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# ON INTEGRALS OF VECTOR-VALUED FUNCTIONS ON TIME SCALES

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**Abstract**

The main goal of the paper is to define new type of integrals for vector-valued functions on time scales. This allows to make possible the advantages of dynamic equations also for vector-valued functions i.e. for dynamic modeling in Banach spaces. To do it we define some appropriate integrals for vector-valued functions on time scales and we prove their properties.

We emphasize on the particular ones, which are useful for solving dynamic equations in Banach spaces.

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## 1 Introduction

Differential and difference equations in infinite dimenensional Banach spaces are intensively studied (at least from the paper of E.H. Moore in 1908). When we consider infinite systems of differential equations, evolution equations or functional-differential equations such problems should be considered as the problems in infinite dimensional spaces. This justify still growing development of equations in Banach spaces. Moreover, in many problems arising, for instance, in the control theory or mathematical economics it is necessary to consider both continuous and discrete models which lead to superfluous duplicating of the theory. One of the procedure of avoiding such problems is based on utilization of the notion of a time scale.

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We need to enlarge this procedure also for problems in infinite dimensional Banach spaces by introducing new type of integrals for vector valued functions defined on a general time scale. Due to equivalence of differential or dynamic problems to the integral form we are able to fully cover all theories for differential and difference equations in Banach spaces including all types of considered solutions (cf. [19]).

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. Thus $\mathbb{R},\mathbb{Z},\mathbb{N}$ or the set of harmonic numbers $\{H_n = \sum_{k=1}^{n}\frac{1}{k} : n\in\mathbb{N}\}\cup\{0\}$ are the examples of time scales while $\mathbb{Q}$, $\{\frac{1}{n} : n\in\mathbb{N}\}$ and $(0,1)$ are not time scales. For simplicity, we will denote by $[a,b]$ a time scale interval i.e. $[a,b]\cap\mathbb{T}$.

The notion of a time scale was introduced by Hilger and allows us to treat by unified manner differential equations, integral equations and difference equations. Moreover, the so-called dynamic equations cover different kind of hybrid equations which do not involve solely continuous aspects or solely discrete aspects. For instance, neither difference equations nor differential equations give a good description of most population growth. We deal with similar problem when we try to describe a population dynamics where nonoverlapping generations occur (cf. [11], [12] or [33]).

The main goal of this paper is to make possible such an advantage of dynamic equations also for vector-valued functions i.e. for dynamic modeling in Banach spaces. To do it we define appropriate integrals on time scales and we prove their properties which are useful for solving dynamic equations. For simplicity, we present some definitions and results in terms of delta derivatives and integrals. The case of nabla-type derivatives and integrals can be obtained in a similar manner (cf. [31] or [11]). We stress also on a possibly complete list of references, in such a way to make the paper useful as a short survey about integrals on time scales.

Let us also note, that by using our new integrals we are able to extend all existing results for dynamic equations in Banach spaces including the latest one [21] for other classes of solutions.

Define the so-called delta derivative and delta Cauchy-Newton integral for Banach valued functions in the same manner as usual $\Delta$-derivative and $\Delta$-integral on time scales.

**Definition 1.1.** Fix an arbitrary $t\in(a,b)\subset\mathbb{T}$. Let $f:[a,b]\to E$. Then we define $\Delta$-derivative $f^{\Delta}(t)$ by

$$f^{\Delta}(t) = \lim_{s\to t}\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

The $\Delta$-derivative turns out that in particular cases we have

**(i)** $f^{\Delta} = f'$ is the usual derivative if $(a,b)\subset\mathbb{T}=\mathbb{R}$ and

**(ii)** $f^{\Delta} = \Delta f$ is the usual forward difference operator if $(a,b)\subset\mathbb{T}=\mathbb{Z}$.

Hence time scale allows us to unify the treatment of differential and difference equations (and not only these cases).

Let $E$ be an arbitrary Banach space and $E^*$ be its topological dual.

**Definition 1.2.** We say that $f:[a,b]\to E$ is right dense continuous (rd-continuous) if $f$ is continuous at every right dense point $t\in[a,b]$ and $\lim_{s\to t^-} f(s)$ exists and is finite at every left dense point $t\in[a,b]$.

The above notion, specific for time scales, is important in view of existence of antiderivatives:

*Remark.* [11] Every rd-continuous function has an antiderivative. In particular, if $t, t_0 \in \mathbb{T}$ then $F$ defined by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau, \;\; t \in \mathbb{T}$$

is an antiderivative of $f$.

**Definition 1.3.** If $F^{\Delta}(t) = f(t)$ for each $t \in \mathbb{T}$ then we define the Cauchy-Newton integral by

$$(\mathrm{CN}) \int_{a}^{t} f(\tau)\Delta\tau = F(t) - F(a).$$

The main disadvantage of the Cauchy-Newton integral is its inapplicability in real problems for differential equations: too small class of integrable functions (cf. [41] or [8]). Moreover, we are unable to check many important properties of such an integral, including convergence theorems.

