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Abstract

We introduce new definitions of semi-continuity for multifunctions, combining the topological and the ordered structure of a Banach space induced by a closed convex cone. We prove two types Nash equilibrium theorems for multifunctions using scalarization and the Ky Fan's inequality. As corollaries we obtain saddle point theorems for convex-concave multifunctions, which can be considered as generalization to the vector-valued set-valued case of the Von Neumann minimax theorem.

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1 Introduction and notions of convexity of multifunctions

Minimax problems and their related extensions have been subject to extensive research of many mathematicians and scientists. For a recent development in this area we refer to [5]. The theory of vector optimization has been intensively developed in recent years, as currently the interest is focused on vector valued multifunctions. Important parts of this theory are the minimax problems and saddle point problems, which have their specific features with respect to the real-valued case. For a development of such vector-valued problems we refer to [2], [8], [9], [10], [11], [12] and references therein. The vector-valued, set-valued case proposes more possibilities for definitions of saddle points. In this paper we prove two types Nash equilibrium theorems for vector-valued multifunctions using scalarization and Ky Fan's inequality. As a corollary we obtain two types saddle point theorems for convex-concave multifunctions (with respect to a specified definition). An advantage in our saddle point theorems with respect to the existing ones in the literature (see [3], [4]) is that our conditions are explicit.

Let E be topological vector space, Z be a Banach space and $C \subset Z$ be a closed convex cone with nonempty interior.

Definition 1.1. The multifunction $F : E \supset X \to 2^Z$, where X is a convex nonempty subset, is called C-convex, if for every $x, y \in X, \lambda \in [0,1], u \in \lambda F(x) + (1-\lambda)F(y)$ there exists $v \in F(\lambda x + (1-\lambda)y)$ such that $u - v \in C$. If F is $(-C)$-convex, then F is called C-concave.

Let $k^0 \in intC$ be fixed. Define the functions

$$h_C(x) = inf\{t \in \mathbf{R} : x \in tk^0 - C\},$$

$$\varphi_C(x) = \inf h_C(F(x)),$$

$$\psi_C(x) = \sup h_C(F(x)).$$

It is easy to see that h_C is continuous and sublinear (see [6], [7]).

Lemma 1.2. *Let the multifunction $F : E \supset X \to 2^Z$ be C-convex. Then the function φ_C is convex.*

Proof. Let $x_1, x_2 \in X$. By definition of φ_C and h_C, for every $\varepsilon > 0$ there exist $z_i \in F(x_i), t_i \in \mathbf{R}, i = 1,2$ such that

$$z_i - t_i k^0 \in -C \tag{1}$$

and

$$t_i < \varphi_C(x_i) + \varepsilon. \tag{2}$$

By definition of C-convex multifunction,

$$\exists v \in F(\lambda x_1 + (1-\lambda)x_2) : \lambda z_1 + (1-\lambda)z_2 \in v + C. \tag{3}$$

By (1) we have

$$-C \ni \lambda(z_1 - t_1k^0) + (1-\lambda)(z_2 - t_2k^0) = \lambda z_1 + (1-\lambda)z_2 - (\lambda t_1 + (1-\lambda)t_2)k^0. \tag{4}$$

By (3) and (4) we have

$$\begin{aligned} v &\in \lambda z_1 + (1-\lambda) z_2 - C \\ &\subset (\lambda t_1 + (1-\lambda)t_2)k^0 - C - C \\ &= (\lambda t_1 + (1-\lambda)t_2)k^0 - C. \end{aligned}$$

Hence

$$\begin{aligned} h_C(v) &\leq \lambda t_1 + (1-\lambda)t_2 \\ &< \lambda \varphi_C(x_1) + (1-\lambda)\varphi_C(x_2) + 2\varepsilon. \end{aligned}$$

Therefore

$$\varphi_C(\lambda x_1 + (1-\lambda)x_2) := \inf_{z \in F(\lambda x_1 + (1-\lambda)x_2)} h_C(z) \leq \lambda \varphi_C(x_1) + (1-\lambda)\varphi_C(x_2) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi_C(\lambda x_1 + (1-\lambda)x_2) \leq \lambda \varphi_C(x_1) + (1-\lambda)\varphi_C(x_2). \blacksquare$$

The proof of the next lemma is similar to that one of Lemma 1.2 and is omitted.

