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A Theorem on Global Regularity for Solutions of Degenerate Elliptic Equations

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Abstract

In this article we establish the global regularity of weak solutions of the Dirichlet problem for a class of degenerate elliptic equations.

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1 Introduction

Let L be a degenerate elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)) - \sum_{i=1}^{n} b_i(x)D_iu(x) \tag{1.1}$$

where the coefficients a_{ij} and b_i are measurable, real-valued functions defined on a bounded open set $\Omega\subset\mathbb{R}^n$, and whose coefficient matrix $A(x) = (a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2\omega(x), \tag{1.2}$$

and

$$|b_i(x)| \le C_1\omega(x),\ i = 1,2...,n, \tag{1.3}$$

for all $\xi\in\mathbb{R}^n$, and a.e. $x\in\Omega$, C_1, λ and Λ are positive constants, and ω is a weight function (that is, ω is a nonnegative locally integrable function on $\mathbb{R}^n$).

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The main purpose of this paper is to establish the global regularity of solutions of the equation $Lu = g$ in Ω, $u-\varphi\in W_0^{1,2}(\Omega,\omega)$ (see Theorem 3.6). Under appropriate smoothness conditions on the boundary $\partial\Omega$ the preceding interior regularity results (see Theorem 2.12) can be extended to all Ω. The global regularity in non-degenerate case (i.e. with $\omega(x)\equiv 1$) have been studied by many authors (see e.g. [6], Theorem 8.12).

2 Definitions and basic results

Let ω be a locally integrable nonnegative function on $\mathbb{R}^n$ and assume that $0<\omega<\infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1<p<\infty$, or that ω is an A_p-weight, if there is a constant $C=C_{p,\omega}$ such that

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\Big)^{p-1}\le C$$

for all balls $B\subset\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\le p$, then $A_q\subset A_p$ (see [4],[9] or [12] for more information about A_p-weights). The weight ω satisfies the doubling condition if $\omega(2B)\le C\omega(B)$, for all balls $B\subset\mathbb{R}^n$, where $\omega(B)=\int_B\omega(x)\,dx$ and $2B$ denotes the ball with the same center as B which is twice as large. If $\omega\in A_p$, then ω is doubling (see Corollary 15.7 in [9]).

Example 2.1. As an example of A_p-weight, the function $\omega(x)=|x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n<\alpha<n(p-1)$ (see Corollary 4.4, Chapter IX in [12]).

If $\omega\in A_p$, then $\Big(\frac{|E|}{|B|}\Big)^p\le C\frac{\omega(E)}{\omega(B)}$, whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [9]). Therefore, if $\omega(E)=0$ then $|E|=0$.

Definition 2.2. Let ω be a weight, and let $\Omega\subset\mathbb{R}^n$ be open. For $0<p<\infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that $\|f\|_{L^p(\Omega,\omega)}=\Big(\int_\Omega|f(x)|^p\omega(x)dx\Big)^{1/p}<\infty$.

If $\omega\in A_p$, $1<p<\infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [13]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.3. Let $\Omega\subset\mathbb{R}^n$ be open, $1<p<\infty$, k a nonnegative integer and $\omega\in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $1\le|\alpha|\le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)}=\Big(\int_\Omega|u(x)|^p\omega(x)dx+\sum_{1\le|\alpha|\le k}\int_\Omega|D^\alpha u(x)|^p\omega(x)dx\Big)^{1/p}. \tag{2.1}$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)}=\Big(\sum_{1\le|\alpha|\le k}\int_\Omega|D^\alpha u(x)|^p\omega(x)dx\Big)^{1/p}.$$

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Corollary 2.1.6 in [13]). The spaces $W^{1,2}(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ are Hilbert spaces. It is evident that the weights ω which satisfy $0 < C_1 \leq \omega(x) \leq C_2$ for $x \in \Omega$ give nothing new (the space $W^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both). For a general theory of weighted Sobolev spaces $W^{k,p}(\Omega,\omega)$ with $\omega \in A_p$ see [4], [9], [12] and [13]. For information about weighted Sobolev spaces with others weights see [14].

In this work we use the following two theorems.

