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Wavelet Packets Associated with a Dunkl Type Operator on the Real Line

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Abstract

We consider a singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the one-dimensional Dunkl operator. Using harmonic analysis associated with Λ we define and study three types of generalized wavelet packets and the corresponding wavelet transforms. As an application, we introduce on $\mathbb{R}$ a new multiresolution analysis tied to Λ.

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1 Introduction

In this paper, we consider the first-order differential-difference operator on $\mathbb{R}$

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2}$, B being a positive C^∞ even function on $\mathbb{R}$.

In this paper we suppose that the function A satisfies the following assumptions:

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1) $A(x)$ is increasing on $[0,\infty[$ and $\lim_{x\to\infty} A(x) = \infty$.

2) $\frac{A'(x)}{A(x)}$ is decreasing $]0,\infty[$ and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 0$.

3) There exists a constant $\delta > 0$ such that for x big enough :

$$\frac{B'(x)}{B(x)} = e^{-\delta x} D(x),$$

where D is a C^∞-function bounded together with its derivatives.

For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, we regain the differential-difference operator

$$D_\alpha f = \frac{df}{dx} + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $(\alpha + \frac{1}{2})$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl in connection with a generalization of the classical theory of spherical harmonics (see [5, 6]). An equally important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland types. These are exactly solvable models in one dimension, which were first studied by Calogero and Sutherland [14]. During the last years, such models have gained considerable interest in mathematical physics. They are, for example, of interest in conformal field theory, and are being used to test the ideas of fractional statistics. The Dunkl operator formalism provides explicit operator solutions for a variety of such systems [12].

In [1] the authors have investigated in depth a continuous wavelet transform on the real line tied to the differential-difference operator Λ. Starting from this continuous wavelet analysis, we provide in this paper a general construction allowing the development of wavelet packets associated with the Dunkl type operator Λ. In our construction here, we adopt the same scheme used by Trimeche in [10, 15]. For other construction scheme we refer the reader to [3, 7, 8] .

Historically, the concept of wavelets started to appear more frequently only in the early 1980's. This new concept can be viewed as a synthesis of various ideas originating from different disciplines including mathematics (Calderon-Zygmund operators and Littlewood-Paley theory), physics (the coherent states formalism in quantum mechanics and the renormalization group), and engineering (quadratic mirror filters, sideband coding in signal processing, and pyramidal algorithms in image processing). Briefly, wavelet theory involves breaking up a complicated function (phenomenon) into many simple pieces at different scales and positions. It allows greater flexibility with more desirable features such as "discretization by wavelet packets", and readiness for better implementation (see [2, 4, 9, 11]) and the references therein.

The content of this paper is as follows. In Sec. 2 we mention some basic facts about harmonic analysis related to the differential-difference operator Λ. Using this harmonic analysis, we introduce in Sec. 3-6 three types of generalized wavelet packets (generalized P-wavelet packets, generalized M-wavelet packets, generalized S-wavelet packets) and we investigate the corresponding wavelet packet transforms, i.e., we establish for these transforms Plancherel and reconstruction formulas. Finally we discuss in Sec. 7 a generalized multiresolution analysis by means of generalized wavelet packets.

2 Preliminaries

In this section we recapitulate some basic harmonic analysis results related to the differential-difference operator Λ. We cite here as briefly as possible, only those properties actually required for the discussion. For more details we refer to [1, 13].

Notation. For a positive Borel measure ν on $\mathbb{R}$, and $p = 1$ or 2, we write $L^p(\mathbb{R}, d\nu)$ for the class of measurable functions f on $\mathbb{R}$ for which

$$||f||_{p,\nu} = \left(\int_{\mathbb{R}} |f(x)|^p d\nu(x) \right)^{\frac{1}{p}} < \infty.$$

Theorem 2.1. *For each $\lambda \in \mathbb{C}$, the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Ψ_λ given by:

$$\Psi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \frac{1}{i\lambda}\frac{d}{dx}\varphi_\lambda(x) & \text{if} \quad \lambda \neq 0, \\ 1 & \text{if} \quad \lambda = 0, \end{cases}$$

where $\varphi_\lambda(x)$ is the solution of the differential equation

$$\Delta u = -\lambda^2 u, \quad u(0) = 1, \quad u'(0) = 0,$$

Δ being the differential operator defined on $\mathbb{R}$ by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

Definition 2.2. The generalized Fourier transform $\mathcal{F}_\Lambda$ is defined for smooth functions f on $\mathbb{R}$ by

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)A(x)dx.$$

Remark 2.3. Let $f \in L^1(\mathbb{R}, A(x)dx)$. By [1] we know that

$$|\Psi_\lambda(x)| \leq 1, \tag{2.1}$$

for all λ, $x \in \mathbb{R}$. So it is not hard to see that $\mathcal{F}_\Lambda(f)$ is continuous on $\mathbb{R}$ and $||\mathcal{F}_\Lambda(f)||_\infty \leq ||f||_{1,A}$.

An outstanding result about the generalized Fourier transform $\mathcal{F}$ is as follows.

Theorem 2.4. *[13] (i)For every f in $L^1 \cap L^2(\mathbb{R}, A(x)dx)$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2},$$

$c(z)$ being a continuous function on $]0,\infty[$ *such that*

$$c(z)^{-1} \sim k_1 z^{\alpha+\frac{1}{2}} \quad as \quad z \to \infty, \tag{2.2}$$

$$c(z)^{-1} \sim k_2 z^{\alpha+\frac{1}{2}} \quad as \quad z \to 0, \tag{2.3}$$

for some $k_1, k_2 \in \mathbb{C}$.

(ii) The generalized Fourier transform $\mathcal{F}_\Lambda$ extends uniquely to a unitary isomorphism from $L^2(\mathbb{R}, A(x)dx)$ onto $L^2(\mathbb{R}, d\sigma)$. The inverse transform is given by

$$\mathcal{F}_\Lambda^{-1} g(x) = \int_{\mathbb{R}} g(\lambda)\Psi_\lambda(x)d\sigma(\lambda)$$

where the integral converges in $L^2(\mathbb{R}, A(x)dx)$.

Remark 2.5. (i) The tempered measure σ is called the spectral measure associated with the differential-difference operator Λ.

(ii) For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, we have

$$c(s) = \frac{2^{\alpha+1}\Gamma(\alpha+1)}{s^{\alpha+1/2}}.$$

Definition 2.6. (i) The generalized translation operators $T^x, x \in \mathbb{R}$, are defined on $L^2(\mathbb{R}, A(x)dx)$ by the relation

$$\mathcal{F}_\Lambda(T^x f)(\lambda) = \Psi_\lambda(x)\mathcal{F}_\Lambda(f)(\lambda). \tag{2.4}$$

(ii) The generalized convolution product of two functions f and g in $L^2(\mathbb{R}, A(x)dx)$ is defined by

$$f\#g(x) = \int_{\mathbb{R}} T^x f(-y)g(y)A(y)dy. \tag{2.5}$$

Remark 2.7. Let f and g be in $L^2(\mathbb{R}, A(x)dx)$. Then

(i) By (2.1), (2.4) and Theorem 2.4, we deduce that

$$||T^x f||_{2,A} \leq ||f||_{2,A}, \tag{2.6}$$

for any $x \in \mathbb{R}$.