Lemma 1.3. *Let $F : E \supset X \to 2^Z$ be $(-C)$-convex. Then the function $\psi_C(x)$ is concave.*

2 Notions of semicontinuity for multifunctions

Here we give some definitions of semicontinuity for multifunctions motivated from the scalar case. Let us first remind the classical definitions of lower semi-continuity and upper semi-continuity of multi-valued mappings, and of real valued functions.

If X and Y are topological spaces, the multi-valued mapping $F : X \to 2^Y$ is called upper semi-continuous at the point $x_0 \in X$, iff for any open set $V \supset F(x_0)$ there exists an open set $U \ni x_0$ such that $F(x) \subset V$ for every $x \in U$. The multi-valued mapping F is called lower semi-continuous at $x_0 \in X$, if for every open set V in Y with $V \cap F(x_0) \neq \emptyset$, there exists an open set $U \ni x_0$ such that $F(x) \cap V \neq \emptyset$ for every $x \in U$.

The real valued function $f : X \to \mathbb{R}$ is called upper semi-continuous (lower semi-continuous) at x_0, if for every $\varepsilon > 0$ there exists an open set $U \ni x_0$ such that $f(x) - f(x_0) < \varepsilon$ ($f(x_0) - f(x) < \varepsilon$) for every $x \in U$.

Definition 2.1. We shall say that the multifunction $F : E \to 2^Z$ is C-lower semi-continuous at x_0, if for every $y \in F(x_0)$ and every open $V \ni 0$ there exists an open $U \ni x_0$ such that $(y + V + C) \cap F(x) = \emptyset$ for every $x \in U$.

When Z is the real line, $Z = \mathbb{R}$, and the cone C is the the cone of all non-negative real numbers, the above definition coincides with the classical definition of lower semi-continuity of real-valued functions.

When the cone C consists only of the zero element of the space E, the above definition reduces to the notion of lower semi-continuity of multi-valued mappings.

Definition 2.2. The multifunction $F : E \to 2^Z$ will be called $(C,k^0)^-$-upper semi-continuous at x_0, if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that

$$[(\varphi_C(x_0) - \varepsilon)k^0 - C] \cap F(x) = \emptyset \quad \forall x \in U.$$

In particular, every upper semi-continuous multi-valued mapping is $(C,k^0)^-$-upper semi-continuous (and this is the motivation of this notion).

When Z is the real line, $Z = \mathbb{R}$, and the cone C is the the cone of all non-negative real numbers, the above definition coincides with the classical definition of lower semi-continuity of real-valued functions.

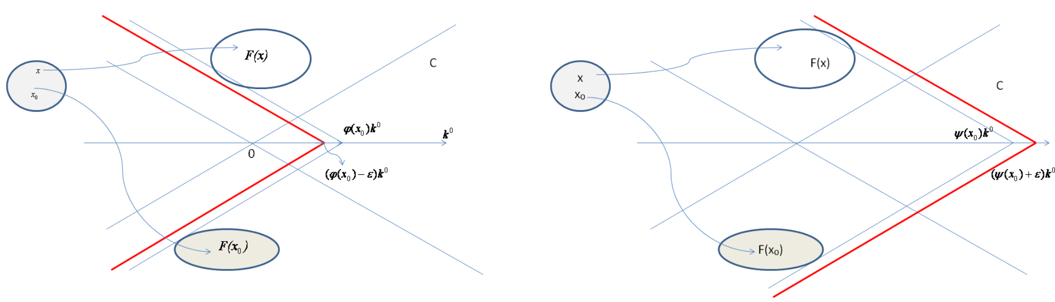

Figure 1. Geometric visualization of $(C,k^0)^-$-upper semicontinuity (left) and $(C,k^0)^+$-upper semicontinuity (right).

