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A SURVEY OF JENSEN TYPE INEQUALITIES FOR LOG-CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract

Some recent Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are surveyed. Applications in relation with some celebrated results due to Hölder-McCarthy and Ky Fan are provided as well.

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1 Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [13, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right)=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

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and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [13] and the references therein. For other results, see [19], [15], [18] and [16]. For recent results, see [2]-[12].

2 Some Jensen's Type Inequalities for Log-convex Functions

2.1 Preliminary Results

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [17] (see also [13, p. 5]):

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If f is a convex function on $[m,M]$, then

$$f(\langle Ax,x\rangle) \leq \langle f(A)x,x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f : I \to (0,\infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1-t)a+tb) \leq [f(a)]^{1-t}[f(b)]^{t} \tag{2.1}$$

for any $a,b \in I$ and $t \in [0,1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

As an imediate consequence of the Mond-Pečarić inequality above we can provide the following result:

Theorem 2.1 (Dragomir, 2010, [11])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If $g : [m,M] \to (0,\infty)$ is log-convex, then*

$$g(\langle Ax,x\rangle) \leq \exp\langle \ln g(A)x,x\rangle \leq \langle g(A)x,x\rangle \tag{2.2}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $f := \ln g$, which is convex on $[m,M]$. Writing (MP) for f we get $\ln[g(\langle Ax,x\rangle)] \leq \langle \ln g(A)x,x\rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.2).

If we also use (MP) for the exponential function, we get

$$\exp\langle \ln g(A)x,x\rangle \le \langle \exp[\ln g(A)]x,x\rangle = \langle g(A)x,x\rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. □

The case of sequences of operators may be of interest and is embodied in the following corollary:

Corollary 2.2 (Dragomir, 2010, [11])**.** *Assume that g is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m,M]$, $j \in \{1,...,n\}$ and $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$g\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \le \exp\left\langle \sum_{j=1}^n \ln g(A_j)x_j, x_j\right\rangle \le \left\langle \sum_{j=1}^n g(A_j)x_j, x_j\right\rangle. \tag{2.3}$$

Proof. Follows from Theorem 2.1and we omit the details. □

In particular we have:

Corollary 2.3 (Dragomir, 2010, [11])**.** *Assume that g is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m,M] \subset \mathring{I}$, $j \in \{1,...,n\}$ and $p_j \ge 0$, $j \in \{1,...,n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right) \le \left\langle \prod_{j=1}^n \left[g(A_j)\right]^{p_j} x, x\right\rangle \le \left\langle \sum_{j=1}^n p_j g(A_j)x, x\right\rangle \tag{2.4}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows from Corollary 2.2 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1,...,n\}$ where $x \in H$ with $\|x\| = 1$. □

It is also important to observe that, as a special case of (MP) we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

Theorem 2.4 (Hölder-McCarthy, 1967, [14])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^{-r} x, x\rangle \ge \langle Ax, x\rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows:

Proposition 2.5. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax, x\rangle^{-r} \le \exp\langle \ln(A^{-r})x, x\rangle \le \langle A^{-r}x, x\rangle \tag{2.5}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [13, p. 57]:

Theorem 2.6. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If f is a convex function on $[m,M]$, then*

$$\langle f(A)x,x\rangle \leq \frac{M-\langle Ax,x\rangle}{M-m} \cdot f(m) + \frac{\langle Ax,x\rangle - m}{M-m} \cdot f(M) \tag{2.6}$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows:

Theorem 2.7 (Dragomir, 2010, [11])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If $g : [m,M] \to (0,\infty)$ is log-convex, then*

$$\begin{aligned} \langle g(A)x,x\rangle &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A-m1_H}{M-m}} \right] x,x \right\rangle \\ &\leq \frac{M-\langle Ax,x\rangle}{M-m} \cdot g(m) + \frac{\langle Ax,x\rangle - m}{M-m} \cdot g(M) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} g(\langle Ax,x\rangle) &\leq [g(m)]^{\frac{M-\langle Ax,x\rangle}{M-m}} [g(M)]^{\frac{\langle Ax,x\rangle - m}{M-m}} \\ &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A-m1_H}{M-m}} \right] x,x \right\rangle \end{aligned} \tag{2.8}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Observe that, by the log-convexity of g, we have

$$g(t) = g\left(\frac{M-t}{M-m} \cdot m + \frac{t-m}{M-m} \cdot M \right) \leq [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \tag{2.9}$$

for any $t \in [m,M]$.

Applying the property (P) for the operator A, we have that

$$\langle g(A)x,x\rangle \leq \langle \Psi(A)x,x\rangle$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}}$, $t \in [m,M]$. This proves the first inequality in (2.7).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have

$$[g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \leq \frac{M-t}{M-m} \cdot g(m) + \frac{t-m}{M-m} \cdot g(M)$$

for any $t \in [m,M]$.

Applying the property (P) for the operator A we deduce the second inequality in (2.7).

Further on, if we use the inequality (2.9) for $t = \langle Ax,x\rangle \in [m,M]$ then we deduce the first part of (2.8).

Now, observe that the function Ψ introduced above can be rearranged to read as

$$\Psi(t) = g(m)\left[\frac{g(M)}{g(m)}\right]^{\frac{t-m}{M-m}}, t \in [m, M]$$

showing that Ψ is a convex function on $[m, M]$.