(ii) It follows from (2.5),(2.6) and Schwarz inequality that $f\#g \in L^\infty(\mathbb{R})$ and

$$||f\#g||_\infty \leq ||f||_{2,A}||g||_{2,A}.$$

(iii) By virtue of (2.4),(2.5) and Theorem 2.4, $f\#g$ may be rewritten as

$$f\#g(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)\Psi_\lambda(x)d\sigma(\lambda).$$

Proposition 2.8. *[1] Let $f \in L^2(\mathbb{R}, A(x)dx)$ and $g \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$. Then $f\#g \in L^2(\mathbb{R}, A(x)dx)$,*

$$||f\#g||_{2,A} \leq ||f||_{2,A}||g||_{1,A},$$

and

$$\mathcal{F}_\Lambda(f\#g) = \mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g) \tag{2.7}$$

The following proposition will play a key role in the sequel.

Proposition 2.9. *[1] Let $f, g \in L^2(\mathbb{R}, A(x)dx)$. Then*

$$\int_{\mathbb{R}} |f\#g(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \, |\mathcal{F}_\Lambda(g)(\lambda)|^2 d\sigma(\lambda), \tag{2.8}$$

where both sides are finite or infinite.

3 Generalized wavelet packets

Our construction of wavelet packets is based on the notion of generalized wavelets studied in depth in [1].

Definition 3.1. We say that a function $g \in L^2(\mathbb{R}, A(x)dx)$ is a generalized wavelet if it satisfies the admissibility condition

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda g(a\lambda)|^2 \frac{da}{a} < \infty, \tag{3.1}$$

for almost all $\lambda \in \mathbb{R}$.

Remark 3.2.

1. The admissibility condition (3.1) can also be written as

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda(g)(\lambda)|^2 \frac{d\lambda}{\lambda} = \int_0^\infty |\mathcal{F}_\Lambda(g)(-\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

2. If g is real-valued we have $\mathcal{F}_\Lambda(g)(-\lambda) = \overline{\mathcal{F}_\Lambda(g)(\lambda)}$, so (3.1) reduces to

$$0 < C_g = \int_0^\infty |\mathcal{F}_\Lambda(g)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

3. If $0 \neq g \in L^2(\mathbb{R}, A(x)dx)$ is real-valued and satisfies

$$\exists\, \eta > 0 \quad \text{such that} \quad \mathcal{F}_\Lambda(g)(\lambda) - \mathcal{F}_\Lambda(g)(0) = O(\lambda^\eta), \quad \text{as} \quad \lambda \to 0^+,$$

then (3.1) is equivalent to $\mathcal{F}_\Lambda(g)(0) = 0$.

The following technical lemma proved in [1] will be of later use.

Lemma 3.3. *Let $a > 0$ and $g \in L^2(\mathbb{R}, A(x)dx)$. Then there exists a function g_a (and only one) in $L^2(\mathbb{R}, A(x)dx)$ such that*

$$\mathcal{F}_\Lambda(g_a)(\lambda) = \mathcal{F}_\Lambda(g)(a\lambda), \tag{3.2}$$

for almost all $\lambda \in \mathbb{R}$. This function is given by the relation

$$g_a = \frac{1}{\sqrt{a}} \mathcal{F}_\Lambda^{-1} \circ H_{a^{-1}} \circ \mathcal{F}_\Lambda(g),$$

and satisfies

$$\|g_a\|_{2,A} \leq \frac{k(a)}{\sqrt{a}} \|g\|_{2,A}, \tag{3.3}$$

where H_a is the dilatation operator defined by

$$H_a(f)(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a}), \quad x \in \mathbb{R},$$

and

$$k(a) = \sup_{\lambda>0} \frac{|c(\lambda)|}{|c(\lambda/a)|}.$$

Remark 3.4. (i) Notice that according to (2) and (3), there exist two positive constants m_1 and m_2 such that

$$\frac{m_1}{a^{\alpha+1/2}} \leq k(a) \leq \frac{m_2}{a^{\alpha+1/2}} \quad \text{for all } a > 0.$$

(ii) For $A(x) = |x|^{2\alpha+1}$, $\alpha > -1/2$, the function g_a, $a > 0$, is given by

$$g_a(x) = \frac{1}{a^{2\alpha+2}} g\left(\frac{x}{a}\right), \quad x \in \mathbb{R}.$$

From now on $g \in L^2(\mathbb{R}, A(x)dx)$ designates a generalized wavelet and $\{r_j\}_{j\in\mathbb{Z}}$ denotes a scale sequence in $]0,\infty[$; that is, strictly decreasing and satisfying

$$\lim_{j\to-\infty} r_j = \infty, \ \lim_{j\to\infty} r_j = 0.$$

Proposition 3.5. *For all $j \in \mathbb{Z}$, there exists a unique function g_j^P in $L^2(\mathbb{R}, A(x)dx)$ such that*

$$\mathcal{F}_\Lambda(g_j^P)(\lambda) = \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathcal{F}_\Lambda(g)(a\lambda)|^2 \frac{da}{a}\right)^{\frac{1}{2}}, \tag{3.4}$$

for almost all $\lambda \in \mathbb{R}$.

Proof. From (10), (11), Theorem 2.4 and Remark 3.2,

$$\begin{aligned}
\frac{1}{C_g}\int_{\mathbb{R}}\int_{r_{j+1}}^{r_j} |\mathcal{F}_\Lambda(g)(a\lambda)|^2 \frac{da}{a} d\sigma(\lambda) &= \frac{1}{C_g}\int_{r_{j+1}}^{r_j} \|\mathcal{F}_\Lambda(g_a)\|_{2,\sigma}^2 \frac{da}{a} \\
&= \frac{1}{C_g}\int_{r_{j+1}}^{r_j} \|g_a\|_{2,A}^2 \frac{da}{a} \\
&\leq \frac{\|g\|_{2,A}^2}{C_g}\int_{r_{j+1}}^{r_j} \left(\frac{k(a)}{a}\right)^2 da \\
&\leq \frac{(m_2)^2 \|g\|_{2,A}^2}{C_g}\int_{r_{j+1}}^{r_j} \frac{da}{a^{2\alpha+3}} < \infty.
\end{aligned}$$

This shows that the function $\lambda \to \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathcal{F}_\Lambda(g)(a\lambda)|^2 \frac{da}{a}\right)^{\frac{1}{2}}$ belongs to $L^2(\mathbb{R}, d\sigma(\lambda))$. The result follows now from Theorem 2.2. □

Definition 3.6. (i) The sequence $\{g_j^P\}_{j\in\mathbb{Z}}$ is called generalized wavelet packet (also called generalized P-wavelet packet).