Definition 2.3. The multifunction $F : E \to 2^Z$ will be called $(C,k^0)^+$-upper semi-continuous at x_0, if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that

$$F(x) \subset (\psi_C(x_0) + \varepsilon)k^0 - intC \quad \forall x \in U.$$

In particular, every upper semi-continuous multi-valued mapping is $(C,k^0)^-$-upper semi-continuous.

Geometric visualizations of $(C,k^0)^-$-upper semicontinuity and $(C,k^0)^+$-upper semicontinuity are shown in Figure 1.

When Z is the real line, $Z = \mathbb{R}$, and the cone C is the the cone of all non-negative real numbers, the above definition coincides with the classical definition of upper semi-continuity of real-valued functions.

It is easy to see that the notion of $(C,k^0)^-$-upper semi-continuity coincides with the notion of $(E \setminus C,k^0)^+$-upper semi-continuity and therefore, the notion of $(C,k^0)^+$-upper semi-continuity coincides with the notion of $(E \setminus C,k^0)^-$-upper semi-continuity.

Lemma 2.4. *If F is $(-C)$-lower semi-continuous, then φ_C is upper semi-continuous.*

Proof. Let $x_0 \in E, \varepsilon > 0$ be fixed and $y_0 \in F(x_0)$ be such that

$$h_C(y_0) < \inf h_C(F(x_0)) + \varepsilon.$$

By continuity of h_C, there exists an open $V \ni 0$ such that

$$h_C(v) < \varepsilon \quad \forall v \in V.$$

By definition of $(-C)$-lower semi-continuity, there exists an open $U \ni x_0$ such that

$$F(x) \cap (y_0 + V - C) \neq \emptyset \quad \forall x \in U.$$

Let $y \in F(x) \cap (y_0 + V - C)$. Then $y = y_0 + v - c$ for some $v \in V, c \in C$ and we can write

$$\begin{aligned}
\varphi_C(x) &= \inf_{y' \in F(x)} h_C(y') \\
&\leq h_C(y) \\
&\leq h_C(y_0) + h_C(v) + h_C(-c) \qquad \text{(by sublinearity of } h_C \text{ and since } h_C(-c) \leq 0) \\
&\leq \varphi_C(x_0) + 2\varepsilon.
\end{aligned}$$

Lemma 2.5. *If F is $(C,k^0)^-$-upper semi-continuous, then φ_C is lower semi-continuous.*

Proof. Let $x_0 \in E, y \in F(x_0)$ and $x \in U$, where U is given by the definition of $(C,k^0)^-$-upper semicontinuity of F at x_0. Let $z \in F(x)$. Then by definition of $(C,k^0)^-$-upper semicontinuity, $z \notin (\varphi_C(x_0) - \varepsilon)k^0 - C$. Now by definition of the function h_C,

$$\varphi_C(x_0) - \varepsilon < h_C(z)$$

and since this is valid for every $z \in F(x)$, we have

$$\varphi_C(x_0) - \varepsilon \leq \varphi_C(x),$$

which proves the lemma. ■

Lemma 2.6. *If F is C-lower semi-continuous, then ψ_C is lower semi-continuous.*

Proof. Similar to that one of Lemma 2.4. ■

Lemma 2.7. *If F is $(C,k^0)^+$-upper semi-continuous, then ψ_C is upper semi-continuous.*

Proof. Similar to that one of Lemma 2.5. ■

3 Saddle points via Nash equilibrium.

In this section we prove two types Nash equilibrium theorems and two types of saddle point theorems. The proofs are based on scalarization via Lemmas 2.4,..., 2.7, and on the Ky Fan inequality.