There exist a few interesting generalizations of the notion of the Cauchy-Newton integral for **real-valued** functions on time scale, namely the Riemann integral ([29], [8], [14]), the improper Riemann integral (or: the Cauchy-Riemann integral)([8]), the Lebesgue integral (Aulbach and Neidhard [8], Guseinov [29], Cabada and Vivero [13], [14], Rzeżuchowski [42], Chyan and Fryszkowski [18] or Agarwal et al. [5]), the Lebesgue-Stjeltjes integral (Deniz and Ufuktepe [24]) or the Henstock-Kurzweil integral (Peterson and Thompson [41] and for unbounded time scales: [10], [30], [9]). Let us stress, that the case of integrals for real-valued functions is still intensively investigated. Each of such extensions of the notion of integral has some important advantages (bigger class of integrable functions, mean-value theorems, convergence theorems etc.), in particular each new definition of the integral has applications in the theory of differential and difference equations.

Nevertheless, for the vector-valued functions this topic is not sufficiently investigated. Such integrals are necessary to unify theories of differential, difference, $q$-difference equations for vector-valued functions. Each of these theories is intensively developed, but the unification is still an open problem, mainly due to lack of research dealing with different kind of derivatives and integrals. Some interesting considerations about derivatives were presented in [45] (in the real-valued context), for instance. We will deal with a new type of integrals.

Even in the case $\mathbb{T} = \mathbb{R}$ new definition of integrals was motivated by some problems arising in the theory of differential equations (the problem of primitives) (cf. Cichoń [19]). In the case $\mathbb{T} = \mathbb{Z}$ our study has some applications in the theory of difference equations in Banach spaces. Since the unification of continuous and discrete case lies at the basis of the theory of time scales we present some results about weak type integral in Banach spaces.

Till now, except the Cauchy-Newton integral and the Riemann integral, only the Bochner integral was described for functions defined on time scales with values in a Banach space. Such an integral was necessary to investigate some problems for differential equations in Banach spaces (see [19], for instance). Note, that the function $u$ is Bochner integrable on

$[a,b]$ iff the real-valued function $\|u\|$ is Lebesgue integrable (cf. [8]) so it is the absolute integration.

In the present paper we will stress on weak type integrals which are important in a new field of research concerned with weak solutions for dynamic equations (covering both differential and difference equations) - cf. [21]. Let us recall selected papers for differential equations: [46] or [40] for the weak Riemann integral, [34] or [20] for the Pettis integral, [37], [38], [7], [43], [16], [17] or [22] for the Henstock-Kurzweil integral (cf. also [44]) or finally [19] for the Henstock-Kurzweil-Pettis integral and comparison results between classes of solutions.

For difference equations in Banach spaces see [2], [4], [6], [39], [23], [26] or [25], for instance. For integral equations on time scales see [36]. In particular, regarding on the papers dealing with continuous and discrete equations as similar results (in the same paper) see [35] or [3], for instance. In this paper, a basis for such a unification is introduced.

All the integrals mentioned above could be defined on time scales and used for solving dynamical equations on time scales.

For all our definitions and results there are corresponding ones for nabla derivatives and integrals. We will not bother to consider that case.

The paper contains results about basic properties of the new integrals. We emphasize the properties which are necessary when we solve some problems for differential, integral or difference equations. In such a case the equivalence of differential Cauchy problems and integral equations is a really standard method. But to take advantage from this equivalence the integral problem should be effectively studied. This depends on the properties of integrals which are used in the proof (cf. [19] for differential equations on compact intervals).

## 2 Integrals.

Now, we define new type of integrals on time scales and we check some of their properties. We will check the new integral to make possible the application of defined integrals for checking the existence of solutions for differential (or: dynamic) equations in Banach spaces. We will define both strong and weak type of integrals.

A function $y : [a,b] \to E$ is said to be weakly-$\Delta$ differentiable at point $t \in [a,b]$ if for each $x^* \in E^*$ the real valued function $x^*y$ is $\Delta$ differentiable in the usual sense on time scales i.e. for any $\varepsilon > 0$ there is a neighborhood $U$ of $t$ with

$$|[x^*y(\sigma(t)) - x^*y(s)] - x^*y^{w\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

where $y^{w\Delta}(t)$ is an element of $E$ called the weak-$\Delta$ derivative of $y$ at $t$.

In this paper we will prove some basic properties of the weak-$\Delta$ derivative. The definition of weak Cauchy-Newton integral follows directly from the real-valued one.

**Definition 2.1.** If $F^{w\Delta}(t) = f(t)$ for each $t \in \mathbb{T}$ then we define the weak Cauchy-Newton integral by

$$\text{(wCN)}\int_a^t f(\tau)\Delta\tau = F(t) - F(a).$$

A function $y : [a,b] \to E$ is said to be weakly rd-continuous iff for each $x^* \in E^*$ the real-valued function is rd-continuous. Thus from this definition it follows that

**Lemma 2.2.** *Every weakly rd-continuous function $f$ has a weak antiderivative i.e. for each $t_0 \in \mathbb{T}$ then*

$$F(t) := (\text{wCN})\int_{t_0}^{t} f(\tau)\Delta\tau, \ \ t \in \mathbb{T}$$

*is a weak antiderivative of $f$.*

Recall the definition of the Riemann integral for vector-valued function on time scales:

**Definition 2.3.** (Aulbach and Neidhart [8], Guseinov [29]) The function $f : [a,b] \to E$ is Riemann integrable on $[a,b]$ if there exists $A \in E$ with the following property: for every $\varepsilon > 0$ there exists a positive constant $\delta$ on $[a,b]$ such that for every partition $\mathcal{D}$ of $[a,b]$ given by $x_0 < x_1 < \ldots < x_n$ which is finer then $\delta$ and any set of points $y_0, y_1, \ldots, y_n \in \mathcal{D}$ with $y_i \in [x_{i-1}, x_i)$ for $i = 1, 2, \ldots, n$ one has

$$\|\sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}) - A\| < \varepsilon \, .$$

We write $(\text{R})\int_a^b f(t)\Delta t = A$. A partition $\mathcal{D}$ is called "finer than $\delta$" if for each $i = 1, 2, ..., n$ we have either $(x_i - x_{i-1}) \leq \delta$ or both $(x_i - x_{i-1}) > \delta$ and $x_i = \sigma(x_{i-1})$.

**Definition 2.4.** The function $f : [a,b] \to E$ is weak Riemann integrable on $[a,b]$ if there exists $A \in E$ with the following property: for every $\varepsilon > 0$ and every $x^* \in E^*$ there exists a positive constant $\delta$ on $[a,b]$ such that for every partition $\mathcal{D}$ of $[a,b]$ given by $x_0 < x_1 < \ldots < x_n$ which is finer then $\delta$ and any set of points $y_0, y_1, \ldots, y_n \in \mathcal{D}$ with $y_i \in [x_{i-1}, x_i)$ for $i = 1, 2, \ldots, n$ one has

$$|x^*(\sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}) - A)| < \varepsilon \, .$$

We write $(\text{wR})\int_a^b f(t)\Delta t = A$.

Note, that for vector-valued functions even in the case $\mathbb{T} = \mathbb{R}$ Riemann-type integrable functions need not be measurable which is a necessary condition for the Bochner integrability. Such functions are also neither continuous almost everywhere nor weakly continuous almost everywhere, in general (cf. [27]). Then we cannot expect that we will be able to characterize a class of all Riemann integrable functions, which make such an integral almost useless in the theory of dynamic equations in Banach spaces. We will prove mainly the properties of another type of integrals.

As in a classical case ([15] or [19], cf. [41] for real-valued functions), we need to introduce two definitions of vector-valued Henstock-Kurzweil integrals.

Let $\delta$ be a pair $(\delta_L, \delta_R)$ of positive functions. We say that a partition $\mathcal{D}$ of $[a,b]$ given by $x_0 \leq \xi_1 < x_1 \leq \xi_2 < \ldots \leq \xi_n < x_n$ is $\delta$-fine and we can write $\mathcal{D} = \{[u,v], \xi\} = \{[x_{i-1}, x_i]; \xi_i \,, i = 1, 2, \ldots n\}$ whenever the following condition is satisfied

$$\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))$$

if only $x_i \neq \sigma(x_{i-1})$. Let us recall that such a partition exists for arbitrary positive pair of functions (Cousin's Lemma i.e. Lemma 1.9 in [41]) (even taking into account some changes in the above definition). We sometimes use the abbreviation $\sum_{\mathcal{D}} f(\xi)(v-u)$ instead of $\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$.

**Definition 2.5.** A function $f : [a,b] \to E$ is (HK) integrable on $[a,b]$ if there exists a function $F : [a,b] \longrightarrow E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$ there exists a pair $\delta$ of positive functions $(\delta_L(\cdot), \delta_R(\cdot))$ on $[a,b]$ such that if $\mathcal{D} = \{[u,v], \xi\}$ is a $\delta$-fine division of $[a,b]$, we have

$$\| \sum_{\mathcal{D}} f(\xi)(v-u) - (F(v) - F(u))]) \| < \varepsilon \, .$$

**Definition 2.6.** A function $f : [a,b] \to E$ is (HL) integrable on $[a,b]$ if there exists a function $F : [a,b] \longrightarrow E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$ there exists a pair $\delta$ of two positive functions $(\delta_L(\cdot), \delta_R(\cdot))$ on $[a,b]$ such that if $\mathcal{D} = \{[u,v], \xi\}$ is a $\delta$-fine division of $[a,b]$, we have

$$\sum_{\mathcal{D}} \| f(\xi)(v-u) - (F(v) - F(u))]) \| < \varepsilon \, .$$

*Remark.* We note that, by the triangle inequality, if $f$ is (HL) integrable it is also (HK) integrable. In general, the converse is not true. For real-valued functions, the two integrals are equivalent. Nevertheless, in [41] for real-valued function is considered a definition of the (HK) type of integral.

The above integrals are really applicable in the theory of differential equations and solves the problem of primitives for the strong (norm) topology (integrability of an arbitrary derivative). Recall that they are nonabsolute integrals, covering as particular cases both Riemann integrals (for the case of constant function $\delta(\cdot)$) and Bochner integrals (for the case of integrability for both $f$ and $\|f\|$). Nevertheless, let us remark, that the existence of the Henstock-Kurzweil integral over $[a,b]$ implies the existence of such integrals over all subintervals of $[a,b]$ but not for all measurable subsets of this interval, so the theory of such integrals on $\mathbb{T}$ does not follows from general theory on $\mathbb{R}$.

We need to define also some integrals which are important when we consider the weak topology on the Banach space $E$. The first one is an adaptation of the definition of Pettis integrals from $\mathbb{R}$.

