Assume that c satisfies (2.5) and $tf(t) > 0 \quad \forall t \neq 0$. Then any bounded solution of*

$$\nabla.\Big\{A(u)\nabla u\Big\} + c(x)u + f(u) = 0 \quad \text{in } \mathbb{R}^n \tag{3.1}$$

is oscillatory in $\mathbb{R}^n$.

Proof. For $a_1 := A(|u|_\infty)$, let v be an oscillatory solution of $\nabla\{a_1\nabla v\} + c(x)v = 0$. If we suppose that u is strictly positive in Ω_R^c for some $R > 0$,let $D_1 \subset \Omega_R^c$ be a nodal set of v.Then as

$$\nabla.\{va_1\nabla v - \frac{v^2}{u}A(u)\nabla u\} = (a_1 - A(u))|\nabla v|^2 +$$
$$+A(u)|\nabla v - \frac{v}{u}\nabla u|^2 + v^2\frac{f(u)}{u} > 0$$

in D_1 and $v|_{\partial D_1} = 0$, we reach an absurdity after integrating both sides of the equation over D_1.Therefore u has a zero in any Ω_R^c. The next results will show that when the condition $tf(t) > 0$ is missing in the equation,the oscillation of solutions can be obtained by imposing a large multiplying coefficient to c. Having in mind the equation (1.1),we now consider for $\lambda > 0$ the equation

$$\nabla\{A(u)\nabla u\} + \lambda c(x)u + h(x,u) = 0 \quad \text{in } \mathbb{R}^n. \tag{3.2}$$

□

Theorem 3.2. *Assume that c satisfies (2.5) and for any* $D \subset \mathbb{R}^n$ *,*

$$\forall t \quad x \mapsto |h(x,t)| \quad \textit{is locally bounded in } \mathbb{R}^n \ ; \tag{3.3}$$

$$\textit{whenever meas.}(D) > 0\ , \quad \lim_{\mu \nearrow \infty} \mu \int_D c(x)dx = \infty. \tag{3.4}$$

Then for large $\lambda > 0$*,any non trivial and bounded solution* u_λ *of (3.2) is oscillatory unless*

$$\liminf_{|x| \nearrow \infty} |u_\lambda(x)| = 0. \tag{3.5}$$

Proof. Assume that there are R_0 , λ_0, v_0 such that $\forall \lambda > \lambda_0$, such a solution $u = u_\lambda$ satisfies $u > v_0$ in $\Omega^c_{R_0}$. For v, an oscillatory solution of $\nabla.[a_0 \nabla v] + c(x)v = 0$,

$$\nabla.\Big\{ v a_0 \nabla v - \frac{v^2}{u} A(u) \nabla u \Big\} = (a_0 - A(u))|\nabla v|^2 +$$
$$+ A(u)|\nabla v - \frac{v}{u}\nabla u|^2 + v^2[\ (1-\lambda)c(x) + \frac{h(x,u)}{u}\ ].$$

Let $G \subset \Omega^c_{R_0}$ be a nodal set of v.Then for any $\lambda > \lambda_0$

$$0 = \int_G \Big\{ (a_0 - A(u))|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2 \Big\} dx +$$
$$+ \int_G v^2 \Big[\ (1-\lambda)c(x) + \frac{h(x,u)}{u} \Big] dx \tag{3.6}$$

which cannot hold when λ is arbitrary large; in fact (3.3) and (3.4) ensure that $| \int_G v^2 \Big[\ (1 - \lambda)c(x) + \frac{h(x,u)}{u} \Big] dx|$ is unbounded in $\lambda > \lambda_0$. Therefore u has to have a zero in G. We then get the following important results: □

Theorem 3.3. *Assume that* $\forall t \neq 0,\ t f(x,t) > 0$ *or* $f(x,t) := f_0(t) + g(x)$ *with* $t f_0(t) > 0$ *. For the equation*

$$\nabla.\Big\{ A(u) \nabla u \Big\} + f(x,u) = 0 \quad in \quad \mathbb{R}^n, \tag{3.7}$$

if $f(x,t)$ *is locally bounded then any bounded solution u of the equation is oscillatory unless u satisfies (3.5).*