Recall that the point $x_0 \in X$, where $X \subset E$ is a compact convex non-empty subset, is called a solution of the *quasi-variational inequality* (F, f, X) [1] for a given multi-valued mapping $F : X \to X$ and a given function $f : X \times X \to \mathbb{R}$, iff

$$x \in F(x) \quad \text{and} \quad \sup_{y \in F(x)} f(x,y) \leq 0.$$

The famous Ky Fan inequality (see, for instance, [1], Theorem 6.3.5) is a particular case of the quasi-variational inequality when F is the constant mapping $F(x) = X$ for every $x \in X$.

Let E_1, E_2 be topological vector spaces, Z be a Banach space, $X \subset E_1, Y \subset E_2$ be convex compact nonempty subsets and $C_i \subset Z$ be closed convex cones with nonempty interiors, $k_i^0 \in intC_i, i = 1,2$.

Theorem 3.1. *(Nash equilibrium I). Let the multifunctions $F_i : X \times Y \to 2^Z$ be $(C_i, k_i^0)^-$-upper semi-continuous. Assume that $F_1(.,y)$ is C_1-convex for every $y \in Y$, $F_1(x,.)$ is $-C_1$-lower semi-continuous for every $x \in X$, $F_2(x,.)$ is C_2-convex for every $x \in X$ and $F_2(.,y)$ is $-C_2$-lower semi-continuous for every $y \in Y$. Then there exists a Nash equilibrium of type I, $(x_0, y_0) \in X \times Y$, which means*

$$F_1(x,y_0) \cap [\inf h_{C_1}(F_1(x_0,y_0))k_1^0 - intC_1] = \emptyset \quad \forall x \in X, \tag{5}$$

$$F_2(x_0,y) \cap [\inf h_{C_2}(F_2(x_0,y_0))k_2^0 - intC_2] = \emptyset \quad \forall y \in Y. \tag{6}$$

Proof. Define

$$f(x,y,\overline{x},\overline{y}) = \inf h_{C_1}(F_1(x,y)) - \inf h_{C_1}(F_1(\overline{x},y)) + \inf h_{C_2}(F_2(x,y)) - \inf h_{C_2}(F_2(x,\overline{y}))$$

By Lemma 1.2, $f(x,y,.,.)$ is concave for every $x \in X, y \in Y$ and by Lemmas 2.4, 2.5, $f(.,.,\overline{x},\overline{y})$ is lower semi-continuous for every $\overline{x} \in X, \overline{y} \in Y$. By Ky Fan's inequality (see [1], Theorem 6.3.5) there exists $(x_0, y_0) \in X \times Y$ such that

$$\sup_{(\overline{x},\overline{y}) \in X \times Y} f(x_0,y_0,\overline{x},\overline{y}) \leq 0$$

Putting $\overline{y} = y_0$ we obtain

$$\inf h_{C_1}(F_1(x_0,y_0)) \leq \inf h_{C_1}(F_1(x,y_0)) \quad \forall x \in X, \tag{7}$$

and putting $\overline{x} = x_0$ we obtain

$$\inf h_{C_2}(F_2(x_0,y_0)) \leq \inf h_{C_2}(F_2(x_0,y)) \quad \forall y \in Y. \tag{8}$$

Now (7) implies

$$F_1(x,y_0) \cap [\inf h_{C_1}(F_1(x_0,y_0))k_1^0 - intC_1] = \emptyset$$

and (8) implies

$$F_2(x_0,y)\cap[\inf h_{C_2}(F_2(x_0,y_0))k_2^0 - intC_2] = \emptyset,$$

which finishes the proof. ■

Theorem 3.2. *(Nash equilibrium II). Let the multifunctions $F_i : X\times Y \to 2^Z$ be $(C_i,k_i^0)^+$-upper semi-continuous. Assume that $F_1(.,y)$ is C_1-concave for every $y\in Y$, $F_1(x,.)$ is C_1-lower semi-continuous for every $x\in X$, $F_2(x,.)$ is C_2-concave for every $x\in X$ and $F_2(.,y)$ is C_2-lower semi-continuous for every $y\in Y$. Then there exists a Nash equilibrium of type II, $(x_0,y_0)\in X\times Y$, which means*

$$F_1(x,y_0)\subset \sup h_{C_1}(F_1(x_0,y_0))k_1^0 - C_1 \quad \forall x\in X,$$

$$F_2(x_0,y)\subset \sup h_{C_2}(F_2(x_0,y_0))k_2^0 - C_2 \quad \forall y\in Y.$$