Theorem 2.4. *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist constants C_Ω and δ positive such that for all $u \in C_0^\infty(\Omega)$ and all k satisfying $1 \leq k \leq n/(n-1)+\delta$, we have $\|u\|_{L^{kp}(\Omega,\omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}$.*

Proof. See [2],Theorem 1.3. □

Theorem 2.5. *Let Ω be an open set in $\mathbb{R}^n$. If $\omega \in A_2$ then the embedding $W_0^{1,2}(\Omega,\omega) \hookrightarrow L^2(\Omega,\omega)$ is compact and $\|u\|_{L^2(\Omega,\omega)} \leq C \|u\|_{W_0^{1,2}(\Omega,\omega)}$.*

Proof. See [3], Theorem 4.6. □

Definition 2.6. Let ω be a weight in $\mathbb{R}^n$. We say that ω is uniformly A_p in each coordinate if
(a) $\omega \in A_p(\mathbb{R}^n)$;
(b) $\omega_i(t) = \omega(x_1,...,x_{i-1},t,x_{i+1},...,x_n)$ is in $A_p(\mathbb{R})$, for $x_1,...,x_{i-1},x_{i+1},...,x_n$ a.e., $1 \leq i \leq n$, with A_p constant is bounded independently of $x_1,...,x_{i-1},x_{i+1},...,x_n$.

Example 2.7. Let $\omega(x,y) = \omega_1(x)\omega_2(y)$, with $\omega_1(x) = |x|^{1/2}$ and $\omega_2(y) = |y|^{1/2}$. We have that ω is uniformly A_2 in each coordinate.

Definition 2.8. We say that an element $u \in W^{1,2}(\Omega,\omega)$ is a weak solution of the equation

$$Lu = g - \sum_{i=1}^n D_i f_i, \quad \text{with } \frac{g}{\omega}, \frac{f_i}{\omega} \in L^2(\Omega,\omega)$$

if

$$\mathcal{B}(u,\varphi) = \sum_{i=1}^n \int_\Omega f_i(x) D_i\varphi(x) + \int_\Omega g(x)\varphi(x)\,dx, \quad \forall \varphi \in W_0^{1,2}(\Omega,\omega),$$

where

$$\mathcal{B}(u,\varphi) = \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j\varphi(x) - \sum_{i=1}^n b_i(x)\varphi(x) D_i u(x) \Big] dx.$$

Theorem 2.9. *(Solvability of the Dirichlet problem) Let L be the operator (1.1) satisfying (1.2) and (1.3). Assume that* $\varphi \in W^{1,2}(\Omega,\omega)$, $g/\omega \in L^2(\Omega,\omega)$, $f_i/\omega \in L^2(\Omega,\omega)$ *and* $\omega \in A_2$. *Then the Dirichlet problem*

$$(P)\begin{cases} Lu = g - \sum_{i=1}^{n} D_i f_i \\ u-\varphi \in W_0^{1,2}(\Omega,\omega) \end{cases}$$

has a unique solution $u \in W^{1,2}(\Omega,\omega)$.

Proof. See [1], Theorem 2.9. □

Definition 2.10. Let u be a function on a bounded open set $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient of u at x in the direction e_i by

$$\Delta_k^h u(x) = \frac{u(x+he_k)-u(x)}{h}\ ,\ (0<|h|<\operatorname{dist}(x,\partial\Omega)). \tag{2.2}$$

Lemma 2.11. *Let* $\Omega' \subset\subset \Omega$ *and* $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$. *If* $u,v \in L^2_{\text{loc}}(\Omega,\omega)$, $\operatorname{supp}(v) \subset \Omega'$ *and* g *is a measurable function with* $|g(x)| \leq C\omega(x)$, *then*
(a) $\Delta_k^h(uv)(x) = u(x+he_k)\Delta_k^h v(x) + v(x)\Delta_k^h u(x)$, *with* $1 \leq k \leq n$;
(b) $\int_\Omega g(x)u(x)\Delta_k^{-h}v(x)\,dx = -\int_\Omega v(x)\Delta_k^h(gu)(x)\,dx$;
(c) $\Delta_k^h(D_j v)(x) = D_j(\Delta_k^h v)(x)$.

Proof. The proof of this lemma follows trivially from the Definition 2.10. □

Our first regularity result provides conditions under which weak solutions of the equation $Lu = g$ are twice weakly differentiable.