**Definition 2.7.** The function $f : [a,b] \to E$ is Pettis integrable ((P) integrable for short) if
(i) $\forall_{x^* \in E^*}$ $x^* f$ is Lebesgue integrable on $[a,b]$,
(ii) $\forall_{\substack{A \subset [a,b] \\ A \text{ measurable}}}$ $\exists g \in E$ $\forall_{x^* \in E^*}$ $x^* g = \text{(L)} \int_A x^* f(s) \Delta(s)$.

Now, we present a new definition of the integral on time scales which is a generalization for both Pettis and Henstock-Kurzweil integrals.

**Definition 2.8.** The function $f : [a,b] \to E$ is Henstock-Kurzweil-Pettis integrable ((HKP) integrable for short) if there exists a function $g : [a,b] \to E$ with the following properties:
(i) $\forall_{x^* \in E^*}$ $x^* f$ is Henstock-Kurzweil integrable on $[a,b]$,
(ii) $\forall_{t \in [a,b]}$ $\forall_{x^* \in E^*}$ $x^* g(t) = (HK) \int_0^t x^* f(s) \Delta(s)$.

This function $g$ we denote by (HKP)$\int f(t)\Delta(t)$ and will be called Henstock-Kurzweil-Pettis integral of $f$ on the time scale interval $[a,b]$.

We have the following diagram of implications for vector-valued integration on time scales which is, in fact, similar to those for integration on $\mathbb{R}$

$$(Bochner) \Rightarrow (HL) \Rightarrow (HK) \Rightarrow (HKP) \Leftarrow (Pettis) \Leftarrow (Bochner)$$

Here, we need to present some **e**xamples. Please find the considered spaces $\mathbb{T}$ as time scales, not simply as subsets of $\mathbb{R}$. In particular, the Lebesgue measure on $\mathbb{T}$ is taken in the sense of time scales.

**1)** Considering the case $\mathbb{T}=\mathbb{R}$. For $[a,b]\subset\mathbb{T}$ we can recall, that all the classes of integrable functions are essentially different. For such examples cf. [19] and references therein.

**2)** For $\mathbb{T}=\mathbb{Z}_+$ we have the function $f$ defined on $\mathbb{T}$ is Bochner integrable if the series $\sum_{t_i\in\mathbb{T}} f(t_i)$ is absolutely convergent, Henstock-Kurzweil integrable if the series is conditionally convergent and finally is Pettis integrable if the series is weakly convergent. Of course, the sum of the series should be an element of the target space for $f$ (cf. an example below).

A definite example is the following: take a sequence $a_1=(1,0,0,0,0,...)$, $a_2=(-1,1,0,0,0,...)$, $a_3=(0,-1,1,0,0,...)$, ... of elements considered as elements of $l_2$. Define a function $f:\mathbb{T}\to l_2$ by the following formula $f(n)=a_n$ for each $n\in\mathbb{T}$.

Consider the series $\sum_{k=1}^{\infty} a_k$ in $l_2$. A partial sum for this series is in the form: $S_1=(1,0,0,0,...)$, $S_2=(0,1,0,0,...)$, $S_3=(0,0,1,0,...)$ ... . Since $\|S_n-S_m\|_2=\sqrt{2}$ for each $n\neq m$, $(S_n)$ cannot be a Cauchy sequence (due to completeness of the space $l_2$) consequently is not (strongly) convergent.

Nevertheless, taking an arbitrary $x^*\in(l_2)^*=l_2$ we have $x^*=(s_k)\in l_2$, so

$$x^*S_n=\sum_{k=1}^{\infty}\delta_{k,n}\cdot s_k=s_n,$$

where $\delta_{k,n}=1$ for $k=n$ and $\delta_{k,n}=0$ for $k\neq n$.

But $x^*=(s_k)\in l_2$, so $s_n\to 0$ as $n\to\infty$. Finally $x^*S_n\to 0$ as $n\to\infty$.

Remark that $\int_{\mathbb{T}} f(t)\Delta(t)=\sum_{k=1}^{\infty} a_k$ when the integral is considered in the sense of Bochner (strong convergence of the series) or in the sense of Pettis (weak convergence of the series). Thus (B)$\int_{\mathbb{T}} f(t)\Delta(t)$ does not exist, but (P)$\int_{\mathbb{T}} f(t)\Delta(t)=0$.

**3)** We need to show an example which clarifies the importance of both conditions in definitions of weak integrals.

For $\mathbb{T}=q^{\mathbb{N}}\cup\{0\}$ with $q=\frac{1}{2}$. Put $A_n=\{\frac{1}{2^k}:k\geq n\}$. Consider a function $f:\mathbb{T}\to c_0$ by the formula

$$f(t)=(\chi_{A_0}(t),2\cdot\chi_{A_1}(t),2^2\cdot\chi_{A_2}(t),2^3\cdot\chi_{A_3}(t),...)\in c_0,$$

where $\chi_A$ denotes a characteristic function of $A$.