Proof. Let u be such a solution of (3.7) and assume that $\exists R_0,\ v_0 > 0$ such that $u(x) > v_0$ in $\Omega^c_{R_0}$.For any $\lambda > 0$ and c which satisfies (2.5) and (3.4), $\nabla[a_0 \nabla v] + \lambda c(x)v = 0$ in $\mathbb{R}^n$ has an oscillatory solution $v = v_\lambda$, say. Let $G \subset \Omega^c_{2R_0}$ be a nodal set of v.For $\Psi_\lambda(x,u) := f(x,u) - \lambda c(x)u$,

$$\nabla.\Big[v a_0 \nabla v - \frac{v^2}{u} A(u) \nabla u \Big] =$$
$$= [a_0 - A(u)]|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2 + v^2 \frac{\Psi(x,u)}{u}$$
$$:= [a_0 - A(u)]|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2 + v^2 \Big[\frac{f(x,u)}{u} - \lambda c(x) \Big].$$

Integrating of the formula over G,we get for $G(x,u) := \left[\frac{f(x,u)}{u} - \lambda c(x)\right]$

$$0 = \int_G \left\{[a_0 - A(u)]|\nabla v|^2 + A(u)|\nabla v - \frac{v}{u}\nabla u|^2 + v^2 G(x,u)\right\} dx \tag{3.8}$$

which would not hold for arbitrary large λ ; this implies that u has a zero in G. □

4 Applications

Because the method calls on bounded solutions and comparison method,the cases with constant coefficients and some one-dimensional cases can be useful.In fact most oscillation criteria in one-dimensional lead to those of some related multi-dimensional cases.When the equation is odd in the unknown function,when a solution is non oscillatory,we can consider it strictly positive eventually.

Application 1

Consider for $\gamma > 0$ the equation

$$y''(x) + e^{-x} \sin x\, |y(x)|^{\gamma-1} y(x) = 0 \quad \text{in } \mathbb{R}. \tag{4.1}$$

In [8] it is shown that this equation is non oscillatory for $\gamma = 1$ but the case $\gamma \neq 1$ was left open.We can now say more about the case $\gamma \neq 1$. For $n > 3$, in the transformation $y(r) = r^{(n-1)/2} u(r)$, (4.1) becomes in $E^* := \mathbb{R}^n \setminus \{0\}$

$$\triangle u + \frac{(n-1)(n-3)}{4r^2} u + r^{(n-1)(\gamma-1)/2}\, e^{-r} \sin r\, |u|^{\gamma-1} u = 0. \tag{4.2}$$

For the associated equation $\triangle v + \frac{(n-1)(n-3)}{4r^2} v = 0$, as $A(u) \equiv 1$ and $c(r) = \frac{(n-1)(n-3)}{4r^2}$, its solutions are oscillatory in E^* by Lemma 2.3. For any bounded solution u of (4.2)

$$\nabla\left[v\nabla v - \frac{v^2}{u}\nabla u\right] = |\nabla v - \frac{v}{u}\nabla u|^2 + v^2 |u|^{\gamma-1}\, r^{(n-1)(\gamma-1)/2}\, e^{-r} \sin r$$

and for any nodal set G of v,

$$0 = \int_G \left\{|\nabla v - \frac{v}{u}\nabla u|^2 + v^2 |u|^{\gamma-1}\, r^{(n-1)(\gamma-1)/2}\, e^{-r} \sin r\right\} r^{(n-1)/2}\, dr. \tag{4.3}$$

But the presence of e^{-r} implies that if v and u are bounded,

$$\forall \varepsilon > 0 \quad \exists R_\varepsilon > 0\,; \quad \int_{\Omega^c_{R_\varepsilon}} v^2 |u|^{\gamma+1}\, r^{(n-1)\gamma/2}\, e^{-r}\, dr < \varepsilon$$

rending (4.3) absurd. We then have the following result:

Theorem 4.1. *Let* $n > 3$.