Proof. Define

$$f(x,y,\overline{x},\overline{y}) = -\sup h_{C_1}(F_1(x,y)) + \sup h_{C_1}(F_1(\overline{x},y)) - \sup h_{C_2}(F_2(x,y)) + \sup h_{C_2}(F_2(x,\overline{y})).$$

By Lemma 1.3, $f(x,y,.,.)$ is concave for every $x\in X, y\in Y$ and by Lemmas 2.6, 2.7, $f(.,.,\overline{x},\overline{y})$ is lower semi-continuous for every $\overline{x}\in X,\overline{y}\in Y$. By Ky Fan's inequality (see [A-E, Theorem 6.3.5]) there exists $(x_0,y_0)\in X\times Y$ such that

$$\sup_{(\overline{x},\overline{y})\in X\times Y} f(x_0,y_0,\overline{x},\overline{y}) \le 0$$

Putting $\overline{y} = y_0$ we obtain

$$\sup h_{C_1}(F_1(x,y_0)) \le \sup h_{C_1}(F_1(x_0,y_0)) \quad \forall x\in X, \tag{9}$$

and putting $\overline{x} = x_0$ we obtain

$$\sup h_{C_2}(F_2(x_0,y)) \le \sup h_{C_2}(F_2(x_0,y_0)) \quad \forall y\in Y \tag{10}$$

Now (9) implies

$$F_1(x,y_0)\subset \sup h_{C_1}(F_1(x_0,y_0))k_1^0 - C_1$$

and (10) implies

$$F_2(x_0,y)\subset \sup h_{C_2}(F_2(x_0,y_0))k_2^0 - C_2,$$

which finishes the proof. ■

Remark 3.3. Here we considered the case of two multifunctions. The general case of finitely many multifunctions is considered analogically.

In the special case when $F_1 = -F_2$ and $C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$, we obtain the following saddle point theorems.

Theorem 3.4. *(Saddle point theorem I). Suppose that the multifunction* $F : X \times Y \to 2^Z$ *have compact images and is* $(C,k^0)^-$*-upper semi-continuous and* $(-C,-k^0)^-$*-upper semi-continuous,* $F(.,y), y \in Y$ *is C-convex and C-upper semi-continuous,* $F(x,.), x \in X$ *is C-concave and* $(-C)$*-lower semi-continuous. Then there exists a saddle point* $(x_0,y_0) \in X \times Y$ *of type I, namely there exist* $z_1, z_2 \in F(x_0,y_0)$*, such that*

$$(z_1 - intC) \cap F(x,y_0) = \emptyset \quad \forall x \in X, \tag{11}$$

$$(z_2 + intC) \cap F(x_0,y) = \emptyset \quad \forall y \in Y. \tag{12}$$

Proof. We apply Theorem 3.1 and obtain a Nash equilibrium of type I, $(x_0,y_0) \in X \times Y$. By continuity of h_C, there exists points $z_1, z_2 \in F(x_0,y_0)$ such that

$$\inf h_C(F(x_0,y_0) = h_C(z_1) \tag{13}$$

and

$$\inf h_{-C}(F(x_0,y_0) = h_{-C}(z_2).$$

By definition of h_C, (13) implies $z_1 \in h_C(z_1)k^0 - C$, whence

$$z_1 - intC \subset h_C(z_1)k^0 - C - intC = h_C(z_1)k^0 - intC.$$

Now (5) implies (11).

Analogically we prove (12). ■

Geometric visualizations of multifunction Nash equilibrium I, when $F_1 = -F_2 = F, C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$ and multifunction saddle point I are shown in Figure 2.