As in a classical case we see, that if $g(s)=(g_1(s),g_2(s),g_3(s),...)\in c_0$ then $\int_A g(t)\Delta(t)=(\int_A g_1(s)\Delta(s),(\int_A g_2(s)\Delta(s),(\int_A g_3(s)\Delta(s),...)$. Hence for any $x^*\in(c_0)^*=l_1$ we have $x^*=(\alpha_1,\alpha_2,...)$ and then $x^*f=\sum_{n=1}^{\infty}\alpha_n\cdot 2^n\cdot\chi_{A_n}$, the series is convergent and this function is

(HK) integrable (or even Lebesgue integrable). However

$$(L)\int_{[0,1]} x^* f(t)\Delta(t) = ((L)\int_{[0,1]} \chi_{A_0}(s)\Delta(s),(L)\int_{[0,1]} 2\chi_{A_1}(s)\Delta(s),(L)\int_{[0,1]} 2^2\chi_{A_2}(s)\Delta(s),\dots)$$
$$= (\sum_{n=0}^{\infty} 1\cdot\frac{1}{2^n}, 2\sum_{n=1}^{\infty} 1\cdot\frac{1}{2^n}, 2^2\sum_{n=2}^{\infty} 1\cdot\frac{1}{2^n},\dots)$$
$$= (2,2,2,\dots) \notin c_0.$$

Thus this function is neither (HKP) nor (P) integrable on $\mathbb{T}$.

## 3 Basic properties.

Now, we prove some basic properties for the integrals defined above, in such a way to explain their usefulness. All of them will be necessary for proving the existence of solutions for the Cauchy problem in Banach spaces. Applicability of the presented results will be shown elsewhere ([21], for instance).

Without repetition of a full theory for Henstock-Kurzweil integrals on $\mathbb{T}$ we refer the reader to [41] for the proofs of indispensable properties of such an integral which will be used in the sequel for the proof of the following lemma:

**Theorem 3.1.** *Let $f$ and $g$ be functions defined on the interval of time scale $[a,b] \subset \mathbb{T}$ into the Banach space $E$ and let $\alpha$ , $\beta \in \mathbb{R}$.*

*a) If $f$ is (HKP) integrable on $[a,c]$ and on $[c,b]$ then this function is (HKP) integrable on $[a,b]$. Moreover*

$$(HKP)\int_a^b f(s)\Delta(s) = (HKP)\int_a^c f(s)\Delta(s) + (HKP)\int_c^b f(s)\Delta(s).$$

*b) If $f$ ang $g$ be (HKP) integrable on $[a,b]$, then $\alpha f + \beta g$ is (HKP) integrable on $[a,b]$ and*

$$(HKP)\int_a^b (\alpha f + \beta g)(s)\Delta(s) = \alpha(HKP)\int_a^b f(s)\Delta(s) + \beta(HKP)\int_a^b f(s)\Delta(s).$$

*Proof.* a) Since $f$ is (HKP) integrable on $[a,c]$ ($[c,b]$), for each $x^* \in E^*$ ,$x^* f$ is (HK) integrable on $[a,c]$ ($[c,b]$, respectively). Such functions are real-valued, so by Theorem 2.12 from [41] $x^* f$ are (HK) integrable on $[a,b]$.

Thus

$$x^*((HKP)\int_a^c f(s)\Delta(s)) = (HK)\int_a^c (x^* f(s))\Delta(s)$$

and

$$x^*((HKP)\int_c^b f(s)\Delta(s)) = (HK)\int_c^b (x^* f(s))\Delta(s).$$

We have

$$x^*((HKP)\int_a^c f(s)\Delta(s) + (HKP)\int_c^b f(s)\Delta(s))$$
$$= (HK)\int_a^c x^* f(s)\Delta(s) + (HK)\int_c^b x^* f(s)\Delta(s)$$
$$= (HK)\int_a^b x^* f(s)\Delta(s)$$

It follows that $f$ is (HKP) integrable on $[a,b]$ and

$$(HKP)\int_a^b f(s)\Delta(s) = (HKP)\int_a^c f(s)\Delta(s) + (HKP)\int_c^b f(s)\Delta(s).$$

b) It follows easily that from the above consideration and from linearity of $x^*$:

$$x^*((HKP)\int_a^b (\alpha f + \beta g)(s)\Delta(s)) = \alpha(HK)\int_a^b x^* f(s)\Delta(s) + \beta(HK)\int_a^b x^* f(s)\Delta(s).$$

□

Since the class of real-valued Lebesgue integrable functions defined on $\mathbb{T}$ is essentially contained in the class of such functions which are Henstock-Kurzweil integrable (cf. [41] Example 2.5 and Theorem 2.19, for instance), the same holds true for vector-valued functions. Thus we do not need to formulate similar lemmas for Pettis, Henstock-Kurzweil or Bochner integrals.