Theorem 2.12. *Let* $u \in W^{1,2}(\Omega,\omega)$ *be a weak solution of the equation* $Lu = g$ *in* Ω, *and assume that*
(a) $g/\omega \in L^2(\Omega,\omega)$;
(b) ω *is a weight uniformly* A_2 *in each coordinate;*
(c) $|\Delta_k^h a_{ij}(x)| \leq C_1\,\omega(x)$, *a.e.* $x \in \Omega' \subset\subset \Omega$, $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$, *with constant* C_1 *is independent of* Ω' *and h.*
Then for any subdomain $\Omega' \subset\subset \Omega$, *we have* $u \in W^{2,2}(\Omega',\omega)$ *and*

$$\|u\|_{W^{2,2}(\Omega',\omega)} \leq \mathbf{C}\Big(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\Big) \tag{2.3}$$

for $\mathbf{C} = \mathbf{C}(n,\lambda,\Lambda,C_1,d')$, *and* $d' = \operatorname{dist}(\Omega',\partial\Omega)$.

Proof. See [1], Theorem 3.8. □

Example 2.13. If $\varphi \in BMO(\mathbb{R}^n)$ then $\omega(x) = \mathrm{e}^{\alpha\varphi(x)} \in A_2$, for some $\alpha > 0$ (see [5] or [11], Chapter V, section 6). Let $\varphi_1, \varphi_2 \in BMO(\mathbb{R})$, with $\varphi_1', \varphi_2' \in L^\infty(\mathbb{R})$ and let α_1, α_2 be constants such that $\omega_1(x) = \mathrm{e}^{\alpha_1\varphi_1(x)}$, $\omega_2(y) = \mathrm{e}^{\alpha_2\varphi_2(y)} \in A_2(\mathbb{R})$. Then the weight $\omega(x,y) = \omega_1(x)\,\omega_2(y)$ is a weight uniformly A_2 in each coordinate.
Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and consider the operator

$$Lu(x,y) = -\frac{\partial}{\partial x}\left[\beta_1\,\omega(x,y)\,\frac{\partial u}{\partial x}\right] - \frac{\partial}{\partial y}\left[\beta_2\,\omega(x,y)\,\frac{\partial u}{\partial y}\right]$$

where β_1 and β_2 are positive constants. By Theorem 2.12 the equation $Lu = g$, with $\dfrac{g}{\omega} \in L^2(\Omega,\omega)$, has a solution $u \in W^{2,2}(\Omega',\omega)$ for all $\Omega' \subset\subset \Omega$.

3 Global Regularity

We recall here that a mapping $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open ($n \geq 2$), is quasiconformal if f is one-to-one, the components, f_i, of f have distributional derivatives belonging to $L^n_{\mathrm{loc}}(\mathbb{R}^n)$, and there is a constant $C > 0$ such that $|Df(x)|^n \leq C\,J_f(x)$ for a.e. $x \in \mathbb{R}^n$, where $Df(x) = (\partial_j f_i(x))$ is the formal differential matrix of f and $J_f(x)$ is the Jacobian determinant of f at x. We have that f^{-1} is a quasiconformal mapping in $f(\Omega)$. For instance, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = x\,|x|^\alpha$, $\alpha > 1$, is a quasiconformal mapping.

In this paper we use the following theorems.

Theorem 3.1. *If $h : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping then* $\log(|J_h(x)|) \in BMO$.

Proof. See [10], Theorem 1. □

Theorem 3.2. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiconformal mapping and $\omega \in A_p$. Then $\omega \circ h \in A_p$ if and only if* $\log(|J_h(x)|) \in BMO$.

Proof. See [7], Theorem at page 96, or Theorem 2.11 in [8]. □

Definition 3.3. Let ω be a weight uniformly A_p in each coordinate. We denote by $\mathcal{A}(\omega)$ the set of all quasiconformal mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $\omega \circ h$ is a weight uniformly A_p in each coordinate.

Example 3.4. Let $\omega_1(x,y) = |x|^{1/2}|y|^{1/2}$ and $\omega_2(x,y) = |x|^{1/2}|y|^{-1/2}$. We have that ω_1 and ω_2 are two weights uniformly $A_2(\mathbb{R}^2)$ in each coordinate. Consider the quasiconformal mapping $h(x,y) = (x,y)\,|(x,y)|^2 = (x(x^2+y^2), y(x^2+y^2))$. We have that $\tilde{\omega}_1(x,y) = \omega_1(h(x,y))$ is not a weight uniformly A_2 in each coordinate, and $\tilde{\omega}_2(x,y) = \omega_2(h(x,y))$ is a weight uniformly A_2 in each coordinate (see Example 2.1). Therefore $h \notin \mathcal{A}(\omega_1)$ and $h \in \mathcal{A}(\omega_2)$.