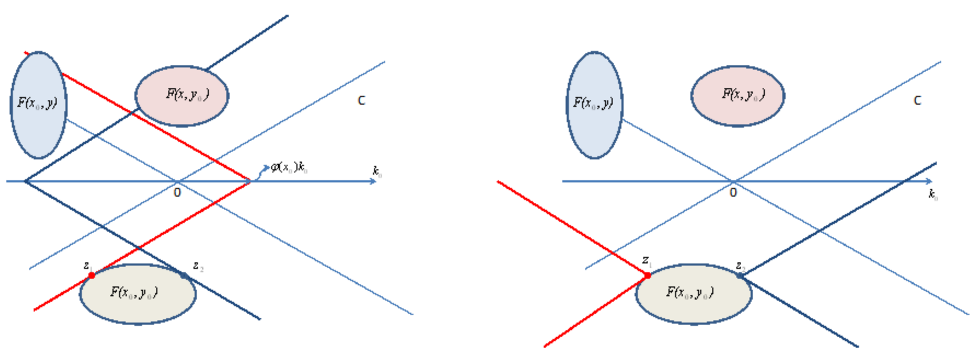

Figure 2. Geometric visualization of multifunction Nash equilibrium I, when $F_1 = -F_2 = F, C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$ (left) and multifunction saddle point I (right).

Theorem 3.5. *(Saddle point theorem II). Suppose that the multifunction* $F : X \times Y \to 2^Z$ *have compact images and is* $(C,k^0)^+$*-upper semi-continuous and* $(-C,-k^0)^+$*-upper semi-continuous,* $F(.,y), y \in Y$ *is* C*-concave and* $(-C)$*-lower semi-continuous,* $F(x,.), x \in X$ *is* C*-convex and* C*-lower semi-continuous. Then there exists a saddle point* $(x_0,y_0) \in X \times Y$*, i.e. there exist* $z_1,z_2 \in F(x_0,y_0)$*, such that satisfying (11) and (12).*

The proof is similar to those of Theorem 3.4, applying Theorem 3.2.

Geometric visualizations of multifunction Nash equilibrium II, when $F_1 = -F_2 = F, C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$ and multifunction saddle point II are shown in Figure 3.

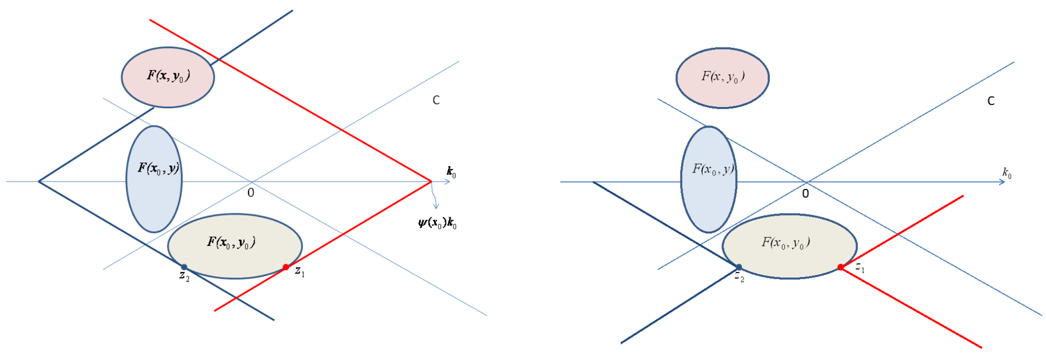

Figure 3. Geometric visualization of multifunction Nash equilibrium II, when $F_1 = -F_2 = F, C_1 = C_2 = C, k_1^0 = k_2^0 = k^0$ (left) and multifunction saddle point II (right).

Remark 3.6. When Z is the real line, $Z = \mathbb{R}$, and the cone C is the the cone of all non-negative real numbers, the above saddle point theorem reduces to the classical Von Neumann minimax theorem [1]. When F is upper semi-continuous, the assumptions (and the conclusions) of the above two saddle point theorems coincide.

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