(ii) The function g_j^P, $j \in \mathbb{Z}$, is called generalized P-wavelet packet member of step j.

Remark 3.7. From (3.1) and (3.4) we deduce that

$$0 \leq \mathcal{F}_\Lambda\left(g_j^P\right)(\lambda) \leq 1 \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \left(\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right)^2 = 1 \tag{3.5}$$

for all $j \in \mathbb{Z}$ and almost all $\lambda \in \mathbb{R}$.

Definition 3.8. Let $\{g_j^P\}_{j\in\mathbb{Z}}$ be a generalized P-wavelet packet. The generalized P-wavelet packet transform Φ_g^P is defined for regular functions f on $\mathbb{R}$ by

$$\Phi_g^P(f)(j,s) = \int_{\mathbb{R}} f(x)\overline{g_{j,s}^P(x)}A(x)dx, \quad j \in \mathbb{Z}, \quad s \in \mathbb{R},$$

where

$$g_{j,s}^P(x) = T^{-s} g_j^P(x)$$

T^{-s} being the generalized translation operators given by (2.4). This transform can also be written as

$$\Phi_g^P(f)(j,s) = f\# g_j^P(s) \tag{3.6}$$

where # is the generalized convolution product given by (2.5).

Theorem 3.9. *Let $\{g_j^P\}_{j\in\mathbb{Z}}$ be a generalized P-wavelet packet. Then for all $f \in L^2(\mathbb{R}, A(x)dx)$ we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |\Phi_g^P(f)(j,s)|^2 A(s)ds.$$

Proof. From (2.8)and (3.6) we have for all $j \in \mathbb{Z}$:

$$\int_{\mathbb{R}} |\Phi_g^P(f)(j,s)|^2 A(s)ds = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2 d\sigma(\lambda).$$

So using (13) and Fubini-Tonelli's theorem we get

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |\Phi_g^P(f)(j,s)|^2 A(s)ds = \\ &= \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \left(\sum_{j=-\infty}^{\infty} (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2 \right) d\sigma(\lambda) \\ &= \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda). \end{aligned}$$

The result follows now from Theorem 2.2. □

Theorem 3.10. *Let $\{g_j^P\}_{j\in\mathbb{Z}}$ be a generalized P-wavelet packet. For all $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ such that $\mathcal{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma(\lambda))$, we have the following reconstruction formula for Φ_g^P :*

$$f(x) = \sum_{j=-\infty}^{\infty} \left(\int_{\mathbb{R}} \Phi_g^P(f)(j,s)\, g_{j,s}^P(x)A(s)ds \right),$$

for almost all $x \in \mathbb{R}$.

Proof. For $j \in \mathbb{Z}$ and $x \in \mathbb{R}$, put

$$\begin{aligned} I(j,x) &= \int_{\mathbb{R}} \Phi_g^P(f)(j,s)\, g_{j,s}^P(x) A(s)ds \\ &= \int_{\mathbb{R}} f\#g_j^P(s)\, \overline{T^{-x} g_j^P(s)} A(s)ds \end{aligned}$$

Using (4), (7) and Theorem 2.2 we obtain

$$I(j,x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)\left[\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right]^2 d\sigma(\lambda). \tag{3.7}$$

By (1) and (13),

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)||\Psi_\lambda(x)|\left[\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right]^2 d\sigma(\lambda) \leq \\ \leq\ & \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|\left[\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right]^2 d\sigma(\lambda) \\ =\ & \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|\left(\sum_{j=-\infty}^{\infty} \left[\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right]^2\right) d\sigma(\lambda) \\ =\ & \|\mathcal{F}_\Lambda(f)\|_{1,\sigma} \quad < \quad \infty. \end{aligned}$$

So using (13), Fubini's theorem and Theorem 2.4 we obtain

$$\begin{aligned} \sum_{j=-\infty}^{\infty} I(j,x) &= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\left(\sum_{j=-\infty}^{\infty} \left[\mathcal{F}_\Lambda\left(g_j^P\right)(\lambda)\right]^2\right)\Psi_\lambda(x)\, d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)\, d\sigma(\lambda) \\ &= f(x), \end{aligned}$$

for almost every $x \in \mathbb{R}$. □

4 Generalized scale discrete scaling function

Proposition 4.1. *Let $\{g_j^P\}_{j\in\mathbb{Z}}$ be a generalized P-wavelet packet. Then*

(i) For all $J \in \mathbb{Z}$,

$$\sum_{j=-\infty}^{J-1} (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2 = \frac{1}{C_g}\int_{r_J}^{\infty} |\mathcal{F}_\Lambda(g)(a\lambda)|^2 \frac{da}{a}. \tag{4.1}$$

(ii) For all $J \in \mathbb{Z}$, there exists a function $G_J^P \in L^2(\mathbb{R}, A(x)dx)$ (and only one) such that

$$\mathcal{F}_\Lambda(G_J^P)(\lambda) = \left(\sum_{j=-\infty}^{J-1} (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2\right)^{\frac{1}{2}}, \quad \lambda \in \mathbb{R}. \tag{4.2}$$

Proof. Assertion (i) follows directly from (12). Let us check (ii). In view of Theorem 2.2, it is sufficient to check that the function $\lambda \to \left(\sum_{j=-\infty}^{J-1} (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2\right)^{\frac{1}{2}}$ belongs to $L^2(\mathbb{R}, d\sigma(\lambda))$. By (10), (11), (16), Theorem 2.4 and Remark 3.2,

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{j=-\infty}^{J-1} (\mathcal{F}_\Lambda(g_j^P)(\lambda))^2 d\sigma(\lambda) &= \frac{1}{C_g} \int_{r_J}^{\infty} \|\mathcal{F}_\Lambda(g_a)\|_{2,\sigma}^2 \frac{da}{a} \\
&= \frac{1}{C_g} \int_{r_J}^{\infty} \|g_a\|_{2,A}^2 \frac{da}{a} \\
&\leq \frac{\|g\|_{2,A}^2}{C_g} \int_{r_J}^{\infty} \left(\frac{k(a)}{a}\right)^2 da \\
&\leq \frac{(m_2)^2 \|g\|_{2,A}^2}{C_g} \int_{r_J}^{\infty} \frac{da}{a^{2\alpha+3}} < \infty,
\end{aligned}$$

which ends the proof. □

Definition 4.2. The sequence $\{G_J^P\}_{J\in\mathbb{Z}}$ is called generalized scale discrete scaling function.