Definition 3.5. Let $\Omega \subset \mathbb{R}^n$ be open bounded set, ω a weight uniformly A_p in each coordinate and k a nonnegative integer. We say that Ω is of class C^k with ω-quasiconformal boundary if for each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a quasiconformal mapping ψ of B onto an open set $D \subset \mathbb{R}^n$ such that
(i) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \ : \ x_n > 0\}$;
(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$;
(iii) $\psi \in C^k(B)$ and $\psi^{-1} \in C^k(D)$;
(iv) $\psi \in \mathcal{A}(\omega)$ and $\psi^{-1} \in \mathcal{A}(\omega)$.

We are able now to prove the main result of this paper.

Theorem 3.6. *Let us assume, in addition to the hypotheses of Theorem 2.12, that*
(a) There exist $D_k a_{ij}$ a.e. $x \in \Omega$, $1 \le k \le n$ (and then we have $|D_k a_{ij}(x)| \le C_2\, \omega(x)$ for any $\Omega' \subset\subset \Omega$);
(b) Ω is of class C^2 with ω-quasiconformal boundary;
(c) There exists a function $\varphi \in W^{2,2}(\Omega, \omega)$ for which $u - \varphi \in W^{1,2}_0(\Omega, \omega)$.
Then we have also $u \in W^{2,2}(\Omega, \omega)$ and

$$||u||_{W^{2,2}(\Omega,\omega)} \le C\Big(||u||_{W^{1,2}(\Omega,\omega)} + ||g/\omega||_{L^2(\Omega,\omega)} + ||\varphi||_{W^{2,2}(\Omega,\omega)}\Big),$$

where $C = C(n, \lambda, \Lambda, C_1, C_2, \partial\Omega)$.

Proof. Replacing u by $u - \varphi$, we see that there is no loss of generality in assuming $\varphi \equiv 0$ and hence $u \in W^{1,2}_0(\Omega, \omega)$. Since Ω is of class C^2 with ω-quasiconformal boundary, then exists for each $x_0 \in \partial\Omega$, a ball $B = B(x_0)$ and a quasiconformal mapping ψ from B onto an open set $D \subset \mathbb{R}^n$ such that $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$, $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$, $\psi \in C^2(B)$ and $\psi^{-1} \in C^2(D)$. Let $B(x_0; R) \subset\subset B$ and set $B^+ = B(x_0; R) \cap \Omega$, $\tilde{D} = \psi(B(x_0; R))$ and $D^+ = \psi(B^+)$.

STEP 1. We set $y = \psi(x) = (y_1, ..., y_n) = (\psi_1(x), ..., \psi_n(x))$ and we define the weight $\tilde{\omega}(y) = \omega(\psi^{-1}(y))$. We have that $\tilde{\omega}$ is uniformly A_2 in each coordinate. If $u \in W^{1,2}(B^+, \omega)$, then $v = u \circ \psi^{-1} \in W^{1,2}(D^+, \tilde{\omega})$.

STEP 2. We define the operator $\tilde{L}v(y) = Lu(\psi^{-1}(y))$.We have

$$(i)\ \frac{\partial}{\partial x_i}(v \circ \psi)(x) = \sum_{k=1}^{n} \frac{\partial v}{\partial y_k} \frac{\partial \psi_k}{\partial x_i},$$

$$(ii)\ \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i}(v \circ \psi)(x)\Big) = \sum_{k=1}^{n} \frac{\partial v}{\partial y_k} \frac{\partial^2 \psi_k}{\partial x_j \partial x_i} + \sum_{k=1}^{n}\Big(\sum_{l=1}^{n} \frac{\partial^2 v}{\partial y_l \partial y_k} \frac{\partial \psi_l}{\partial x_j}\Big)\frac{\partial \psi_k}{\partial x_i}.$$