Applying Mond-Pečarić's inequality for Ψ we deduce the second part of (2.8) and the proof is complete. □

The case of sequences of operators is as follows:

Corollary 2.8 (Dragomir, 2010, [11])**.** *Assume that g is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then*

$$\begin{aligned} &\sum_{j=1}^{n} \left\langle g(A_j) x_j, x_j \right\rangle \qquad (2.10) \\ &\le \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} \right] x_j, x_j \right\rangle \\ &\le \frac{M - \sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle}{M-m} \cdot g(m) + \frac{\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle - m}{M-m} \cdot g(M) \end{aligned}$$

and

$$\begin{aligned} &g\left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle\right) \qquad (2.11) \\ &\le [g(m)]^{\frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M-m}} [g(M)]^{\frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M-m}} \\ &\le \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} \right] x_j, x_j \right\rangle. \end{aligned}$$

In particular we have:

Corollary 2.9 (Dragomir, 2010, [11])**.** *Assume that g is as in the Theorem 2.1. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \qquad (2.12) \\ &\le \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} x, x \right\rangle \\ &\le \frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M-m} \cdot g(m) + \frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M-m} \cdot g(M) \end{aligned}$$

and

$$g\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) \tag{2.13}$$
$$\leq [g(m)]^{\frac{M-\left\langle\sum_{j=1}^{n} p_j A_j x,x\right\rangle}{M-m}} [g(M)]^{\frac{\left\langle\sum_{j=1}^{n} p_j A_j x,x\right\rangle-m}{M-m}}$$
$$\leq \left\langle\sum_{j=1}^{n} p_j [g(m)]^{\frac{M1_H-A_j}{M-m}} [g(M)]^{\frac{A_j-m1_H}{M-m}} x, x\right\rangle.$$

The above result from Theorem 2.7 can be utilized to produce the following reverse inequality for negative powers of operators:

Proposition 2.10. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M]\,(0 < m < M)$, then*

$$\langle A^{-r}x, x\rangle \leq \left\langle\left[m^{\frac{M1_H-A}{M-m}} M^{\frac{A-m1_H}{M-m}}\right]^{-r} x, x\right\rangle \tag{2.14}$$
$$\leq \frac{M-\langle Ax, x\rangle}{M-m}\cdot m^{-r} + \frac{\langle Ax, x\rangle - m}{M-m}\cdot M^{-r}$$

and

$$\langle Ax, x\rangle^{-r} \leq \left[g(m)^{\frac{M-\langle Ax,x\rangle}{M-m}} g(M)^{\frac{\langle Ax,x\rangle-m}{M-m}}\right]^{-r} \tag{2.15}$$
$$\leq \left\langle\left[m^{\frac{M1_H-A}{M-m}} M^{\frac{A-m1_H}{M-m}}\right]^{-r} x, x\right\rangle$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

2.2 Jensen's Inequality for Differentiable Log-convex Functions

The following result provides a reverse for the Jensen type inequality (MP):

Theorem 2.11 (Dragomir, 2008, [5])**.** *Let J be an interval and $f : J \to \mathbb{R}$ be a convex and differentiable function on $\mathring{J}$ (the interior of J) whose derivative f' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(0 \leq)\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{2.16}$$

for any $x \in H$ with $\|x\| = 1$.

The following result may be stated:

Proposition 2.12 (Dragomir, 2010, [11])**.** *Let J be an interval and $g : J \to \mathbb{R}$ be a differentiable log-convex function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$(1 \leq)\frac{\exp\langle \ln g(A)x, x\rangle}{g(\langle Ax, x\rangle)} \tag{2.17}$$
$$\leq \exp\left[\left\langle g'(A)[g(A)]^{-1}Ax, x\right\rangle - \langle Ax, x\rangle \cdot \left\langle g'(A)[g(A)]^{-1}x, x\right\rangle\right]$$

for each $x \in H$ with $\|x\| = 1$.

Proof. It follows by the inequality (2.16) written for the convex function $f = \ln g$ that

$$\begin{aligned}\langle \ln g(A)x,x\rangle &\le \ln g(\langle Ax,x\rangle)\\ &\quad+\left\langle g'(A)[g(A)]^{-1}Ax,x\right\rangle-\langle Ax,x\rangle\cdot\left\langle g'(A)[g(A)]^{-1}x,x\right\rangle\end{aligned}$$

for each $x\in H$ with $\|x\|=1$.

Now, taking the exponential and dividing by $g(\langle Ax,x\rangle)>0$ for each $x\in H$ with $\|x\|=1$, we deduce the desired result (2.17). □

Corollary 2.13 (Dragomir, 2010, [11]). *Assume that g is as in the Proposition 2.12 and A_j are selfadjoint operators with $Sp\left(A_j\right)\subseteq[m,M]\subset\mathring{J}$, $j\in\{1,...,n\}$.*

If and $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^n\left\|x_j\right\|^2=1$, then

$$\begin{aligned}(1\le)\,&\frac{\exp\left\langle\sum_{j=1}^n\ln g\left(A_j\right)x_j,x_j\right\rangle}{g\left(\sum_{j=1}^n\left\langle A_jx,x_j\right\rangle\right)} \qquad (2.18)\\ &\le\exp\left[\left\langle\sum_{j=1}^n g'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}A_jx_j,x_j\right\rangle\right.\\ &\left.-\sum_{j=1}^n\left\langle A_jx_j,x_j\right\rangle\cdot\sum_{j=1}^n\left\langle g'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}x_j,x_j\right\rangle\right].\end{aligned}$$

If $p_j\ge0$, $j\in\{1,...,n\}$ with $\sum_{j=1}^n p_j=1$, then

$$\begin{aligned}(1\le)\,&\frac{\left\langle\prod_{j=1}^n\left[g\left(A_j\right)\right]^{p_j}x,x\right\rangle}{g\left(\left\langle\sum_{j=1}^n p_jA_jx,x\right\rangle\right)} \qquad (2.19)\\ &\le\exp\left[\left\langle\sum_{j=1}^n p_jg'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}A_jx,x\right\rangle\right.\\ &\left.-\sum_{j=1}^n p_j\left\langle A_jx,x\right\rangle\cdot\sum_{j=1}^n p_j\left\langle g'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}x,x\right\rangle\right]\end{aligned}$$

for each $x\in H$ with $\|x\|=1$.