1) If $\gamma > 1$ *then (4.2) has oscillatory solutions in* $\mathbb{R}^n$.

2) If $0 < \gamma < 1$ then (4.2) has oscillatory solutions u in $\mathbb{R}^n$ unless u satisfies (3.5).

Therefore for the solutions $y(r)$ of (4.1) with $u(r) :=:= r^{(1-n)/2}y(r);\ n > 3$, we can say that

(i) if $\gamma > 1$ then $u(r)$ is oscollatory in $\mathbb{R}^n$;

((ii) if $0 < \gamma \leq 1$, $u(r)$ either is oscillatory in any exterior domain Ω_R^c ; $R > 0$ or $\liminf_{r \nearrow \infty} \{|u(r)|\} = 0$.

Application 2

From [9] we consider for $\phi(t) := |t|^{\sigma-1}t$ where $\sigma > 0$, the equation

$$\Big[a(t)\phi(y')\Big]' + q(t)f(y(t)) = 0. \tag{4.4}$$

We assume that (i) $a \in C^1(\mathbb{R},\ (0,\ \infty));\ q \in C(\mathbb{R},\ \mathbb{R});\ f \in C(\mathbb{R},\ \mathbb{R})$ with $sf(s) > 0 \quad \forall s \neq 0$ and (ii) $\dfrac{f(s)}{\phi(s)}$ is non decreasing and strictly positive for $s > 0$. Assume that for some positive function $\mu \in C(\mathbb{R},\ (0,\ \infty))$,

$$0 < M < y \Longrightarrow 0 < \mu(M) < \frac{f(y)}{\phi(y)}. \tag{4.5}$$

Then from the equation (4.4), for $t_1 < t$

$$a(t)\phi(y'(t)) - a(t_1)\phi(y'(t_1)) = -\int_{t_1}^{t} q(s)f(y(s))ds. \tag{4.6}$$

From (4.5),if $y > 0$ and bounded in $I_{t_0} := [t_0,\ \infty)$ then $y' > 0$ there as well (as it decreases to 0 there). In that case the function $\omega(t) := a(t)\dfrac{\phi(y')}{\phi(y)}$ satisfies

$$\omega(t) \leq \omega(t_0) - \mu(y(t_0)) \int_{t_0}^{t} q(s)ds.$$

We then have the following result:

Theorem 4.2. *Under the hypotheses (i) and (ii) displayed above,any bounded solution of the equation (4.4) is oscillatory if for some $T > 0$*

$$q \text{ is eventually positive and } \qquad \int_T^{\infty} q(t)dt = \infty.$$

Setting $a(t) := t^{n-1} \max_{|x|=t} A(x)$ and $q(t) := t^{n-1} \min_{|x|=t} Q(x)$,Theorem 4.2 applies to the equation

$$\nabla\Big\{A(x)|\nabla u|^{\sigma-1}\nabla u\Big\} + Q(x)f(u) = 0 \quad \text{in } \mathbb{R}^n. \tag{4.7}$$

(see e.g Theorem 3.1 of [3]). Consider the equation

$$
\begin{aligned}
&(i) \qquad \Big\{a(t,y)y'\Big\}' + q(t)f(y) = 0 \\
&\text{where} \quad q(t) \text{ and } tf(t) > 0 \quad \forall t \in \mathbb{R};\ q,f \in C(\mathbb{R}) \text{ and} \\
&(ii) \qquad a(t,u) = a_0 + a_1(t,u); \quad a_0 > 0; \\
&\frac{\partial a_1}{\partial t},\ \frac{\partial a_1}{\partial u} \text{ and } a_1 \in C^1(\mathbb{R}\times\mathbb{R},\ (0,\ +\infty)).
\end{aligned}
\tag{4.8}
$$