Hence, by condition (a), we obtain

$$\begin{aligned}
\tilde{L}v(y) &= L(v\circ\psi)(x)\\
&= -\sum_{i,j=1}^{n}\Big(a_{ij}(x)D_{ij}(v(y))+D_i a_{ij}(x)D_j(v(y))\Big)-\sum_{j=1}^{n}b_j(x)D_j(v(y))\\
&= -\sum_{k,l=1}^{n}\Big(\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\Big)D_{kl}v(y)\\
&- \sum_{k=1}^{n}\Big(\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2\psi_k}{\partial x_j\partial x_i}+D_i a_{ij}(x)\frac{\partial\psi_k}{\partial x_j}+\sum_{j=1}^{n}b_j\frac{\partial\psi_k}{\partial x_j}\Big)D_k v(y)\\
&= -\sum_{k,l=1}^{n}\tilde{a}_{kl}(y)D_{kl}v(y)-\sum_{k=1}^{n}\tilde{b}_k(y)D_k v(y),
\end{aligned}$$

where

$$\tilde{a}_{kl}(y)=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\text{ and }\tilde{b}_k(y)=\sum_{i,j=1}^{n}\Big(a_{ij}(x)\frac{\partial^2\psi_k}{\partial x_j\partial x_i}+D_i a_{ij}(x)\frac{\partial\psi_k}{\partial x_j}\Big)+\sum_{j=1}^{n}b_j(x)\frac{\partial\psi_k}{\partial x_j}.$$

Therefore, under the mapping ψ the equation $Lu=g$ in B^+ is transformed to an equation of the same form in D^+.
Since $\psi\in C^2$ and by conditions about the coefficients a_{ij} e b_i, we obtain the following estimates.
(1) $|\Delta_p^h\tilde{a}_{kl}(y)|\le\tilde{C}_2\,\tilde{\omega}(y)$, $y\in D^+$, for all $1\le p\le n$. In fact, using the Mean Value Theorem we obtain $\left|\Delta_p^h\Big(\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\Big)\right|\le C_\psi$. By (1.2), $\lambda\,\omega(x)\le a_{ij}(x)\le\Lambda\,\omega(x)$ a.e. $x\in\Omega$. Hence, by Lemma 2.11(a) we have

$$\begin{aligned}
|\Delta_p^h\tilde{a}_{kl}(y)|&\le\sum_{i,j=1}^{n}\left|\Delta_p^h\Big(a_{ij}\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\Big)(x)\right|\\
&=\sum_{i,j=1}^{n}\left|\Big(\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\Big)(x+he_p)\,\Delta_p^h a_{ij}(\psi^{-1}(y))+a_{ij}(\psi^{-1}(y))\Delta_p^h\Big(\frac{\partial\psi_l}{\partial x_j}\frac{\partial\psi_k}{\partial x_i}\Big)(x)\right|\\
&\le\sum_{i,j=1}^{n}\Big(C_\psi|\Delta_p^h a_{ij}(\psi^{-1}(y))|+C_\psi|a_{ij}(\psi^{-1}(y))|\Big)\\
&\le\sum_{i,j}^{n}C_\psi\Big(C_2\omega(\psi^{-1}(y))+\Lambda\,\omega(\psi^{-1}(y))\Big)\\
&=\Big(\sum_{i,j=1}^{n}C_\psi(C_2+\Lambda)\Big)\omega(\psi^{-1}(y))\\
&=\tilde{C}_2\,\omega(\psi^{-1}(y))=\tilde{C}_2\,\tilde{\omega}(y).
\end{aligned}$$

(2) $|\tilde{b}_k(y)|\le\tilde{C}_1\,\tilde{\omega}(y)$, $y\in D^+$. In fact,

$$
\begin{aligned}
|\tilde{b}_k(y)| \le & \sum_{i,j=1}^{n} |a_{ij}(\psi^{-1}(y))| C_\psi + |D_i a_{ij}(\psi^{-1}(y))| C_\psi + \sum_{j=1}^{n} |b_j(\psi^{-1}(y))| C_\psi \\
\le & C_\psi \Big(\Lambda\, \omega(\psi^{-1}(y)) + C_2 \omega(\psi^{-1}(y)) + C_1 \omega(\psi^{-1}(y)) \Big) \\
= & C_\psi (\Lambda + C_2 + C_1) \omega(\psi^{-1}(y)) = \tilde{C}_1 \tilde{\omega}(y).
\end{aligned}
$$