Remark 2.14. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$(1\le)\langle Ax,x\rangle^r\exp\langle\ln(A^{-r})x,x\rangle\le\exp\left[r\left(\langle Ax,x\rangle\cdot\left\langle A^{-1}x,x\right\rangle-1\right)\right] \qquad (2.20)$$

for all $r>0$ and $x\in H$ with $\|x\|=1$.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality can be stated as well:

Theorem 2.15 (Dragomir, 2010, [11])**.** *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$1 \leq \left\langle \exp\left[\frac{g'(\langle Ax,x\rangle)}{g(\langle Ax,x\rangle)}(A - \langle Ax,x\rangle 1_H)\right]x,x\right\rangle \tag{2.21}$$
$$\leq \frac{\langle g(A)x,x\rangle}{g(\langle Ax,x\rangle)} \leq \left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax,x\rangle 1_H)\right]x,x\right\rangle$$

for each $x \in H$ with $\|x\| = 1$, where 1_H denotes the identity operator on H.

Proof. It is well known that if $h : J \to \mathbb{R}$ is a convex differentiable function on $\mathring{J}$, then the following *gradient inequality* holds

$$h(t) - h(s) \geq h'(s)(t - s)$$

for any $t, s \in \mathring{J}$.

Now, if we write this inequality for the convex function $h = \ln g$, then we get

$$\ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t - s) \tag{2.22}$$

which is equivalent with

$$g(t) \geq g(s)\exp\left[\frac{g'(s)}{g(s)}(t - s)\right] \tag{2.23}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax,x\rangle \in [m,M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (2.23), then we get

$$g(t) \geq g(\langle Ax,x\rangle)\exp\left[\frac{g'(\langle Ax,x\rangle)}{g(\langle Ax,x\rangle)}(t - \langle Ax,x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

$$\langle g(A)y,y\rangle \geq g(\langle Ax,x\rangle)\left\langle \exp\left[\frac{g'(\langle Ax,x\rangle)}{g(\langle Ax,x\rangle)}(A - \langle Ax,x\rangle 1_H)\right]y,y\right\rangle \tag{2.24}$$
$$\geq g(\langle Ax,x\rangle)\exp\left[\frac{g'(\langle Ax,x\rangle)}{g(\langle Ax,x\rangle)}(\langle Ay,y\rangle - \langle Ax,x\rangle)\right]$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Further, if we put $y = x$ in (2.24), then we deduce the first and the second inequality in (2.21).

Now, if we replace s with t in (2.23) we can also write the inequality

$$g(t)\exp\left[\frac{g'(t)}{g(t)}(s - t)\right] \leq g(s)$$

which is equivalent with

$$g(t) \le g(s)\exp\left[\frac{g'(t)}{g(t)}(t-s)\right] \tag{2.25}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (2.25), then we get

$$g(t) \le g(\langle Ax, x\rangle)\exp\left[\frac{g'(t)}{g(t)}(t-\langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A, then we can state the following inequality that is of interest in itself as well:

$$\langle g(A)y, y\rangle \le g(\langle Ax, x\rangle)\left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right]y, y\right\rangle \tag{2.26}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (2.26), then we deduce the last inequality in (2.21). □

The case of operator sequences is embodied in the following corollary:

Corollary 2.16 (Dragomir, 2010, [11])**.** *Assume that g is as in the Proposition 2.12 and A_j are selfadjoint operators with $Sp\left(A_j\right) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \left\|x_j\right\|^2 = 1$, then

$$\begin{aligned} 1 &\le \left\langle \sum_{j=1}^n \exp\left[\frac{g'\left(\sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle\right)}{g\left(\sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle\right)}\left(A_j - \sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle 1_H\right)\right] x_j, x_j\right\rangle \\ &\le \frac{\sum_{j=1}^n \left\langle g\left(A_j\right)x_j, x_j\right\rangle}{g\left(\sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle\right)} \\ &\le \left\langle \sum_{j=1}^n \exp\left[g'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}\left(A_j - \sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle 1_H\right)\right] x_j, x_j\right\rangle. \end{aligned} \tag{2.27}$$

If $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$\begin{aligned} 1 &\le \left\langle \sum_{j=1}^n p_j \exp\left[\frac{g'\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)}\right.\right. \\ &\quad \times \left.\left.\left(A_j - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle 1_H\right)\right] x, x\right\rangle \\ &\le \frac{\left\langle \sum_{j=1}^n p_j g\left(A_j\right)x, x\right\rangle}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle\right)} \\ &\le \left\langle \sum_{j=1}^n p_j \exp\left[g'\left(A_j\right)\left[g\left(A_j\right)\right]^{-1}\left(A_j - \left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle 1_H\right)\right] x, x\right\rangle. \end{aligned} \tag{2.28}$$

Remark 2.17. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$\begin{aligned} 1 &\le \left\langle \exp\left[r\left(1_H - \langle Ax,x\rangle^{-1} A\right)\right] x,x\right\rangle \\ &\le \langle A^{-r}x,x\rangle \langle Ax,x\rangle^{r} \le \left\langle \exp\left[r\left(1_H - \langle Ax,x\rangle A^{-1}\right)\right] x,x\right\rangle \end{aligned} \tag{2.29}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well:

Theorem 2.18 (Dragomir, 2010, [11]). *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m,M] \subset \mathring{J}$, then*

$$\begin{aligned} (1 \le) &\frac{\left\langle [g(M)]^{\frac{A-m1_H}{M-m}} [g(m)]^{\frac{M1_H-A}{M-m}} x,x\right\rangle}{\langle g(A)x,x\rangle} \\ &\le \frac{\left\langle g(A)\exp\left[\frac{(M1_H-A)(A-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]x,x\right\rangle}{\langle g(A)x,x\rangle} \\ &\le \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \end{aligned} \tag{2.30}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Utilising the inequality (2.22) we have successively

$$\frac{g((1-\lambda)t+\lambda s)}{g(s)} \ge \exp\left[(1-\lambda)\frac{g'(s)}{g(s)}(t-s)\right] \tag{2.31}$$

and

$$\frac{g((1-\lambda)t+\lambda s)}{g(t)} \ge \exp\left[-\lambda\frac{g'(t)}{g(t)}(t-s)\right] \tag{2.32}$$

for any $t,s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we take the power λ in the inequality (2.31) and the power $1-\lambda$ in (2.32) and multiply the obtained inequalities, we deduce

$$\begin{aligned} &\frac{[g(t)]^{1-\lambda}[g(s)]^{\lambda}}{g((1-\lambda)t+\lambda s)} \\ &\le \exp\left[(1-\lambda)\lambda\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t-s)\right] \end{aligned} \tag{2.33}$$

for any $t,s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Further on, if we choose in (2.33) $t = M, s = m$ and $\lambda = \frac{M-u}{M-m}$, then, from (2.33) we get the inequality

$$\begin{aligned} &\frac{[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}}}{g(u)} \\ &\le \exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \end{aligned} \tag{2.34}$$

which, together with the inequality

$$\frac{(M-u)(u-m)}{M-m} \le \frac{1}{4}(M-m)$$

produce

$$\begin{aligned}
&[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}} \\
&\le g(u)\exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] \\
&\le g(u)\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]
\end{aligned} \tag{2.35}$$

for any $u \in [m,M]$.

If we apply the property (P) to the inequality (2.35) and for the operator A we deduce the desired result. □

Corollary 2.19 (Dragomir, 2010, [11]). *Assume that g is as in the Theorem 2.18 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m,M] \subset \mathring{J}$, $j \in \{1,...,n\}$.*

If $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$\begin{aligned}
(1 \le)\, &\frac{\sum_{j=1}^n \left\langle [g(M)]^{\frac{A_j - m1_H}{M-m}}[g(m)]^{\frac{M1_H - A_j}{M-m}} x_j, x_j \right\rangle}{\sum_{j=1}^n \left\langle g(A_j) x_j, x_j\right\rangle} \\
\le\, &\frac{\sum_{j=1}^n \left\langle g(A_j)\exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] x_j, x_j\right\rangle}{\sum_{j=1}^n \left\langle g(A_j) x_j, x_j\right\rangle} \\
\le\, &\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right].
\end{aligned} \tag{2.36}$$

If $p_j \ge 0$, $j \in \{1,...,n\}$ with $\sum_{j=1}^n p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$\begin{aligned}
(1 \le)\, &\frac{\left\langle \sum_{j=1}^n p_j [g(M)]^{\frac{A_j - m1_H}{M-m}}[g(m)]^{\frac{M1_H - A_j}{M-m}} x, x \right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x\right\rangle} \\
\le\, &\frac{\left\langle \sum_{j=1}^n p_j g(A_j)\exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] x, x\right\rangle}{\left\langle \sum_{j=1}^n p_j g(A_j) x, x\right\rangle} \\
\le\, &\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right].
\end{aligned} \tag{2.37}$$

Remark 2.20. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m,M]\,(0<m<M)$, then

$$(1 \leq) \frac{\left\langle [g(M)]^{\frac{r(m1_H - A)}{M-m}} [g(m)]^{\frac{r(A-M1_H)}{M-m}} x, x \right\rangle}{\langle A^{-r}x, x\rangle} \leq \frac{\left\langle A^{-r} \exp\left[\frac{r(M1_H - A)(A-m1_H)}{Mm}\right] x, x\right\rangle}{\langle A^{-r}x, x\rangle} \leq \exp\left[\frac{1}{4} r \frac{(M-m)^2}{mM}\right] \tag{2.38}$$

2.3 Applications for Ky Fan's Inequality

Consider the function $g:(0,1)\to\mathbb{R}$, $g(t)=\left(\frac{1-t}{t}\right)^r$, $r>0$. Observe that for the new function $f:(0,1)\to\mathbb{R}$, $f(t)=\ln g(t)$ we have

$$f'(t) = \frac{-r}{t(1-t)} \text{ and } f''(t) = \frac{2r\left(\frac{1}{2}-t\right)}{t^2(1-t)^2} \text{ for } t\in(0,1)$$

showing that the function g is log-convex on the interval $\left(0,\frac{1}{2}\right)$.

If $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $t_i\in\left(0,\frac{1}{2}\right)$ for $i\in\{1,...,n\}$, then by applying the Jensen inequality for the convex function f (with $r=1$) on the interval $\left(0,\frac{1}{2}\right)$ we get

$$\frac{\sum_{i=1}^n p_i t_i}{1-\sum_{i=1}^n p_i t_i} \geq \prod_{i=1}^n \left(\frac{t_i}{1-t_i}\right)^{p_i}, \tag{2.39}$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [1, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^n \left(\frac{1-t_i}{t_i}\right)^{p_i} \geq \frac{1-\sum_{i=1}^n p_i t_i}{\sum_{i=1}^n p_i t_i},$$

where $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $t_i\in\left(0,\frac{1}{2}\right)$ for $i\in\{1,...,n\}$.