Remark 4.3. It follows from (13) and (17) that

$$0 \leq \mathcal{F}_\Lambda(G_J^P)(\lambda) \leq 1; \quad \lim_{J\to\infty} \mathcal{F}_\Lambda(G_J^P)(\lambda) = 1; \tag{4.3}$$

$$\left(\mathcal{F}_\Lambda(g_J^P)(\lambda)\right)^2 = \left(\mathcal{F}_\Lambda(G_{J+1}^P)(\lambda)\right)^2 - \left(\mathcal{F}_\Lambda(G_J^P)(\lambda)\right)^2; \tag{4.4}$$

$$\sum_{j=-\infty}^{\infty} \left[\left(\mathcal{F}_\Lambda(G_{j+1}^P)(\lambda)\right)^2 - \left(\mathcal{F}_\Lambda(G_j^P)(\lambda)\right)^2\right] = 1; \tag{4.5}$$

for all $J \in \mathbb{Z}$ and almost all $\lambda \in \mathbb{R}$.

Notation. We denote by $\langle .,. \rangle_A$ the scalar product of the Hilbert space $L^2(\mathbb{R}, A(x)dx)$, i.e.,

$$\langle f, g \rangle_A = \int_{\mathbb{R}} f(x)\overline{g(x)}A(x)dx \tag{4.6}$$

for all $f, g \in L^2(\mathbb{R}, A(x)dx)$.

Theorem 4.4. *Let $\{G_J^P\}_{J\in\mathbb{Z}}$ be a generalized scale discrete scaling function. Then for all $f \in L^2(\mathbb{R}, A(x)dx)$ we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \lim_{J\to\infty} \int_{\mathbb{R}} |\langle f, G_{J,s}^P \rangle_A|^2 A(s)ds,$$

where

$$G_{J,s}^P(x) = T^{-s}(G_J^P)(x), \quad x \in \mathbb{R}. \tag{4.7}$$

Proof. From (21) and (22) we have

$$\langle f, G_{J,s}^P \rangle_A = \int_{\mathbb{R}} f(x) T^s G_J^P(-x) A(x)dx = f\#G_J^P(s). \tag{4.8}$$

By (8) we obtain

$$\begin{aligned}\int_{\mathbb{R}} |\langle f, G^P_{J,s}\rangle_A|^2 A(s)ds &= \int_{\mathbb{R}} |f \# G^P_J(s)|^2 A(s)ds \\ &= \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 [\mathcal{F}_\Lambda(G^P_J)(\lambda)]^2 d\sigma(\lambda).\end{aligned}$$

Using (18), Theorem 2.4 and the dominated convergence theorem we get

$$\lim_{J\to\infty} \int_{\mathbb{R}} |\langle f, G^P_{J,s}\rangle_A|^2 A(s)ds = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda) = \|f\|^2_{2,A}\ .$$

□

The next theorem provides a Plancherel formula involving both $\{G^P_J\}_{J\in\mathbb{Z}}$ and Φ^P_g.

Theorem 4.5. *For all $f \in L^2(\mathbb{R}, A(x)dx)$ and all $J \in \mathbb{Z}$ we have*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\langle f, G^P_{J,s}\rangle_A|^2 A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} |\Phi^P_g(f)(j,s)|^2 A(s)ds.$$

Proof. From (8), (17) and (23) and we have

$$\int_{\mathbb{R}} |\langle f, G^P_{J,s}\rangle_A|^2 A(s)ds = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2 d\sigma(\lambda).$$

On the other hand, using (8), (14) and Fubini-Tonelli's theorem we obtain

$$\sum_{j=J}^{\infty} \int_{\mathbb{R}} |\Phi^P_g(f)(j,s)|^2 A(s)ds = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \left(\sum_{j=J}^{\infty} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2 \right) d\sigma(\lambda).$$

Thus

$$\int_{\mathbb{R}} |\langle f, G^P_{J,s}\rangle_A|^2 A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} |\Phi^P_g(f)(j,s)|^2 A(s)ds =$$

$$= \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \left(\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2 \right) d\sigma(\lambda) = \|f\|^2_{2,A},$$

by virtue of (13) and Theorem 2.4. □

Theorem 4.6. *Let $\{G^P_J\}_{J\in\mathbb{Z}}$ be the generalized scale discrete scaling function corresponding to a generalized P-wavelet packet $\{g^P_j\}_{j\in\mathbb{Z}}$. For $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ such that $\mathcal{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma(\lambda))$, we have the following reconstruction formulas :*

(i) For almost all $x \in \mathbb{R}$,

$$f(x) = \lim_{J\to\infty} \int_{\mathbb{R}} \langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) A(s)ds \tag{4.9}$$

(ii) For almost all $x \in \mathbb{R}$ and all $J \in \mathbb{Z}$,

$$f(x) = \int_{\mathbb{R}} \langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} \Phi^P_g(f)(j,s)\, g^P_{j,s}x) A(s)ds$$

Proof. (i) From (22) and (23), we have

$$\langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) = f\# G^P_J(s)\, T^x G^P_J(-s).$$

Using Theorem 2.2 we get

$$\int_{\mathbb{R}} \langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) A(s)ds = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)[\mathcal{F}_\Lambda(G^P_J)(\lambda)]^2\, \Psi_\lambda(x) d\sigma(\lambda). \tag{4.10}$$

Identity (24) follows then by using (18), Theorem 2.4 and the dominated convergence theorem.

(ii) By (17) and (25),
$\int_{\mathbb{R}} \langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) A(s)ds = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\left(\sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2\right) \Psi_\lambda(x) d\sigma(\lambda).$
From this, (13), (15) and Theorem 2.4 it follows that

$$\begin{aligned} &\int_{\mathbb{R}} \langle f, G^P_{J,s}\rangle_A\, G^P_{J,s}(x) A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} \Phi^P_g(f)(j,s)\, g^P_{j,s}x) A(s)ds = \\ = &\int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\left(\sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2 + \sum_{j=J}^{\infty} [\mathcal{F}_\Lambda(g^P_j)(\lambda)]^2\right) \Psi_\lambda(x) d\sigma(\lambda) \\ = &\int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\, \Psi_\lambda(x) d\sigma(\lambda) = f(x), \end{aligned}$$

for almost every $x \in \mathbb{R}$. □

5 Generalized modified wavelet packets

Let $\{G^P_J\}_{J\in\mathbb{Z}}$ be a generalized scale discrete scaling function corresponding to a generalized P-wavelet packet $\{g^P_j\}_{j\in\mathbb{Z}}$. Let g^M_j (resp. $\widetilde{g}^M_j$), $j \in \mathbb{Z}$, be the function defined by

$$g^M_j = G^P_{j+1} - G^P_j. \tag{5.1}$$

$$\left(\text{resp. } \widetilde{g}^M_j = G^P_{j+1} + G^P_j\right) \tag{5.2}$$

Remark 5.1. According to (18) and (20), we have

$$\|\mathcal{F}_\Lambda(g^M_j)\|_\infty \le 2, \qquad \|\mathcal{F}_\Lambda(\widetilde{g}^M_j)\|_\infty \le 2, \tag{5.3}$$

and

$$\sum_{j=-\infty}^{\infty} \mathcal{F}_\Lambda(g^M_j)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}^M_j)(\lambda) = 1, \tag{5.4}$$

for all $j \in \mathbb{Z}$ and almost all $\lambda \in \mathbb{R}$.