(3) Let $\tilde{A}(y) = (\tilde{a}_{kl}(y)) = TA(x)T^t$, where $T = \left(\dfrac{\partial \psi_k}{\partial x_j} \right)_{1 \le j,k \le n}$. Since $\overline{B(x_0;R)}$ is compact and T is invertible, there exists a constant C_3 independent of x such that $\|T^t(x)\xi\| \ge C_3 \|\xi\|$. Hence, by condition (1.2) we obtain

$$
\begin{aligned}
\langle \tilde{A}\xi, \xi \rangle &= \langle TAT^t\xi, \xi \rangle = \langle AT^t\xi, T^t\xi \rangle \\
&\ge \lambda \|T^t\xi\|^2 \omega(x) \\
&\ge \lambda C_3^2 \|\xi\|^2 \omega(\psi^{-1}(y)) \\
&= \tilde{\lambda} \|\xi\|^2 \tilde{\omega}(y),
\end{aligned}
$$

and we also have

$$
\begin{aligned}
\langle \tilde{A}\xi, \xi \rangle &= \langle AT^t\xi, T^t\xi \rangle \\
&\le \Lambda \|T^t\xi\|^2 \omega(x) \\
&\le \Lambda \|T^t\|^2 \|\xi\|^2 \omega(\psi^{-1}(y)) \\
&= \tilde{\Lambda} \|\xi\|^2 \tilde{\omega}(y).
\end{aligned}
$$

Hence we have

$$
\tilde{\lambda} |\xi|^2 \tilde{\omega}(y) \le \sum_{k,l=1}^{n} \tilde{a}_{kl}(y) \xi_k \xi_l \le \tilde{\Lambda} |\xi|^2 \tilde{\omega}(y).
$$

Moreover, if $u \in W_0^{1,2}(B^+, \omega)$ is a solution of $Lu = g$, then $v = u \circ \psi^{-1} \in W_0^{1,2}(D^+, \tilde{\omega})$ is a solution of $\tilde{L}v(y) = \tilde{g}(y) = g(\psi^{-1}(y))$ and satisfies $\eta v \in W_0^{1,2}(D^+, \omega)$, for all $\eta \in C_0^\infty(\tilde{D})$.

Accordingly, let us now suppose that $u \in W_0^{1,2}(D^+, \omega)$ satisfies $Lu = g$ in D^+. Following the lines of Theorem 3.8 in [1], for any $\eta \in C_0^\infty(\tilde{D})$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $\Omega' \subset\subset \tilde{D}$, $\Omega' = \psi(B_r(x_0) \cap \Omega)$ where $0 < r < R$ and $\|\eta\|_{L^\infty} \le 2/d'$, $d' = \mathrm{dist}(\Omega', \partial\tilde{D})$, if $0 < |h| < \mathrm{dist}(\mathrm{supp}(\eta), \partial\tilde{D})$ and $1 \le k \le (n-1)$, we have

$$
\eta^2 \Delta_k^h u \in W_0^{1,2}(D^+, \omega).
$$

Analogously, from Theorem 3.8 (in [1]) we obtain (for any $0 < r < R$ and $B_r = B(x_0, r)$)

$$
D_{ij} u \in L^2(\psi(B_r \cap \Omega), \omega)
$$

with $(i, j) \ne (n, n)$, and

$$\|D_{ij}u\|_{L^2(\psi(B_r\cap\Omega),\omega)} \le C\Big(\|u\|_{W^{1,2}(D^+,\omega)} + \|g/\omega\|_{L^2(D^+,\omega)}\Big). \tag{3.1}$$

STEP 3. We can now estimate the second derivative $D_{nn}u$. Remembering the definition of L, we can rewrite the equation $Lu = g$ as

$$\begin{aligned} g &= Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}D_iu) - \sum_{i=1}^{n} b_iD_iu \\ &= -\sum_{i,j=1}^{n}\Big(D_ja_{ij}D_iu + a_{ij}D_{ij}u\Big) - \sum_{i=1}^{n} b_iD_iu. \end{aligned}$$

So we discover

$$a_{nn}D_{nn}u = -g - \sum_{i,j=1}^{n} D_ja_{ij}D_iu - \sum_{i=1}^{n} b_iD_iu - \sum_{\substack{0\le i,j\le n\\(i,j)\ne(n,n)}} a_{ij}D_{ij}u.$$

Therefore

$$\frac{a_{nn}}{\omega}(D_{nn}u) = -\frac{g}{\omega} - \sum_{i,j=1}^{n}\frac{D_ja_{ij}}{\omega}D_iu - \sum_{i=1}^{n}\frac{b_i}{\omega}D_iu - \sum_{\substack{0\le i,j\le n\\(i,j)\ne(n,n)}}\frac{a_{ij}}{\omega}D_{ij}u.$$