A few necessary results about weak differentiability:

**Lemma 3.2.** *If $y$ is weakly continuous at the point $t \in [a,b]$ and $t$ is right dense then $y$ is weakly $\Delta$ differentiable at $t$ iff its weak $\Delta$ derivative $y^{w\Delta} \in E$ satisfies for each $x^* \in E^*$*

$$x^* y^{w\Delta}(t) = \lim_{s \to t^+} \frac{x^* y(t) - x^* y(s)}{t - s}$$

*provided this limit exists as a finite number (i.e. $y^{w\Delta}$ is a weak $\Delta$ derivative).*

*Proof.* Recall that a function $y$ is weakly $\Delta$ differentiable at $t$ iff for every $\varepsilon > 0$ and every $x^* \in E^*$ we have:

$$|[x^* y(\sigma(t)) - x^* y(s)] - x^* y^{w\Delta}(t)[\sigma(t) - s]| < \varepsilon |\sigma(t) - s|.$$

Thus

$$\frac{|[x^* y(\sigma(t)) - x^* y(s)] - x^* y^{w\Delta}(t)[\sigma(t) - s]|}{|\sigma(t) - s|} < \varepsilon$$

so

$$|\frac{[x^* y(\sigma(t)) - x^* y(s)]}{[\sigma(t) - s]} - x^* y^{w\Delta}(t)| < \varepsilon.$$

Since $t$ is right dense, $\sigma(t) = t$ and

$$|\frac{[x^* y(t) - x^* y(s)]}{[t - s]} - x^* y^{w\Delta}(t)| < \varepsilon.$$

The thesis easily follows from the above inequality. □

**Lemma 3.3.** *If $y$ is weakly continuous at the point $t \in [a,b]$ and $t$ is right-scattered then $y$ is weakly $\Delta$ differentiable at $t$ iff its weak $\Delta$ derivative $y^{w\Delta} \in E$ satisfies*

$$y^{w\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.$$

*Proof.* As in the previous lemma: $y$ is weakly $\Delta$ differentiable at $t$ iff for every $\varepsilon > 0$ and every $x^* \in E^*$ we have:

$$|[x^*y(\sigma(t)) - x^*y(s)] - y^{w\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|.$$

But by the weak continuity of $y$ at $t$:

$$\lim_{s\to t} \frac{[x^*y(\sigma(t)) - x^*y(s)]}{[\sigma(t) - s]} = \frac{x^*y(\sigma(t)) - x^*y(t)}{[\sigma(t) - t]}.$$

Thus for a fixed $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ such that for every $s \in U$

$$|\frac{[x^*y(\sigma(t)) - x^*y(s)]}{[\sigma(t) - s]} - \frac{[x^*y(\sigma(t)) - x^*y(t)]}{[\sigma(t) - t]} < \varepsilon.$$

By multiplying the two side of this inequality by $|\sigma(t) - s|$ we get

$$|[x^*y(\sigma(t)) - x^*y(t)] - \frac{x^*y(\sigma(t)) - x^*y(t)}{\sigma(t) - t}[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|.$$

Finally, the definition of the $\Delta$ derivative for $x^*y$ holds true with $\frac{x^*y(\sigma(t)) - x^*y(t)}{\sigma(t) - t}$. Then

$$x^*y^{w\Delta}(t) = \frac{x^*y(\sigma(t)) - x^*y(t)}{\sigma(t) - t}.$$

From the linearity of $x^*$ it follows that $x^*y^{w\Delta}(t) = x^*(y^{w\Delta}(t) = x^*(\frac{y(\sigma(t)) - y(t)}{\sigma(t) - t})$. The last equality holds for each $x^* \in E^*$ so the weak $\Delta$ derivatives exist iff $y^{w\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}$. □

We will check some properties of the (HKP) integral considered as a function of the right endpoint of integration. Since (HK) integrable functions need not to be bounded nor absolutely integrable, we are unable to repeat a simple proof from [29] (Th. 4.3), so a full proof using the Saks-Henstock lemma instead will be presented. Similar statements hold true for another weak type integrals.

**Theorem 3.4.** *If $f : [a,b] \to E$ is (HKP) integrable, then the function $F(t) = (HKP)\int_0^t f(s)\Delta(s)$ is weakly continuous at each point $t \in [a,b]$. Moreover, for every point $t$ of the weak continuity of $f$ we have $F^{w\Delta}(t) = f(t)$.*

*Proof.* Fix an arbitrary $x^* \in E^*$ , $\varepsilon > 0$ and $\tau \in [a,b]$. If $\tau$ is isolated, then $x^*F$ is continuous at $\tau$. The function $x^*f$ is (HK) integrable on $[a,b]$ and by the Saks-Henstock lemma ([41], Th. 2.14) we ensure that there exists a $\Delta$-gauge $\delta_0 = (\delta_L, \delta_R)$ of $[a,b]$ such that

$$\sum_{i=1}^{n} |(HK)\int_{t_{i-1}}^{t_i} x^*f(s)\Delta(s) - x^*f(\xi_i)(t_i - t_{i-1})| \leq \varepsilon$$

for all $\delta_0$-fine partitions of $[a,b]$.

But for all $[\alpha,\beta] \subset [a,b]$ and $\tau \in [a,b]$ such that $\tau \in [\alpha,\beta] \subset (\tau - \delta_L(\tau), \tau + \delta_R(\tau))$ we have

$$|\int_{\alpha}^{\beta} x^*f(s)\Delta(s)| \leq \varepsilon + |x^*f(\tau)|(\beta - \alpha).$$