Definition 5.2. The sequence $\{g^M_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{g}^M_j\}_{j\in\mathbb{Z}}$ are called respectively generalized modified wavelet packet (or generalized M-wavelet packet) and generalized dual modified wavelet packet (or generalized dual M-wavelet packet).

Definition 5.3. Let $\{g_j^M\}_{j\in\mathbb{Z}}$ be a generalized M-wavelet packet and $\{\widetilde{g}_j^M\}_{j\in\mathbb{Z}}$ the corresponding generalized dual M-wavelet packet. The generalized M-wavelet packet transform Φ_g^M (resp. The generalized dual M-wavelet packet transform $\widetilde{\Phi}_g^M$) is defined for regular functions f on $\mathbb{R}$ by

$$\Phi_g^M(f)(j,s) = \int_{\mathbb{R}} f(x)\overline{g_{j,s}^M(x)}A(x)dx, \quad j\in\mathbb{Z},\ s\in\mathbb{R},$$

$$\left(\text{resp. } \widetilde{\Phi}_g^M(f)(j,s) = \int_{\mathbb{R}} f(x)\overline{\widetilde{g}_{j,s}^M(x)}A(x)dx\right)$$

where

$$g_{j,s}^M(x) = T^{-s}g_j^M(x). \tag{5.5}$$

$$\left(\text{resp. } \widetilde{g}_{j,s}^M(x) = T^{-s}\widetilde{g}_j^M(x)\right) \tag{5.6}$$

The transform Φ_g^M (resp. $\widetilde{\Phi}_g^M$) can also be written in the form

$$\Phi_g^M(f)(j,s) = f\#g_j^M(s), \tag{5.7}$$

$$\left(\text{resp. } \widetilde{\Phi}_g^M(f)(j,s) = f\#\widetilde{g}_j^M(s)\right). \tag{5.8}$$

Theorem 5.4. *Let $\{g_j^M\}_{j\in\mathbb{Z}}$ be a generalized M-wavelet packet and $\{\widetilde{g}_j^M\}_{j\in\mathbb{Z}}$ the corresponding generalized dual M-wavelet packet . For all f in $L^2(\mathbb{R},A(x)dx)$, we have the following Plancherel formula :*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \sum_{j=-\infty}^{\infty}\int_{\mathbb{R}} \Phi_g^M(f)(j,s)\,\overline{\widetilde{\Phi}_g^M(f)(j,s)}A(s)ds.$$

Proof. Notice first that by (30), (32),(33) and Proposition 2.2, the function $s\to\Phi_g^M(f)(j,s)$ (resp. $s\to\overline{\widetilde{\Phi}_g^M(f)(j,s)}$) is in $L^2(\mathbb{R},A(x)dx)$ and $\mathcal{F}_\Lambda(\Phi_g^M(f)(j,\cdot)) = \mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g_j^M)$ (resp. $\mathcal{F}_\Lambda(\widetilde{\Phi}_g^M(f)(j,\cdot)) = \mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(\widetilde{g}_j^M)$). From this and Theorem 2.2, it follows that

$$\int_{\mathbb{R}} \Phi_g^M(f)(j,s)\,\overline{\widetilde{\Phi}_g^M(f)(j,s)}A(s)ds = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2\mathcal{F}_\Lambda(g_j^M)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^M)(\lambda)d\sigma(\lambda).$$

for all $j\in\mathbb{Z}$. But

$$\mathcal{F}_\Lambda(g_j^M)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^M)(\lambda) = \left(\mathcal{F}_\Lambda(g_j^P)(\lambda)\right)^2 \geq 0 \tag{5.9}$$

by virtue of (19), (26) and (27). Then using (29), Theorem 2.4 and Fubini-Tonelli's theorem we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}} \Phi_g^M(f)(j,s)\,\overline{\widetilde{\Phi}_g^M(f)(j,s)}A(s)ds &= \\ &= \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2\left(\sum_{j=-\infty}^{\infty}\mathcal{F}_\Lambda(g_j^M)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^M)(\lambda)\right)d\sigma(\lambda) \\ &= \|\mathcal{F}_\Lambda(f)\|_{2,\sigma}^2 = \|f\|_{2,A}^2.\end{aligned}$$

□

Theorem 5.5. *Let* $\{g_j^M\}_{j\in\mathbb{Z}}$ *be a generalized M-wavelet packet and* $\{\widetilde{g}_j^M\}_{j\in\mathbb{Z}}$ *the corresponding generalized dual M-wavelet packet. For all* $f \in (L^1 \cap L^2)(\mathbb{R}, A(x)dx)$ *such that* $\mathcal{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma(\lambda))$, *we have the following reconstruction formulas for* Φ_g^M *and* $\widetilde{\Phi}_g^M$:

$$\begin{aligned} f(x) &= \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} \Phi_g^M(f)(j,s)\widetilde{g}_{j,s}^M(x)A(s)ds \\ &= \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} \widetilde{\Phi}_g^M(f)(j,s)\, g_{j,s}^M(x)A(s)ds, \end{aligned}$$

for almost all $x \in \mathbb{R}$.

The theorem above is a direct consequence of Theorem 3.10 and the next Lemma .

Lemma 5.6. *Let* $f \in (L^1 \cap L^2)(\mathbb{R}, A(x)dx)$ *such that* $\mathcal{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma(\lambda))$. *Then*

$$\begin{aligned} \int_{\mathbb{R}} \Phi_g^P(f)(j,s)\, g_{j,s}^P(x)A(s)ds &= \int_{\mathbb{R}} \widetilde{\Phi}_g^M(f)(j,s)\, g_{j,s}^M(x)A(s)ds \\ &= \int_{\mathbb{R}} \Phi_g^M(f)(j,s)\widetilde{g}_{j,s}^M(x)A(s)ds \end{aligned}$$

for all $j \in \mathbb{Z}$ *and* $x \in \mathbb{R}$.

Proof. By (15), (30)-(34) and Theorem 2.2, it follows that

$$\begin{aligned} \int_{\mathbb{R}} \Phi_g^M(f)(j,s)\widetilde{g}_{j,s}^M(x)A(s)ds &= \int_{\mathbb{R}} \widetilde{\Phi}_g^M(f)(j,s)\, g_{j,s}^M(x)A(s)ds \\ &= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g_j^M)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^M)(\lambda)\,\Psi_\lambda(x)\,d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)[\mathcal{F}_\Lambda(g_j^P)(\lambda)]^2\,\Psi_{i\lambda}(x)\,d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \Phi_g^P(f)(j,s)\, g_{j,s}^P(x)A(s)ds. \end{aligned}$$

□

A combination of Theorem 4.3 and Lemma 5.1 yields the following result.