Now, we have the following estimates.
(1) $g/\omega\in L^2(\psi(B_r\cap\Omega),\omega)$ (by condition (a) in Theorem 2.12).
(2) $(a_{ij}D_{ij}u)/\omega\in L^2(\psi(B_r\cap\Omega),\omega)$ (with $(i,j)\ne(n,n)$). In fact, if $(i,j)\ne(n,n)$, by (1.2) and (3.1) we obtain

$$\begin{aligned} \int_{\psi(B_r\cap\Omega)}\left(\frac{|a_{ij}D_{ij}u|}{\omega}\right)^2\omega\,dx &= \int_{\psi(B_r\cap\Omega)}\left(\frac{|a_{ij}|}{\omega}\right)^2|D_{ij}u|^2\omega\,dx \\ &\le \Lambda^2\int_{\psi(B_r\cap\Omega)}|D_{ij}u|^2\omega\,dx < \infty. \end{aligned}$$

(3) $(D_ja_{ij}D_iu)/\omega\in L^2(\psi(B_r\cap\Omega),\omega)$. In fact, by condition (a) we have

$$\begin{aligned} \int_{\psi(B_r\cap\Omega)}\left(\frac{|D_ja_{ij}D_iu|}{\omega}\right)^2\omega\,dx &= \int_{\psi(B_r\cap\Omega)}\left(\frac{|D_ja_{ij}|}{\omega}\right)^2|D_iu|^2\omega\,dx \\ &\le C_2^2\int_{\psi(B_r\cap\Omega)}|D_iu|^2\omega dx < \infty. \end{aligned}$$

(4) $(b_iD_iu)/\omega\in L^2(\psi(B_r\cap\Omega),\omega)$. In fact, using (1.3) we have

$$\begin{aligned} \int_{\psi(B_r\cap\Omega)}\left(\frac{|b_iD_iu|}{\omega}\right)^2\omega\,dx &= \int_{\psi(B_r\cap\Omega)}\left(\frac{b_i}{\omega}\right)^2|D_iu|^2\omega\,dx \\ &\le C_1^2\int_{\psi(B_r\cap\Omega)}|D_iu|^2\omega\,dx < \infty. \end{aligned}$$

Therefore $(a_{nn}/\omega)D_{nn}u \in L^2(\psi(B_r \cap \Omega), \omega)$. Since $|a_{nn}/\omega| \geq \lambda$, we conclude

$$D_{nn}u \in L^2(\psi(B_r \cap \Omega), \omega),$$

and using (3.1) we obtain

$$\begin{aligned}
\lambda \|D_{nn}u\|_{L^2(\psi(B_r\cap\Omega),\omega)} &\leq \|g/\omega\|_{L^2(\psi(B_r\cap\Omega),\omega)} + C_2 \sum_{j=1}^{n} \|D_j u\|_{L^2(\psi(B_r\cap\Omega),\omega)} \\
&+ \sum_{\substack{0\leq i,j\leq n \\ (i,j)\neq(n,n)}} \Lambda \|D_{ij}u\|_{L^2(\psi(B_r\cap\Omega),\omega)} + \sum_{i=1}^{n} C_1 \|D_i u\|_{L^2(\psi(B_r\cap\Omega),\omega)} \\
&\leq \|g/\omega\|_{L^2(\psi(B_r\cap\Omega),\omega)} + (C_1 + C_2) \sum_{j=1}^{n} \|D_j u\|_{L^2(\psi(B_r\cap\Omega),\omega)} \\
&+ \Lambda C\Big(\|g/\omega\|_{L^2(\psi(B_r\cap\Omega),\omega)} + \|u\|_{W_0^{1,2}(D^+,\omega)} \Big) \\
&\leq C\Big(\|g/\omega\|_{L^2(D^+,\omega)} + \|u\|_{W_0^{1,2}(D^+,\omega)} \Big).
\end{aligned}$$

Then we obtain

$$\|D_{nn}u\|_{L^2(\psi(B_r\cap\Omega),\omega)} \leq \frac{C}{\lambda}\Big(\|u\|_{W_0^{1,2}(D^+,\omega)} + \|g/\omega\|_{L^2(D^+,\omega)} \Big).$$

Hence, returning to the original domain Ω with the mapping $\psi^{-1} \in C^2$ we obtain that $u \in W^{2,2}(B(x_0, r) \cap \Omega, \omega)$ (for all $0 < r < R$). Since x_0 is an arbitrary point of $\partial\Omega$ and $u \in W^{2,2}(\Omega', \omega)$ for all $\Omega' \subset\subset \Omega$ (by Theorem 2.12) we have that $u \in W^{2,2}(\Omega, \omega)$.