Indeed, taking $t_0 = \alpha$, $t_1 = \beta$ and $\xi_1 = \tau \in [\alpha,\beta] \subset ((\tau - \delta_L(\tau), (\tau + \delta_R(\tau)) \cup [a,b]$ in the inequality from the Saks-Henstock lemma, we have $|x^* f(\tau)(\beta-\alpha) - (\mathrm{HK}) \int_\alpha^\beta x^* f(s)\Delta(s)| \leq \varepsilon$. Consequently

$$
\begin{aligned}
|(\mathrm{HK}) \int_\alpha^\beta x^* f(s)\Delta(s)| &\leq |x^* f(\tau)(\beta-\alpha) - (\mathrm{HK}) \int_\alpha^\beta x^* f(s)\Delta(s)| + |x^* f(\tau)(\beta-\alpha)| \\
&\leq \varepsilon + |x^* f(\tau)|(\beta-\alpha).
\end{aligned}
$$

It is well-known that for every $\delta_0$-fine partition, by adding a point $\tau$ as a tag point we still have $\delta_0$-fine partition ([41], Remark 2.10). Then without loss of generality let us assume, that the points $t$ and $\tau$ are tag points of $\delta_0$.

From the above consideration it follows that for $t \in (\tau - \delta_L(\tau), \tau + \delta_R(\tau)) \cup [a,b]$ we have $|t-\tau| < \delta_L(\tau)$, $|t-\tau| < \delta_R(\tau)$ and

$$
\begin{aligned}
|x^* F(t) - x^* F(\tau)| &= |(\mathrm{HK}) \int_\tau^t x^* f(s)\Delta(s)| \\
&\leq \varepsilon + |x^* f(\tau)|(t-\tau)
\end{aligned}
$$

Taking a $\varepsilon_1$ in such a way that $0 < \varepsilon_1 < \min(\delta_L(\tau), \delta_R(\tau), \frac{\varepsilon}{1+|x^* f(\tau)|})$, we obtain $(\tau - \varepsilon_1, \tau + \varepsilon_1) \subset [a,b]$. Thus for $|t-\tau| < \varepsilon_1$

$$
|x^* F(t) - x^* F(\tau)| \leq \varepsilon + |x^* f(\tau)|(t-\tau) \leq \varepsilon + |x^* f(\tau)| \frac{\varepsilon}{1+|x^* f(\tau)|} < 2\varepsilon.
$$

This means that for every $\varepsilon > 0$ there exists a $\varepsilon_1 > 0$ such that $|t-\tau| < \varepsilon_1$ implies $|x^* F(t) - x^* F(\tau)| < \varepsilon$ i.e. $x^* F$ is continuous at $t$. This means that $F$ is weakly continuous at this point. The last thesis can be proved as in a classical case $\mathbb{T} = \mathbb{R}$. □

As a corollary we obtain also a new result for real-valued functions:

**Corollary 3.5.** *If $f : [a,b] \to \mathbb{R}$ is (HK) integrable, then the function $F(t) =$ (HK)$\int_0^t f(s)\Delta(s)$ is continuous at each point $t \in [a,b]$. Moreover, for every point $t$ of the continuity of $f$ we have $F^\Delta(t) = f(t)$.*

A second statement of the lemma follows easily from our previous Lemmas 3.2 and 3.3 about weak $\Delta$-differentiability.

The crucial role of the Saks-Henstock lemma for unbounded, nonabsolutely integrable functions is clear. Since exactly the same proof as in [15] for the usual (HL) integral holds true for the case of time scales, for the (HL) integral we have (cf. also [28, Theorem 9.12] or [41]):

**Lemma 3.6.** *If $f : [a,b] \to E$ is (HL) integrable, then the function $F(t) =$(HL)$\int_0^t f(s)\Delta(s)$ is continuous at each point $t \in \mathbb{T}$. Moreover, $F^{\,Delta}(t) = f(t)$ for almost all $t$ in the sense of Lebesgue $\Delta$-measure on $\mathbb{T}$. In particular, if $t$ is right-scattered then $y$ is $\Delta$ differentiable at $t$ and its $\Delta$ derivative $y^\Delta \in E$ satisfies*

$$
y^\Delta(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.
$$

From the Lebesgue criterion for Riemann $\Delta$-integrability ([29] Th. 5.8) it follows that for bounded real-valued functions a class of Riemann integrable functions on $[a,b)$ coincide with a class of functions for which the set of all right-dense points of discontinuity of $f$ on $[a,b)$ is a set of $\Delta$-measure zero. In particular, this means that weakly Riemann $\Delta$-integrable functions are $\Delta$ almost everywhere weakly continuous on the set of right dense points of $\mathbb{T}$. Due to Lemma 3.4 for a particular case of weakly Riemann $\Delta$-integrable functions we can ensure that $F$ is weakly $\Delta$-differentiable $\Delta$ a.e.

It might be expected, that $F(t) = \text{(HKP)} \int_a^t f(s)\Delta(s)$ if exists, is also weakly $\Delta$ differentiable (at least $\Delta$ almost everywhere). Unfortunately, this is not true, in general. Such a property does not holds even in the case $\mathbb{T} = \mathbb{R}$ and $F(t) = \text{(P)} \int_a^t f(s)\Delta(s) = \text{(P)} \int_a^t f(s)ds$. Such functions may be even nowhere weakly differentiable (cf. examples in the case $\mathbb{T} = \mathbb{R}$). To characterize such a primitive $F$ for arbitrary integrable function $f$ another notion of weak differentiability was introduced by Pettis in 1938, i.e. so-called pseudo-differentiability. This exceeds the scope of this paper and will be presented elsewhere (cf. [19] for ordinary calculus).