By the weighted arithmetic mean - geometric mean inequality we also have that

$$\sum_{i=1}^n p_i(1-t_i)t_i^{-1} \geq \prod_{i=1}^n \left(\frac{1-t_i}{t_i}\right)^{p_i}$$

giving the double inequality

$$\sum_{i=1}^n p_i(1-t_i)t_i^{-1} \geq \prod_{i=1}^n \left((1-t_i)t_i^{-1}\right)^{p_i} \geq \sum_{i=1}^n p_i(1-t_i)\left(\sum_{i=1}^n p_i t_i\right)^{-1}. \tag{2.40}$$

The following operator inequalities generalizing (2.40) may be stated:

Proposition 2.21. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0, \frac{1}{2}\right)$, then*

$$\left\langle \left(A^{-1}(1_H - A)\right)^r x, x \right\rangle \geq \exp\left\langle \ln\left(A^{-1}(1_H - A)\right)^r x, x \right\rangle \geq \left(\langle (1_H - A)x, x\rangle \langle Ax, x\rangle^{-1}\right)^r \tag{2.41}$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

In particular,

$$\left\langle A^{-1}(1_H - A)x, x \right\rangle \geq \exp\left\langle \ln\left(A^{-1}(1_H - A)\right)x, x \right\rangle \geq \langle (1_H - A)x, x\rangle \langle Ax, x\rangle^{-1} \tag{2.42}$$

for each $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 2.1 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$.

Proposition 2.22. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M] \subset \left(0, \frac{1}{2}\right)$, then*

$$\begin{aligned} &\left\langle \left((1_H - A)A^{-1}\right)^r x, x \right\rangle \\ &\leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A - m1_H)}{M-m}}\right] x, x \right\rangle \\ &\leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot \left(\frac{1-m}{m}\right)^r + \frac{\langle Ax, x\rangle - m}{M - m} \cdot \left(\frac{1-M}{M}\right)^r \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} &\left(\frac{1 - \langle Ax, x\rangle}{\langle Ax, x\rangle}\right)^r \\ &\leq \left(\frac{1-m}{m}\right)^{\frac{r(M - \langle Ax, x\rangle)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(\langle Ax, x\rangle - m)}{M-m}} \\ &\leq \left\langle \left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H - A)}{M-m}} \left(\frac{1-M}{M}\right)^{\frac{r(A - m1_H)}{M-m}}\right] x, x \right\rangle \end{aligned} \tag{2.44}$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Theorem 2.7 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$.

Finally we have:

Proposition 2.23. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subset \left(0, \frac{1}{2}\right)$, then*

$$(1 \leq) \frac{\exp\left\langle \ln\left((1_H - A)A^{-1}\right)^r x, x\right\rangle}{\left((1 - \langle Ax, x\rangle)\langle Ax, x\rangle^{-1}\right)^r} \tag{2.45}$$
$$\leq \exp\left[r\left(\langle Ax, x\rangle \cdot \left\langle A^{-1}(1_H - A)^{-1} x, x\right\rangle - \left\langle (1_H - A)^{-1} x, x\right\rangle\right)\right]$$

and

$$1 \leq \left\langle \exp\left[r(1 - \langle Ax, x\rangle)^{-1}\left(1_H - \langle Ax, x\rangle^{-1} A\right)\right] x, x\right\rangle \tag{2.46}$$
$$\leq \frac{\left\langle \left((1_H - A)A^{-1}\right)^r x, x\right\rangle}{\left((1 - \langle Ax, x\rangle)\langle Ax, x\rangle^{-1}\right)^r}$$
$$\leq \left\langle \exp\left[r(1_H - A)^{-1}\left(\langle Ax, x\rangle A^{-1} - 1_H\right)\right] x, x\right\rangle$$

for each $x \in H$ with $\|x\| = 1$ and $r > 0$.

The proof follows by Proposition 2.12 and Theorem 2.15 applied for the log-convex function $g(t) = \left(\frac{1-t}{t}\right)^r, r > 0, t \in \left(0, \frac{1}{2}\right)$. The details are omitted.

2.4 More Inequalities for Differentiable Log-convex Functions

The following results providing companion inequalities for the Jensen inequality for differentiable log-convex functions hold:

Theorem 2.24 (Dragomir, 2010, [12])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\exp\left[\frac{\langle g'(A)Ax, x\rangle}{\langle g(A)x, x\rangle} - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \cdot \frac{\langle g'(A)x, x\rangle}{\langle g(A)x, x\rangle}\right] \tag{2.47}$$
$$\geq \frac{\exp\left[\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\right]}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)} \geq 1$$

for each $x \in H$ with $\|x\| = 1$.

If

$$\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle} \in \mathring{J} \text{ for each } x \in H \text{ with } \|x\| = 1, \tag{C}$$

then

$$\exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle} - \frac{\langle Ag(A)x, x\rangle}{\langle g(A)x, x\rangle}\right)\right] \tag{2.48}$$
$$\geq \frac{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{\exp\left(\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\right)} \geq 1,$$

for each $x \in H$ with $\|x\| = 1$.

Proof. By the gradient inequality for the convex function $\ln g$ we have

$$\frac{g'(t)}{g(t)}(t-s) \geq \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t-s) \tag{2.49}$$

for any $t, s \in \mathring{J}$, which by multiplication with $g(t) > 0$ is equivalent with

$$g'(t)(t-s) \geq g(t)\ln g(t) - g(t)\ln g(s) \geq \frac{g'(s)}{g(s)}(tg(t) - sg(t)) \tag{2.50}$$

for any $t, s \in \mathring{J}$.

Fix $s \in \mathring{J}$ and apply the property (P) to get that

$$\begin{aligned} \langle g'(A)Ax, x\rangle - s\langle g'(A)x, x\rangle &\geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g(s) \\ &\geq \frac{g'(s)}{g(s)}(\langle Ag(A)x, x\rangle - s\langle g(A)x, x\rangle) \end{aligned} \tag{2.51}$$

for any $x \in H$ with $\|x\| = 1$, which is an inequality of interest in itself as well.