Theorem 5.7. *Let $\{g_j^M\}_{j\in\mathbb{Z}}$ be a generalized M-wavelet packet and $\{\widetilde{g}_j^M\}_{j\in\mathbb{Z}}$ the corresponding generalized dual M-wavelet packet. Let f be in $L^1\cap L^2(\mathbb{R},A(x)dx)$ such that $\mathcal{F}_\Lambda(f)$ belongs to $L^1(\mathbb{R},d\sigma(\lambda))$. Then*

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} \langle f, G_{J,s}^P\rangle_A \, G_{J,s}^P(x) A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} \Phi_g^M(f)(j,s)\,\widetilde{g}_{j,s}^M(x) A(s)ds \\ &= \int_{\mathbb{R}} \langle f, G_{J,s}^P\rangle_A \, G_{J,s}^P(x) A(s)ds + \sum_{j=J}^{\infty} \int_{\mathbb{R}} \widetilde{\Phi}_g^M(f)(j,s)\,g_{j,s}^M(x) A(s)ds, \end{aligned}$$

for all $J\in\mathbb{Z}$ and almost every $x\in\mathbb{R}$.

6 Generalized S-wavelet packets

Definition 6.1. A sequence $\{g_j^S\}_{j\in\mathbb{Z}}\subset L^2(\mathbb{R},A(x)dx)$ is called a generalized S-wavelet packet if it satisfies the following conditions :

1. For all $j\in\mathbb{Z}$, $\mathcal{F}_\Lambda(g_j^S)$ is real-valued.
2. For almost all $\lambda\in\mathbb{R}$,

$$a \le \sum_{j=-\infty}^{\infty} (\mathcal{F}_\Lambda(g_j^S)(\lambda))^2 \le b, \tag{6.1}$$

with $0<a\le b<\infty$. The constants a and b are called the stability constants.

Definition 6.2. Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet. We define the corresponding generalized dual S-wavelet packet $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ by

$$\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda) = \frac{\mathcal{F}_\Lambda(g_j^S)(\lambda)}{\sum_{j=-\infty}^{\infty}(\mathcal{F}_\Lambda(g_j^S)(\lambda))^2}, \quad \lambda\in\mathbb{R}. \tag{6.2}$$

Remark 6.3. (i) If $a=b$, then $\widetilde{g}_j^S = \frac{1}{a} g_j^S$.
(ii) By (36), it is easily seen that

$$\sum_{j=-\infty}^{\infty} \mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda) = 1; \tag{6.3}$$

$$\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)]^2 = \left(\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2\right)^{-1}; \tag{6.4}$$

$$\frac{\sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)]^2}{\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)]^2} = \frac{\sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2}{\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2}; \tag{6.5}$$

$$\sum_{j=-\infty}^{J-1} \mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda) = \frac{\sum_{j=-\infty}^{J-1} [\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2}{\sum_{j=-\infty}^{\infty} [\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2}; \tag{6.6}$$

for all $J \in \mathbb{Z}$ and almost all $\lambda \in \mathbb{R}$.

(iii) For almost all $\lambda \in \mathbb{R}$,

$$\frac{1}{b} \leq \sum_{j=-\infty}^{\infty} \left[\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)\right]^2 \leq \frac{1}{a}, \tag{6.7}$$

where a and b are the stability constants given by the relation (35).

Definition 6.4. Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet and $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ the corresponding generalized dual S-wavelet packet. The generalized S-wavelet packet transform Φ_g^S (resp. the generalized dual S-wavelet packet transform $\widetilde{\Phi}_g^S$) is defined for regular functions f on $\mathbb{R}$ by

$$\Phi_g^S(f)(j,t) = \int_{\mathbb{R}} f(x)\overline{g_{j,t}^S(x)}A(x)dx, \quad j \in \mathbb{Z},\ t \in \mathbb{R},$$

$$\left(\text{resp. } \widetilde{\Phi}_g^S(f)(j,t) = \int_{\mathbb{R}} f(x)\overline{\widetilde{g}_{j,t}^S(x)}A(x)dx\right)$$

where

$$g_{j,t}^S(x) = T^{-t}g_j^S(x) \tag{6.8}$$

$$\left(\text{resp. } \widetilde{g}_{j,t}^S(x) = T^{-t}\widetilde{g}_j^S(x)\right) \tag{6.9}$$

The transform Φ_g^S $\left(\text{resp. } \widetilde{\Phi}_g^S\right)$ can be reformulated as

$$\Phi_g^S(f)(j,t) = f\#g_j^S(t) \tag{6.10}$$

$$\left(\text{resp. } \widetilde{\Phi}_g^S(f)(j,t) = f\#\widetilde{g}_j^S(t)\right) \tag{6.11}$$

Theorem 6.5. *Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet and $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ the corresponding generalized dual S-wavelet packet. For all $f \in L^2(\mathbb{R}, A(x)dx)$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} \Phi_g^S(f)(j,t)\overline{\widetilde{\Phi}_g^S(f)(j,t)}A(t)dt.$$

Proof. Observe first that by (35), (41), (44), (45) and Proposition 2.2, the function $t \to \Phi_g^S(f)(j,t)$ (resp. $t \to \overline{\widetilde{\Phi}_g^S(f)(j,t)}$) belongs to $L^2(\mathbb{R}, A(x)dx)$ and $\mathcal{F}_\Lambda(\Phi_g^S(f)(j,\cdot)) = \mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(g_j^S)$ (resp. $\mathcal{F}_\Lambda(\widetilde{\Phi}_g^S(f)(j,\cdot)) = \mathcal{F}_\Lambda(f)\mathcal{F}_\Lambda(\widetilde{g}_j^S)$). From this and Theorem 2.2, it follows that

$$\int_{\mathbb{R}} \Phi_g^S(f)(j,t)\overline{\widetilde{\Phi}_g^S(f)(j,t)}A(t)dt = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 \mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)d\sigma(\lambda).$$

By use of Fubini-Tonelli's theorem, Theorem 2.2 and (37) we obtain

$$
\begin{aligned}
&\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\overline{\widetilde{\Phi}_g^S(f)(j,t)}A(t)dt = \\
&= \int_{\mathbb{R}}|\mathcal{F}_\Lambda(f)(\lambda)|^2\left(\sum_{j=-\infty}^{\infty}\mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)\right)d\sigma(\lambda)\\
&= \int_{\mathbb{R}}|\mathcal{F}_\Lambda(f)(\lambda)|^2 d\sigma(\lambda)\\
&= \|f\|_{2,A}^2.
\end{aligned}
$$

□

Theorem 6.6. *Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet and $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ the corresponding generalized dual S-wavelet packet. For $f\in L^1\cap L^2(\mathbb{R},A(x)dx)$ such that $\mathcal{F}_\Lambda(f)\in L^1(\mathbb{R},d\sigma(\lambda))$, we have the following reconstruction formulas:*

$$
\begin{aligned}
f(x) &= \sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\widetilde{g}_{j,t}^S(x)A(t)dt\\
&= \sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\widetilde{\Phi}_g^S(f)(j,t)\,g_{j,t}^S(x)A(t)dt
\end{aligned}
$$

for almost every $x\in\mathbb{R}$.