For (i),if we assume that a solution y is strictly positive in some I_T then with $y_i := y(t_i)$,

$$
\forall t_1, t_2 \in I_T,\ t_1 < t_2, \quad a(t_2,y_2)y_2' \le a(t_1,y_1)y_1' - \int_{t_1}^{t_2} q(s)f(y(s))ds \tag{4.9}
$$

and $t \mapsto a(t,y(t))y'(t)$ is decreasing in I_T. If the numbers t_i above are such that $y'(t_1) = y'(t_2)$ then $a(t_2,y_2) \le a(t_1,y_1)$ violating the monotonicity of a.Therefore y' is monotone in I_T and so is y.If y is a bounded function in I_T, y' has then to decrease to zero at ∞ therefore $y' > 0$ in I_T.This again contradict (4.9).This states the following result:

Theorem 4.3. *Assume that the functions a, q and f satisfy the hypotheses in (4.8), for some $\mu, T_0 > 0$, $\quad f(t) > \mu \quad \forall t > T_0$ and*

$$
t \mapsto \int_T^t q(s)ds \quad \textit{increases to } \infty\ . \tag{4.10}
$$

Then a) the equation (4.8)(i) is oscillatory ; b) if the functions $a(t) := t^{n-1}\max_{|x|=t} A(x)$ and $q(t) := t^{n-1}\min_{|x|=t} Q(x)$ satisfy the hypotheses displayed above then the equation

$$
\nabla\Big\{A(x,u)\nabla u\Big\} + Q(x)f(u) = 0 \tag{4.11}
$$

is also oscillatory.

Application 3

Consider a version of the equation (4.4) in which a damping term $q(t)\phi(y')$ is used.We have the following result:

Theorem 4.4. *Assume that for some $t_0 > 0$ the continuous function $q > 0$ for $t > t_0$ and*

$$
\lim_{t\to+\infty} \int_{t_0}^t \frac{q(s)}{a(s)}ds = \infty. \tag{4.12}
$$

Then any non trivial and bounded solution v of

$$
\Big\{a(t)\phi(v')\Big\}' + q(t)\phi(v') = 0 \tag{4.13}
$$

is oscillatory unless $\liminf_{t\nearrow\infty} |v(t)| = 0$.

Before the proof of the theorem,we need an auxillary lemma.

Lemma 4.5. *If such a solution v of Theorem 4.4 is not oscillatory then assuming $v > 0$ in some I_T, v' is non oscillatory and $v' > 0$ in I_T as well.*

Proof. From the equation, for $t_2 > t_1$

$$a(t_2)\phi(v'(t_2)) - a(t_1)\phi(v'(t_1)) = -\int_{t_1}^{t_2} q(s)\phi(v'(s))ds. \tag{4.14}$$

If we assume that v' is oscillatory, we chose t_1 and t_2 in a nodal set $D((v')^-)$ such that $v'(t_2) < v'(t_1) \leq 0$; in that case the equality (4.14) is absurd because its left hand side is negative while the right is positive . This shows at the same time that $v' > 0$ in I_T. □

Proof. (Proof of Theorem 4.4) Assume that such a solution v of (4.13) is non oscillatory and satisfies $v(t) > \mu > 0$ in I_T for some $T, \mu > 0$.In I_T, from Lemma 4.5, $v' > 0$ whence for any $k > 0$, the function

$$\begin{aligned} &W(t) := k + \frac{a\phi(v')}{\phi(v)} > k, \quad \text{satisfies} \\ &W'(t) = -a(t)\phi\Big(\frac{v'}{v}\Big)\Big\{\frac{q(t)}{a(t)} + \sigma\frac{v'}{v}\Big\} \\ \text{and} \quad &\frac{W'}{(W-k)} < -\frac{q(t)}{a(t)}. \end{aligned} \tag{4.15}$$

This leads to

$$0 < W(t) - k < \Big\{W(t_0) - k\Big\}\exp\{-\int_{t_0}^{t} \frac{q(s)}{a(s)}ds,$$

which cannot hold for large $t > t_0 > T$. □

Acknowlegments

To my late son Nkayum Tadie Abissi and cousin Tagne David Pierre: requiescate in pacem.

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