Since

$$\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1$$

then on choosing $s := \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}$ in (2.51) we get

$$\begin{aligned} &\langle g'(A)Ax, x\rangle - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\langle g'(A)x, x\rangle \\ &\geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right) \geq 0, \end{aligned}$$

which, by division with $\langle g(A)x, x\rangle > 0$, produces

$$\begin{aligned} &\frac{\langle g'(A)Ax, x\rangle}{\langle g(A)x, x\rangle} - \frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle} \cdot \frac{\langle g'(A)x, x\rangle}{\langle g(A)x, x\rangle} \\ &\geq \frac{\langle g(A)\ln g(A)x, x\rangle}{\langle g(A)x, x\rangle} - \ln g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right) \geq 0 \end{aligned} \tag{2.52}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the exponential in (2.52) we deduce the desired inequality (2.47).

Now, assuming that the condition (C) holds, then by choosing $s := \frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}$ in (2.51) we get

$$\begin{aligned} 0 &\geq \langle g(A)\ln g(A)x, x\rangle - \langle g(A)x, x\rangle \ln g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right) \\ &\geq \frac{g'\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}\left(\langle Ag(A)x, x\rangle - \frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\langle g(A)x, x\rangle\right) \end{aligned}$$

which, by dividing with $\langle g(A)x,x\rangle > 0$ and rearranging, is equivalent with

$$\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}-\frac{\langle Ag(A)x,x\rangle}{\langle g(A)x,x\rangle}\right) \tag{2.53}$$
$$\geq \ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)-\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\geq 0$$

for any $x \in H$ with $\|x\| = 1$.

Finally, on taking the exponential in (2.53) we deduce the desired inequality (2.48). □

Remark 2.25. We observe that a sufficient condition for (C) to hold is that either $g'(A)$ or $-g'(A)$ is a positive definite operator on H.

Corollary 2.26 (Dragomir, 2010, [12])**.** *Assume that A and g are as in Theorem 2.24. If the condition (C) holds, then we have the double inequality*

$$\ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)\geq\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\geq\ln g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right), \tag{2.54}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 2.27. Assume that A is a positive definite operator on H. Since for $r > 0$ the function $g(t) = t^{-r}$ is log-convex on $(0,\infty)$ and

$$\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}=\frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle}>0$$

for any $x \in H$ with $\|x\| = 1$, then on applying the inequality (2.54) we deduce the following interesting result

$$\ln\left(\frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle}\right)\leq\frac{\langle A^{-r}\ln Ax,x\rangle}{\langle A^{-r}x,x\rangle}\leq\ln\left(\frac{\left\langle A^{-r+1}x,x\right\rangle}{\langle A^{-r}x,x\rangle}\right) \tag{2.55}$$

for any $x \in H$ with $\|x\| = 1$.

The details of the proof are left to the interested reader.

The case of sequences of operators is embodied in the following corollary:

Corollary 2.28 (Dragomir, 2010, [12])**.** *Let A_j, $j \in \{1,...,n\}$ be selfadjoint operators on the Hilbert space H and assume that $Sp\left(A_j\right) \subseteq [m,M]$ for some scalars m, M with $m < M$ and each $j \in \{1,...,n\}$. If $g : J \to (0,\infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m,M] \subset \mathring{J}$, then*

$$\exp\left[\frac{\sum_{j=1}^{n}\left\langle g'\left(A_j\right)A_jx_j,x_j\right\rangle}{\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle}\right. \tag{2.56}$$
$$\left.-\frac{\sum_{j=1}^{n}\left\langle g\left(A_j\right)A_jx_j,x_j\right\rangle}{\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle}\cdot\frac{\sum_{j=1}^{n}\left\langle g'\left(A_j\right)x_j,x_j\right\rangle}{\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle}\right]$$
$$\geq\frac{\exp\left[\frac{\sum_{j=1}^{n}\langle g(A_j)\ln g(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right]}{g\left(\frac{\sum_{j=1}^{n}\langle g(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)}\geq 1$$

for each $x_j \in H,\ j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} \left\|x_j\right\|^2 = 1.$

If

$$\frac{\sum_{j=1}^{n} \left\langle g'\left(A_j\right) A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g'\left(A_j\right) x_j, x_j \right\rangle} \in \mathring{J} \tag{2.57}$$

for each $x_j \in H, j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} \left\|x_j\right\|^2 = 1$*, then*

$$\begin{aligned} &\exp\left[\frac{g'\left(\frac{\sum_{j=1}^{n} \left\langle g'(A_j) A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g'(A_j) x_j, x_j \right\rangle} \right)}{g\left(\frac{\sum_{j=1}^{n} \left\langle g'(A_j) A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g'(A_j) x_j, x_j \right\rangle} \right)} \right. \\ &\left. \times \left(\frac{\sum_{j=1}^{n} \left\langle g'\left(A_j\right) A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g'\left(A_j\right) x_j, x_j \right\rangle} - \frac{\sum_{j=1}^{n} \left\langle A_j g\left(A_j\right) x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g\left(A_j\right) x_j, x_j \right\rangle} \right) \right] \\ &\geq \frac{g\left(\frac{\sum_{j=1}^{n} \left\langle g'(A_j) A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g'(A_j) x_j, x_j \right\rangle} \right)}{\exp\left(\frac{\sum_{j=1}^{n} \left\langle g(A_j) \ln g(A_j) x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle g(A_j) x_j, x_j \right\rangle} \right)} \geq 1, \end{aligned} \tag{2.58}$$

for each $x_j \in H,\ j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} \left\|x_j\right\|^2 = 1.$