Proof. From (36), (42)-(45) and Theorem 2.2 we have

$$
\begin{aligned}
\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\widetilde{g}_{j,t}^S(x)A(t)dt &= \int_{\mathbb{R}}\widetilde{\Phi}_g^S(f)(j,t)\,g_{j,t}^S(x)A(t)dt\\
&= \int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\frac{(\mathcal{F}_\Lambda(g_j^S)(\lambda))^2}{\sum_{j=-\infty}^{\infty}(\mathcal{F}_\Lambda(g_j^S)(\lambda))^2}\Psi_\lambda(x)d\sigma(\lambda).
\end{aligned}
$$

Applying Fubini's theorem we get

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\widetilde{g}_{j,t}^S(x)A(t)dt &= \sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\widetilde{\Phi}_g^S(f)(j,t)g_{j,t}^S(x)A(t)dt\\
&= \int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\Psi_\lambda(x)d\sigma(\lambda)\\
&= f(x),
\end{aligned}
$$

again by Theorem 2.4 . □

Remark 6.7. Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet and $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ the corresponding generalized dual S-wavelet packet. Assume that

$$
\int_{\mathbb{R}}\left(\sum_{j=-\infty}^{J-1}[\mathcal{F}_\Lambda(g_j^S)(\lambda)]^2\right)d\sigma(\lambda)<\infty,\quad \text{for all } J\in\mathbb{Z}. \tag{6.12}
$$

According to (35) and (39)-(41), this ensures that the functions $\lambda \mapsto \left(\sum_{j=-\infty}^{J-1}[\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)]^2\right)^{\frac{1}{2}}$ and $\lambda \mapsto \left(\sum_{j=-\infty}^{J-1}\mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)\right)^{\frac{1}{2}}$ are in $L^2(\mathbb{R}, d\sigma(\lambda))$, and enables us to state the following definition.

Definition 6.8. Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet satisfying (46), and let $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ be the corresponding generalized dual S-wavelet packet. We define the generalized scale discrete scaling function $\{G_J^S\}_{J\in\mathbb{Z}}$ corresponding to $\{g_j^S\}_{j\in\mathbb{Z}}$ by

$$\mathcal{F}_\Lambda(G_J^S)(\lambda) = \left(\sum_{j=-\infty}^{J-1}\mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda)\right)^{\frac{1}{2}}, \quad \lambda \in \mathbb{R}.$$

Remark 6.9. By (40),

$$0 \le \mathcal{F}_\Lambda(G_J^S)(\lambda) \le 1 \quad \text{and} \quad \lim_{J\to\infty}\mathcal{F}_\Lambda(G_J^S)(\lambda) = 1$$

for all $J \in \mathbb{Z}$ and almost all $\lambda \in \mathbb{R}$.

The next statements may be proved in the same way as for Theorems 4.1, 4.2 and 4.3.

Theorem 6.10. *Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet satisfying (46), and let $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ be the corresponding generalized dual S-wavelet packet. For all $f \in L^2(\mathbb{R}, A(x)dx)$ and all $J \in \mathbb{Z}$, we have the following Plancherel formulae :*

$$\begin{aligned}\int_{\mathbb{R}}|f(x)|^2A(x)dx &= \lim_{J\to\infty}\int_{\mathbb{R}}|\langle f, G_{J,t}^S\rangle_A|^2A(t)dt \\ &= \int_{\mathbb{R}}|\langle f, G_{J,t}^S\rangle_A|^2A(t)dt \\ &+ \sum_{j=J}^{\infty}\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\overline{\widetilde{\Phi}_g^S(f)(j,t)}A(t)dt,\end{aligned}$$

with $G_{J,t}^S(x) = T^{-t}(G_J^S)(x)$.

Theorem 6.11. *Let $\{g_j^S\}_{j\in\mathbb{Z}}$ be a generalized S-wavelet packet satisfying (46), and let $\{\widetilde{g}_j^S\}_{j\in\mathbb{Z}}$ be the corresponding generalized dual S-wavelet packet. For $f \in L^1 \cap L^2(\mathbb{R}, A(x)dx)$ such that $\mathcal{F}_\Lambda(f) \in L^1(\mathbb{R}, d\sigma(\lambda))$, we have the following reconstruction formulae :*

(i) For almost all $x \in \mathbb{R}$,

$$f(x) = \lim_{J\to\infty}\int_{\mathbb{R}}\langle f, G_{J,t}^S\rangle_A\, G_{J,t}^S(x)A(t)dt.$$

(ii) For almost all $x \in \mathbb{R}$ and all $J \in \mathbb{Z}$,

$$\begin{aligned}f(x) &= \int_{\mathbb{R}}\langle f, G_{J,t}^S\rangle_A\, G_{J,t}^S(x)A(t)dt + \sum_{j=J}^{\infty}\int_{\mathbb{R}}\Phi_g^S(f)(j,t)\,\widetilde{g}_{j,t}^S(x)A(t)dt \\ &= \int_{\mathbb{R}}\langle f, G_{J,t}^S\rangle_A\, G_{J,t}^S(x)A(t)dt + \sum_{j=J}^{\infty}\int_{\mathbb{R}}\widetilde{\Phi}_g^S(f)(j,t)\, g_{j,t}^S(x)A(t)dt\end{aligned}$$

7 Generalized multiresolution analysis by means of generalized wavelet packets

All generalized wavelet packets discussed in the previous sections show the fact that efficient algorithms for reconstruction and decomposition can be given by using the notion of multiresolution analysis. This will be the theme of the present section.

Definition 7.1. (i) For all $j, J \in \mathbb{Z}$ and $E \in \{P, S\}$, define the convolution operators R_j^E and P_J^E on $L^2(\mathbb{R}, A(x)dx)$ by

$$R_j^E(f) = \begin{cases} g_j^P \# g_j^P \# f = g_j^M \# \widetilde{g}_j^M \# f & \text{for} \quad E = P, \\ g_j^S \# \widetilde{g}_j^S \# f & \text{for} \quad E = S, \end{cases}$$

$$P_J^E(f) = \begin{cases} G_J^P \# G_J^P \# f & \text{for} \quad E = P, \\ G_J^S \# G_J^S \# f & \text{for} \quad E = S. \end{cases}$$

Or equivalently,

$$\mathcal{F}_\Lambda(R_j^E)(f)(\lambda) = \begin{cases} \mathcal{F}_\Lambda(f)(\lambda)\Big(\mathcal{F}_\Lambda(g_j^P)(\lambda)\Big)^2 & \text{for} \quad E = P, \\ \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g_j^M)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^M)(\lambda) & \text{for} \quad E = P, \\ \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g_j^S)(\lambda)\mathcal{F}_\Lambda(\widetilde{g}_j^S)(\lambda) & \text{for} \quad E = S, \end{cases}$$

$$\mathcal{F}_\Lambda(P_J^E)(f)(\lambda) = \begin{cases} \mathcal{F}_\Lambda(f)(\lambda)\Big(\mathcal{F}_\Lambda(G_J^P)(\lambda)\Big)^2 & \text{for} \quad E = P, \\ \mathcal{F}_\Lambda(f)(\lambda)\Big(\mathcal{F}_\Lambda(G_J^S)(\lambda)\Big)^2 & \text{for} \quad E = S. \end{cases}$$

(ii) The scale spaces V_J^E and the detail spaces W_j^E, $E \in \{P, S\}$, are defined respectively by

$$V_J^E = P_J^E\Big(L^2(\mathbb{R}, A(x)dx)\Big)$$

and

$$W_j^E = R_j^E\Big(L^2(\mathbb{R}, A(x)dx)\Big).$$

(iii) The collection $\{V_J^E\}_{J\in\mathbb{Z}}$, $E \in \{P, S\}$, of all the spaces V_J^E is called generalized E-multiresolution analysis of $L^2(\mathbb{R}, A(x)dx)$.