STEP 4. Finally by choosing a finite number of points $x_i \in \partial\Omega$ such that the balls $O_i = B(x_i, R)$ cover $\partial\Omega$. There exist ψ_i such that $\psi_i : O_i \to D$, $\psi_i(O_i \cap D) = D^+$, where each ψ_i satisfies Definition 3.5. We can suppose that $O_1, ..., O_k$ cover Ω. Choosing $\rho_i \in C^2$, $i = 1, 2, ..., k$, such that

$$\mathrm{supp}\,(\rho_i) \subset O_i \ \text{ and } \ \sum_{i=1}^{k} \rho_i(x) = 1, \ \forall\, x \in \bar{\Omega}.$$

Since $u \in W_0^{1,2}(\Omega, \omega)$, we have that $\mathrm{supp}(\rho_i\, u) \subset O_i \subset\subset \Omega$. If $g_i = L(\rho_i u)$, we have (for $i = 1$),

$$\begin{aligned}
g_1 = L(\rho_1 u) &= \sum_{i,j=1}^{n} a_{ij} D_{ij}(\rho_1 u) + D_i a_{ij} D_j(\rho u) + \sum_{j=1}^{n} b_j D_j(\rho_1 u) \\
&= \rho_1 L u + u L\rho_1 + a_{ij} D_i\rho_1 D_j u + a_{ij} D_i u D_j \rho_1 \\
&= \rho_1 g + u L\rho_1 + a_{ij} D_i\rho_1 D_j u + a_{ij} D_i u D_j \rho_1 .
\end{aligned}$$

Since ρ_1 is of class C^2 (in O_1) and by the assumptions about the coefficients a_{ij} e b_j we have

(a) $\rho_1 g/\omega \in L^2(\Omega,\omega)$;
(b) $uL\rho_1/\omega \in L^2(\Omega,\omega)$;
(c) $a_{ij}D_i\rho_1 D_j u/\omega \in L^2(\Omega,\omega)$.
Hence, we have that $L(\rho_1 u) = g_1$, with $g_1/\omega \in L^2(\Omega,\omega)$ and

$$\|g_1/\omega\|_{L^2(\Omega,\omega)} \leq \tilde{C}_1\left(\|g/\omega\|_{L^2(\Omega,\omega)} + \|u\|_{W^{1,2}(\Omega,\omega)}\right).$$

Then we obtain

$$\begin{aligned}
\|\rho_1 u\|_{W^{2,2}(O_1,\omega)} &= \|\rho_1 u\|_{W^{2,2}(\Omega,\omega)} \\
&\leq C\left(\|\rho_1 u\|_{W^{1,2}(\Omega,\omega)} + \|g_1/\omega\|_{L^2(\Omega,\omega)}\right) \\
&\leq C\left[\|\rho_1\|_{L^\infty}\|u\|_{W^{1,2}(\Omega,\omega)} + \tilde{C}_1\left(\|g/\omega\|_{L^2(\Omega,\omega)} + \|u\|_{W^{1,2}(\Omega,\omega)}\right)\right] \\
&\leq \mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\right).
\end{aligned}$$

Analogously we have $g_i/\omega \in L^2(\Omega,\omega)$ $(1 \leq i \leq k)$ and

$$\|g_i/\omega\|_{L^2(\Omega,\omega)} \leq \tilde{C}_i\left(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\right).$$

We also have

$$\|\rho_i u\|_{W^{2,2}(O_i,\omega)} = \|\rho_i u\|_{W^{2,2}(\Omega,\omega)} \leq \mathbf{C}\left(\|u\|_{W^{2,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\right).$$

Therefore we obtain

$$\begin{aligned}
\|u\|_{W^{2,2}(\Omega,\omega)} &= \left\|\sum_{i=1}^{k}\rho_i u\right\|_{W^{2,2}(\Omega,\omega)} \leq \sum_{i=1}^{k}\|\rho_i u\|_{W^{2,2}(\Omega,\omega)} \\
&\leq C\left(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\right).
\end{aligned}$$

□

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