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On the Stability of a Functional Equation in Topological Vector Spaces

Abbas Najati*
Department of Mathematics
University of Mohaghegh Ardabili
Ardabil, Iran

Themistocles M. Rassias†
Department of Mathematics
National Technical University of Athens
Zografou Campus
15780 Athens, Greece

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Abstract

In this work the Hyers–Ulam type stability of the functional equation $f(x+y+xy) = f(x)+f(y+xy)$ is proved.

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1 Introduction

The functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). The stability of functional equations was first introduced by S. M. Ulam [8] in 1940. More precisely, Ulam proposed the following problem: Given a group G_1, a metric group (G_2,d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x,y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x),T(x)) < \epsilon$ for all $x \in G_1$? As it is mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [3] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption

*E-mail address: a.nejati@yahoo.com
†E-mail address: trassias@math.ntua.gr

that G_1 and G_2 are Banach spaces. T. Aoki [1] and Th.M. Rassias [6] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. During the last decades several stability problems of functional equations have been investigated by several mathematicians. A large list of references concerning the stability of functional equations can be found in [2, 4, 5, 7].

In this paper, we deal with the functional equation

$$f(x+y+xy) = f(x) + f(y+xy). \tag{1.1}$$

2 Solution of functional equation (1.1)

Theorem 2.1. *Let X be a vector space. A function $f : \mathbb{R} \to X$ satisfies (1.1) if and only if f is additive.*

Proof. Let f satisfy (1.1) and $a, b \in \mathbb{R}$ with $a \neq -1$. Let $x, y \in \mathbb{R}$ such that $x = a$ and $y = \frac{b}{1+a}$. Since f satisfies (1.1), we get

$$f(a+b) = f(a) + f(b) \tag{2.1}$$

for all $a, b \in \mathbb{R}$ with $a \neq -1$. It is clear that $f(0) = 0$. Letting $a = 1$ and $b = -1$ in (2.1), we get $f(-1) = -f(1)$. Letting $a = 1$ in (2.1), we get

$$f(1+b) = f(1) + f(b) \tag{2.2}$$

for all $b \in \mathbb{R}$. Replacing b by $b-1$ in (2.2), we get $f(b) = f(1) + f(b-1)$ for all $b \in \mathbb{R}$. So (2.1) holds for all $a, b \in \mathbb{R}$. Therefore f is additive.

Conversely, if f is additive, it is easy to check that f satisfies (1.1). □

3 Hyers–Ulam stability of functional equation (1.1)

In this section, we investigate the Hyers–Ulam stability problem for the functional equation (1.1). In this section X is a Banach space.

Theorem 3.1. *Let $\varepsilon \geqslant 0$ be fixed and let $f : \mathbb{R} \to X$ be a mapping satisfying*

$$\|f(x+y+xy) - f(x) - f(y+xy)\| \leqslant \varepsilon \tag{3.1}$$

for all $x, y \in \mathbb{R}$. Then there exists a unique additive mapping $A : \mathbb{R} \to X$ satisfying

$$\|f(x) - A(x)\| \leqslant 3\varepsilon \tag{3.2}$$

for all $x, y \in \mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ with $a \neq -1$. Setting $x = a$ and $y = \frac{b}{1+a}$ in (3.1), we get

$$\|f(a+b) - f(a) - f(b)\| \leqslant \varepsilon \tag{3.3}$$

for all $a, b \in \mathbb{R}$ with $a \neq -1$. Putting $a = 1$ and $b = -1$ in (3.3), yields

$$\|f(0) - f(1) - f(-1)\| \leqslant \varepsilon. \tag{3.4}$$

Letting $a = 1$ in (3.3), we get

$$\|f(1+b) - f(1) - f(b)\| \leqslant \varepsilon \tag{3.5}$$

for all $b \in \mathbb{R}$. Replacing b by $b-1$ in (3.5) and using (3.4), we get

$$\|f(b-1) - f(b) - f(-1) + f(0)\| \leqslant 2\varepsilon \tag{3.6}$$

for all $b \in \mathbb{R}$. Since $\|f(0)\| \leqslant \varepsilon$, it follows from (3.3) and (3.6) that

$$\|f(a+b) - f(a) - f(b)\| \leqslant 3\varepsilon \tag{3.7}$$

for all $a, b \in \mathbb{R}$. By the Hyers' theorem the limit $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in \mathbb{R}$ and A is the unique additive mapping satisfying (3.2). □

Proposition 3.2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } |x| \geqslant 1. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x).$$

Then f satisfies

$$|f(x+y+xy) - f(x) - f(y+xy)| \leqslant 12(|x| + |y|) \tag{3.8}$$

for all $x, y \in \mathbb{R}$, and the range of $|f(x) - A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive mapping $A : \mathbb{R} \to \mathbb{R}$.

Proof. It is clear that f is bounded by 2 on $\mathbb{R}$. If $|x| + |y| = 0$ or $|x| + |y| \geqslant \frac{1}{2}$, then

$$|f(x+y+xy) - f(x) - f(y+xy)| \leqslant 6 \leqslant 12(|x| + |y|).$$

Now suppose that $0 < |x| + |y| < \frac{1}{2}$. Then there exists an integer $k \geqslant 1$ such that

$$\frac{1}{2^{k+1}} \leqslant |x| + |y| < \frac{1}{2^k}. \tag{3.9}$$

Therefore

$$2^m|x+y+xy|, 2^m|x|, 2^m|y+xy| < 1$$

for all $m = 0, 1, ..., k-1$. From the definition of f and (3.9), we have

$$\begin{aligned}
&|f(x+y+xy) - f(x) - f(y+xy)| \\
&\leqslant \sum_{n=k}^{\infty} 2^{-n} \Big[|\phi(2^n(x+y+xy))| + |\phi(2^n(x))| + |\phi(2^n(y+xy))|\Big] \\
&\leqslant \frac{6}{2^k} \leqslant 12(|x| + |y|).
\end{aligned}$$