Remark 7.2. (i) According to our definition, the scale space V_J^E is the image of $L^2(\mathbb{R}, A(x)dx)$ by P_J^E. Loosely spoken, V_J^E contains all J-scale smooth functions of $L^2(\mathbb{R}, A(x)dx)$. The notion "detail space" means that W_j^E contains the "detail" information needed to go from an approximation at resolution j to an approximation at resolution $j+1$.

(ii) W_J^E denotes the complementary space of V_J^E in V_{J+1}^E; that is,

$$V_{J+1}^E = V_J^E + W_J^E.$$

This indicates that the definition of detail spaces is independent of the choice of generalized P-wavelet packets or generalized S-wavelet packets.

Proposition 7.3. *The collection* $\{V_J^E\}_{J\in\mathbb{Z}}$, $E \in \{P, S\}$, *enjoys the following properties :*

(i) $V_J^E \subset V_{J'}^E \subset L^2(\mathbb{R}, A(x)dx)$ *for* $-\infty < J \le J' < \infty$*;*

(ii) $\lim_{J\to-\infty} V_J^E = \{0\}$*;*

(iii) $\lim_{J\to\infty} V_J^E = L^2(\mathbb{R}, A(x)dx)$*.*

Observe that a generalized multiresolution analysis allows us to decompose $L^2(\mathbb{R}, A(x)dx)$ by a nested sequence of subspaces which in turn illustrates how to approximate a given function f in $L^2(\mathbb{R}, A(x)dx)$ by an approximation in each of the spaces V_J^E.

More precisely, put

$$\widetilde{h}_j^E = \begin{cases} \Phi_g^P(f)(j,.) & \text{for} \quad E = P, \\ \Phi_g^M(f)(j,.) & \text{for} \quad E = M, \\ \Phi_g^S(f)(j,.) & \text{for} \quad E = S, \end{cases}$$

$$h_j^E = \begin{cases} (\widetilde{h}_j^P, g_j^P)_A & \text{for} \quad E = P, \\ (\widetilde{h}_j^M, g_j^M)_A & \text{for} \quad E = M, \\ (\widetilde{h}_j^S, g_j^S)_A & \text{for} \quad E = S, \end{cases}$$

$$\widetilde{f}_J^E = \begin{cases} (f, G_J^P)_A & \text{for} \quad E = P, \\ (f, G_J^M)_A & \text{for} \quad E = M, \\ (f, G_J^S)_A & \text{for} \quad E = S, \end{cases}$$

$$f_J^E = \begin{cases} (\widetilde{f}^P, G_J^P)_A & \text{for} \quad E = P, \\ (\widetilde{f}^M, G_J^M)_A & \text{for} \quad E = M, \\ (\widetilde{f}^S, G_J^S)_A & \text{for} \quad E = S, \end{cases}$$

with $j, J \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}, A(x)dx)$.

Then we have

$$f_J^E \in V_J^P \text{ for } E \in \{P, M\}; \qquad f_J^S \in V_J^S;$$
$$h_j^E \in W_j^P \text{ for } E \in \{P, M\}; \qquad h_j^S \in W_j^S.$$

Theorem 7.4. *Each function* f *in* $L^2(\mathbb{R}, A(x)dx)$ *can be approximated in twofold sense :*

$$f = \lim_{J\to\infty} f_J^E, \qquad E \in \{P, M, S\}$$

and

$$f = \lim_{J\to\infty} \sum_{j=-\infty}^{J-1} h_j^E, \qquad E \in \{P, M, S\}.$$

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References

[1] W. Chabeh and M. A. Mourou, Inversion of the generalized Dunkl intertwining operator on $\mathbb{R}$ and its dual using generalized wavelets. *Eur. J. Pure Appl. Math.* **3** (2010), pp 958-979.

[2] C. K. Chui, An introduction to wavelets. *Academic Press, New York* (1992).

[3] R. R. Coifman and M.V. Wickerhauser, Entropy-Based Algorithms for Best Basis Selection. *IEEE Transactions on Information Theory* **38** (1992), pp 1241-1243.

[4] I. Daubechies, Ten lectures on wavelets. *CBMS-NSF regional Conf. Ser. Appl. Math.* **61** (1992).

[5] C. F. Dunkl, Integral kernels with reflection group invariance. *Can. J. Math.* **43** (1991), pp 1213-1227.

[6] C. F. Dunkl, Hankel transforms associated to finite reflection groups. *Contemp. Math.* **138** (1992), pp 128-138.

[7] M. Duval-Destin, M.N. Muschietti and M. Torresani, Continuous wavelet decomposition , multiresolution and contrast analysis. *SIAM J. Math. Anal.* **24** (1993), pp 739-755.

[8] W. Freeden and U. Windheuser, Spherical wavelet transform and its discretization. *Advances in comput. Math.* **5** (1996), pp 51-94.

[9] M. Holschneider, Wavelets: An Analysis Tool. *Clarendon Press, Oxford* (1995).

[10] A. Jouini and K. Trimeche, Two Versions of Wavelets and Applications. *Alpha Sciences* (2006).

[11] T. H. Koornwinder, The continuous wavelet transform. Wavelets : An Elementary Treatement of Theory and Applications. Edited by T. H. Koornwinder, World Scientific **1** (1993), pp 27-48.

[12] L. Lapointe and L. Vinet, Exact operator solution of the Calogero-Sutherland model. *Comm. Math. Phys.* **178** (1996), pp 425-452.

[13] M. A. Mourou and K. Trimeche, Transmutation operators and Paley-Wiener theorem associated with a singular differential-difference operator on the real line. *Anal. Appl.* **1** (2003), pp 43-69.

[14] B. Sutherland, Exact results for a quantum many-body problem in one dimension. *Phys. Rep.* **A5** (1972), pp 1372-1376.

[15] K. Trimeche, Generalized Harmonic Analysis and Wavelet Packets. *Ann. Probab.* Gordon and Breach Publishing group (2001).