The following particular case for sequences of operators also holds:

Corollary 2.29 (Dragomir, 2010, [12])**.** *With the assumptions of Corollary 2.28 and if* $p_j \geq 0,\ j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$*, then*

$$\begin{aligned} &\exp\left[\frac{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle} \right. \\ &\left. - \frac{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle} \cdot \frac{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x \right\rangle} \right] \\ &\geq \frac{\exp\left[\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \right]}{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \right)} \geq 1 \end{aligned} \tag{2.59}$$

for each $x \in H$*, with* $\|x\| = 1$*.*

If

$$\frac{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) A_j x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) x, x \right\rangle} \in \mathring{J} \tag{2.60}$$

for each $x \in H$, with $\|x\| = 1$, then

$$\exp\left[\frac{g'\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)} \times\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'\left(A_j\right) x, x\right\rangle} - \frac{\left\langle \sum_{j=1}^{n} p_j A_j g\left(A_j\right) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right) x, x\right\rangle}\right)\right] \geq \frac{g\left(\frac{\left\langle \sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{\exp\left(\frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x\right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)} \geq 1, \tag{2.61}$$

for each $x \in H$, with $\|x\| = 1$.

Proof. Follows from Corollary 2.28 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. □

The following result providing different inequalities also holds:

Theorem 2.30 (Dragomir, 2010, [12])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\left\langle \exp\left[g'(A)\left(A - \frac{\langle g(A) Ax, x\rangle}{\langle g(A) x, x\rangle} 1_H\right)\right] x, x\right\rangle \geq \left\langle \left(\frac{g(A)}{g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right)}\right)^{g(A)} x, x\right\rangle \geq \left\langle \exp\left[\frac{g'\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right)}{g\left(\frac{\langle g(A)Ax, x\rangle}{\langle g(A)x, x\rangle}\right)}\left(Ag(A) - \frac{\langle g(A) Ax, x\rangle}{\langle g(A) x, x\rangle} g(A)\right)\right] x, x\right\rangle \geq 1 \tag{2.62}$$

for each $x \in H$ with $\|x\| = 1$.

If the condition (C) from Theorem 2.24 holds, then

$$\left\langle \exp\left[\frac{g'\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax, x\rangle}{\langle g'(A)x, x\rangle}\right)}\left(\frac{\langle g'(A) Ax, x\rangle}{\langle g'(A) x, x\rangle} g(A) - Ag(A)\right)\right] x, x\right\rangle \geq \left\langle \left(g\left(\frac{\langle g'(A) Ax, x\rangle}{\langle g'(A) x, x\rangle}\right)[g(A)]^{-1}\right)^{g(A)} x, x\right\rangle \geq \left\langle \exp\left[g'(A)\left(\frac{\langle g'(A) Ax, x\rangle}{\langle g'(A) x, x\rangle} 1_H - A\right)\right] x, x\right\rangle \geq 1 \tag{2.63}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. By taking the exponential in (2.50) we have the following inequality

$$\exp\left[g'(t)(t-s)\right] \geq \left(\frac{g(t)}{g(s)}\right)^{g(t)} \geq \exp\left[\frac{g'(s)}{g(s)}(tg(t)-sg(t))\right] \tag{2.64}$$

for any $t,s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.64), we deduce

$$\begin{aligned}\langle \exp\left[g'(A)(A-s1_H)\right]x,x\rangle &\geq \left\langle \left(\frac{g(A)}{g(s)}\right)^{g(A)} x,x\right\rangle \\ &\geq \left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A)-sg(A))\right]x,x\right\rangle\end{aligned} \tag{2.65}$$

for each $x \in H$ with $\|x\| = 1$, where 1_H is the identity operator on H.

By Mond-Pečarić's inequality applied for the convex function exp we also have

$$\begin{aligned}&\left\langle \exp\left[\frac{g'(s)}{g(s)}(Ag(A)-sg(A))\right]x,x\right\rangle \\ &\geq \exp\left(\frac{g'(s)}{g(s)}(\langle Ag(A)x,x\rangle - s\langle g(A)x,x\rangle)\right)\end{aligned} \tag{2.66}$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Now, if we choose $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} \in [m,M]$ in (2.65) and (2.66) we deduce the desired result (2.62).

Observe that, the inequality (2.64) is equivalent with

$$\exp\left[\frac{g'(s)}{g(s)}(sg(t)-tg(t))\right] \geq \left(\frac{g(s)}{g(t)}\right)^{g(t)} \geq \exp\left[g'(t)(s-t)\right] \tag{2.67}$$

for any $t,s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.67) we deduce

$$\begin{aligned}\left\langle \exp\left[\frac{g'(s)}{g(s)}(sg(A)-Ag(A))\right]x,x\right\rangle &\geq \left\langle \left(g(s)\left[g(A)\right]^{-1}\right)^{g(A)} x,x\right\rangle \\ &\geq \langle \exp\left[g'(A)(s1_H - A)\right]x,x\rangle\end{aligned} \tag{2.68}$$

for each $x \in H$ with $\|x\| = 1$.