Therefore f satisfies (3.8). Let $A:\mathbb{R}\to\mathbb{R}$ be an additive function such that

$$|f(x)-A(x)|\leqslant\beta|x|$$

for all $x\in\mathbb{R}$, where $\beta>0$ is a constant. Then there exists a constant $c\in\mathbb{R}$ such that $A(x)=cx$ for all rational numbers x. So we have

$$|f(x)|\leqslant(\beta+|c|)|x| \tag{3.10}$$

for all rational numbers x. Let $m\in\mathbb{N}$ with $m>\beta+|c|$. If x is a rational number in $(0,2^{1-m})$, then $2^n x\in(0,1)$ for all $n=0,1,...,m-1$. So

$$f(x)\geqslant\sum_{n=0}^{m-1}2^{-n}\phi(2^n x)=mx>(\beta+|c|)x$$

which contradicts (3.10). □

4 Stability of functional equation (1.1) in topological vector spaces

In this section E is a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers.

Theorem 4.1. *Let V be a nonempty bounded convex subset of E containing the origin. Suppose that $f:\mathbb{R}\to E$ satisfies*

$$f(x+y+xy)-f(x)-f(y+xy)\in V \tag{4.1}$$

for all $x,y\in\mathbb{R}$. Then there exists a unique additive mapping $A:\mathbb{R}\to E$ such that

$$A(x)-f(x)\in\overline{2V-V} \tag{4.2}$$

for all $x\in\mathbb{R}$, where $\overline{2V-V}$ denotes the sequential closure of $2V-V$.

Proof. Using the proof of Theorem 3.1, we get

$$f(a+b)-f(a)-f(b)\in V \tag{4.3}$$

for all $a,b\in\mathbb{R}$ with $a\neq-1$. Putting $a=1$ and $b=-1$ in (4.3), yields

$$f(0)-f(1)-f(-1)\in V. \tag{4.4}$$

Letting $a=1$ in (4.3), we get

$$f(1+b)-f(1)-f(b)\in V \tag{4.5}$$

for all $b\in\mathbb{R}$. Replacing b by $b-1$ in (4.5) and using (4.4), we get

$$f(b-1)-f(b)-f(-1)+f(0)\in V-V \tag{4.6}$$

for all $b \in \mathbb{R}$. Since $-f(0) \in V$, V is convex and contains the origin, it follows from (4.3) and (4.6) that

$$f(a+b) - f(a) - f(b) \in 2V - V \tag{4.7}$$

for all $a, b \in \mathbb{R}$. It is easy to prove that

$$\frac{f(2^{n+1}a)}{2^{n+1}} - \frac{f(2^n a)}{2^n} \in \frac{1}{2^{n+1}} W \subseteq W, \tag{4.8}$$

$$\frac{f(2^n a)}{2^n} - f(a) \in \sum_{k=1}^{n} \frac{1}{2^k} W \subseteq W \tag{4.9}$$

for all $a \in \mathbb{R}$ and all integers $n \geqslant 1$, where $W = 2V - V$. Since V is a nonempty bounded convex subset of E containing the origin, W is a nonempty bounded convex subset of E containing the origin. It follows from (4.8) that

$$\frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m} = \sum_{k=m}^{n-1} \Big[\frac{f(2^{k+1}a)}{2^{k+1}} - \frac{f(2^k a)}{2^k} \Big] \in \sum_{k=m}^{n-1} \frac{1}{2^{k+1}} W \subseteq \frac{1}{2^m} W \tag{4.10}$$

for all $a \in \mathbb{R}$ and all integers $n > m \geqslant 0$. Let U be an arbitrary neighborhood of the origin in E. Since W is bounded, there exists a rational number $t > 0$ such that $tW \subseteq U$. Choose $n_0 \in \mathbb{N}$ such that $2^{n_0} t > 1$. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$ with $n \geqslant m \geqslant n_0$. Then (4.10) implies that

$$\frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m} \in U. \tag{4.11}$$

Thus, the sequence $\{2^{-n} f(2^n a)\}$ form a Cauchy sequence in E. By the sequential completeness of E, the limit $A(a) = \lim_{n\to\infty} 2^{-n} f(2^n a)$ exists for each $a \in \mathbb{R}$. So (4.2) follows from (4.9).

To show that $A : \mathbb{R} \to E$ is additive, replace a and b by $2^n a$ and $2^n b$, respectively, in (4.7) and then divide by 2^n to obtain

$$\frac{f(2^n(a+b))}{2^n} - \frac{f(2^n a)}{2^n} - \frac{f(2^n b)}{2^n} \in \frac{1}{2^n} W$$

for all $a \in \mathbb{R}$ and all integers $n \geqslant 0$. Since W is bounded, on taking the limit as $n \to \infty$, we get that A is additive.

To prove the uniqueness of A, assume on the contrary that there is another additive mapping $T : \mathbb{R} \to E$ satisfying (4.2) and there is a $a \in \mathbb{R}$ such that $x = T(a) - A(a) \neq 0$. So there is a neighborhood U of the origin in E such that $x \notin U$, since E is Hausdorff. Since A and T satisfy (4.2), we get $T(b) - A(b) \in \overline{W} - \overline{W}$ for all $b \in \mathbb{R}$. Since W is bounded, $\overline{W} - \overline{W}$ is bounded. Hence there exists a positive integer m such that $\overline{W} - \overline{W} \subseteq mU$. Therefore $mx = T(ma) - A(ma) \in mU$ which is a contradiction with $x \notin U$. This completes the proof. □

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