By Mond-Pečarić's inequality we also have

$$\langle \exp\left[g'(A)(s1_H - A)\right]x,x\rangle \geq \exp\left[s\langle g'(A)x,x\rangle - \langle g'(A)Ax,x\rangle\right] \tag{2.69}$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Taking into account that the condition (C) is valid, then we can choose in (2.68) and (2.69) $s := \frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}$ to get the desired result (2.63). □

Remark 2.31. If we apply, for instance, the inequality (2.62) for the log-convex function $g(t) = t^{-1}, t > 0$, then, after simple calculations, we get the inequality

$$\begin{aligned}\left\langle \exp\left(\frac{A^{-2} - \left\langle A^{-1}x, x\right\rangle A^{-1}}{A^{-2} - \langle A^{-1}x, x\rangle}\right) x, x\right\rangle &\geq \left\langle \left(\left\langle A^{-1}x, x\right\rangle A^{-1}\right)^{A^{-1}} x, x\right\rangle \\ &\geq \left\langle \exp\left(\frac{A^{-1} - \left\langle A^{-1}x, x\right\rangle 1_H}{\langle A^{-1}x, x\rangle^2}\right) x, x\right\rangle \\ &\geq 1\end{aligned} \tag{2.70}$$

for each $x \in H$ with $\|x\| = 1$.

Other similar results can be obtained from the inequality (2.63), however the details are left to the interested reader.

2.5 A Reverse Inequality

The following reverse inequality is also of interest:

Theorem 2.32 (Dragomir, 2010, [12]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\begin{aligned}(1 \leq) &\frac{[g(m)]^{\frac{M - \langle Ax, x\rangle}{M - m}} [g(M)]^{\frac{\langle Ax, x\rangle - m}{M - m}}}{\exp\langle \ln g(A)x, x\rangle} \\ &\leq \exp\left[\frac{\langle (M1_H - A)(A - m1_H)x, x\rangle}{M - m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ &\leq \exp\left[\frac{(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)}{M - m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ &\leq \exp\left[\frac{1}{4}(M - m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]\end{aligned} \tag{2.71}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Utilising the inequality (2.49) we have successively

$$\ln g((1-\lambda)t + \lambda s) - \ln g(s) \geq (1-\lambda)\frac{g'(s)}{g(s)}(t - s) \tag{2.72}$$

and

$$\ln g((1-\lambda)t + \lambda s) - \ln g(t) \geq -\lambda \frac{g'(t)}{g(t)}(t - s) \tag{2.73}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we multiply (2.72) by λ and (2.73) by $1 - \lambda$ and sum the obtained inequalities, we deduce

$$\begin{aligned}&(1-\lambda)\ln g(t) + \lambda \ln g(s) - \ln g((1-\lambda)t + \lambda s) \\ &\leq (1-\lambda)\lambda\left[\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t - s)\right]\end{aligned} \tag{2.74}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we choose $\lambda := \frac{M-u}{M-m}$, $s := m$ and $t := M$ in (2.74) then we get the inequality

$$\frac{u-m}{M-m}\ln g(M) + \frac{M-u}{M-m}\ln g(m) - \ln g(u) \le \left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \tag{2.75}$$

for any $u \in [m, M]$.

If we use the property (P) for the operator A we get

$$\frac{\langle Ax,x\rangle - m}{M-m}\ln g(M) + \frac{M-\langle Ax,x\rangle}{M-m}\ln g(m) - \langle \ln g(A)x,x\rangle \le \left[\frac{\langle (M1_H - A)(A - m1_H)x,x\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \tag{2.76}$$

for each $x \in H$ with $\|x\| = 1$.

Taking the exponential in (2.76) we deduce the first inequality in (2.71).

Now, consider the function $h : [m, M] \to \mathbb{R}$, $h(t) = (M-t)(t-m)$. This function is concave in $[m, M]$ and by Mond-Pečarić's inequality we have

$$\langle (M1_H - A)(A - m1_H)x,x\rangle \le (M - \langle Ax,x\rangle)(\langle Ax,x\rangle - m)$$

for each $x \in H$ with $\|x\| = 1$, which proves the second inequality in (2.71).

For the last inequality, we observe that

$$(M - \langle Ax,x\rangle)(\langle Ax,x\rangle - m) \le \frac{1}{4}(M-m)^2,$$

and the proof is complete. □

Corollary 2.33 (Dragomir, 2010, [12]). *Assume that g is as in Theorem 2.32 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$\begin{aligned}(1 \le)\ & \frac{[g(m)]^{\frac{M - \sum_{j=1}^n \langle A_j x_j, x_j\rangle}{M-m}} [g(M)]^{\frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle - m}{M-m}}}{\exp\left(\sum_j^n \left\langle \ln g(A_j)x_j, x_j\right\rangle\right)} \\ &\le \exp\left[\frac{\sum_{j=1}^n \left\langle (M1_H - A_j)(A_j - m1_H)x_j, x_j\right\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ &\le \exp\left[\frac{\left(M - \sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle - m\right)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ &\le \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right].\end{aligned} \tag{2.77}$$

If $p_j \geq 0,\ j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$(1 \leq) \frac{[g(m)]^{\frac{M-\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle}{M-m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle - m}{M-m}}}{\left\langle \prod_{j=1}^{n} \left[g\left(A_j\right)\right]^{p_j} x, x\right\rangle} \tag{2.78}$$

$$\leq \exp\left[\frac{\sum_{j=1}^{n} p_j \left\langle \left(M1_H - A_j\right)\left(A_j - m1_H\right) x_j, x_j\right\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

$$\leq \exp\left[\frac{\left(M - \left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle - m\right)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

$$\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]$$

for each $x \in H$ *with* $\|x\| = 1$.

Remark 2.34. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$(1 \leq) \frac{m^{\frac{\langle Ax,x\rangle - M}{M-m}} M^{\frac{m-\langle Ax,x\rangle}{M-m}}}{\exp\langle \ln A^{-1} x, x\rangle} \leq \exp\left[\frac{\langle (M1_H - A)(A - m1_H) x, x\rangle}{Mm}\right] \tag{2.79}$$

$$\leq \exp\left[\frac{(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)}{Mm}\right]$$

$$\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right]$$

for all $x \in H$ with $\|x\| = 1$.

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