Finally, let us present some mean-value theorems for vector-valued integrals.

**Theorem 3.7.** (mean-value theorem) *For each measurable subset $A \subset \mathbb{T}$ of $[a,b]$ the integral (int)$\int_A y(s)\Delta(s)$ we have*

$$\text{(int)} \int_A y(s)\Delta(s) \in \mu_\Delta(A) \cdot \overline{\text{conv}}\, y(A),$$

*where the integral (int)$\int_a^t y(s)\Delta(s)$ is understood in the sense of weak Riemann or Pettis integral.*

*Proof.* It is clear that we can restrict our attention to the Pettis integral. Fix arbitrary $x^* \in E^*$. From the definition of the (HKP) integral it follows, that a real-valued function $x^*y$ is Lebesgue integrable on $A$. Moreover, we have the decomposition ([13] Th. 5.2):

$$\int_A x^*y(s)\Delta(s) = \int_A x^*y(s)ds + \sum_{t_i \in I_E} x^*y(t_i)\mu(t_i).$$

Thus by Proposition 3.1 and Theorem 5.2 in ([13]):

$$\begin{aligned}
\int_A x^*y(s)\Delta(s) &= \int_A x^*y(s)ds + \sum_{t_i \in I_E} x^*y(t_i)\mu(t_i) \\
&\in \text{mes(A)} \cdot \overline{\text{conv}}(x^*y)(A) + \sum_{t_i \in I_E} x^*y(t_i)\mu(t_i) \\
&\subset \text{mes(A)} \cdot \overline{\text{conv}}(x^*y)(A) + (x^*y)(A) \cdot \sum_{t_i \in I_E} \mu(t_i) \\
&\subset (\text{mes(A)} + \sum_{t_i \in I_E} \mu(t_i)) \cdot \overline{\text{conv}}(x^*y)(A) \\
&= \mu_\Delta(A) \cdot \overline{\text{conv}}(x^*y)(A).
\end{aligned}$$

Put $v = \text{(P)} \int_A y(s)\Delta(s)$ and $W = \mu_\Delta(A) \cdot \overline{\text{conv}}\, y(A)$. Suppose, contrary to our claim, that $v \notin W$. As $W$ is closed and convex, by separation theorem, there exists $z^* \in E^*$ such that

$\sup_{x\in W} z^*W = \alpha < z^*(v)$. Thus $z^*(v) = \int_A z^*(y(s)\Delta(s) > \alpha$. On the other hand, from the above consideration, $\mu_\Delta(A)z^*(y(s)) \in z^*(W)$, so $\mu_\Delta(A)z^*(y(s)) \leq \alpha$. Hence

$$\text{(P)}\int_A z^*y(s)\Delta(s) \leq \mu_\Delta(A)\frac{\alpha}{\mu_\Delta(A)} = \alpha,$$

a contradiction. □

By the same manner it can be easily proved, that for (HKP) integrable functions we have a little weaker result:

**Theorem 3.8.** *For each interval $[c,d] \subset [a,b]$ if the integral (HKP)$\int_c^d y(s)\Delta(s)$ exists, then we have*

$$\text{(HKP)}\int_c^d y(s)\Delta(s) \in \mu_\Delta([c,d]) \cdot \overline{\text{conv}}y([c,d]).$$

The aim of the paper is to extend the notions of vector-valued integrals for the functions defined on time scales. Applicability of such definitions in the theory of dynamic equations on Banach spaces is evident and exceed the scope of this paper. Some examples of applications can be found in [21]. Now, let us restrict ourselves to the one of the basic problems.

Consider a boundary value problem:

$$y^{\Delta\Delta} = f(t, y^\sigma(t)), \quad y(a) = 0\ ,\ y(b) = 0 \tag{3.1}$$

provided $a, b \in \mathbb{T}$.

Such a problem is important from an application point of view and was investigated separately for two most typical cases: $\mathbb{T} = \mathbb{R}$ (differential equations) or $\mathbb{T} = \mathbb{Z}$ (difference equations, cf. [23] or [2]) for both strong and weak topologies. Nevertheless, by using the dynamic equation (3.1) rewritten in the integral form with just defined integrals, it is possible to unify all existing results and to extend these theorems for another problems. Namely, as special cases we have the following equations:

1. Difference equation: $\mathbb{T} = \mathbb{Z}$, $y^\Delta(t) = \Delta y(t) = y(t+1) - y(t)$
$$\Delta^2 y(n) = f(n, y(n+1)),$$
2. Differential equation: $\mathbb{T} = \mathbb{R}$, $y^\Delta(t) = y'(t)$
$$y''(t) = f(t, y(t)),$$
3. Generalized difference equation: $\mathbb{T} = h\mathbb{Z}$, for $h > 0$, $y^\Delta(t) = \Delta_h y(t) = \frac{y(t+h)-y(t)}{h}$
$$\Delta_h^2 y(n) = f(t, y(t+h)),$$
4. $q$-difference equation: $\mathbb{T} = \{0\} \cup q^{\mathbb{N}}$, $y^\Delta(t) = \Delta_q y(t) = \frac{y(qt)-y(t)}{(q-1)t}$
$$\Delta_q^2 y(n) = f(t, y(qt)).$$

5. Difference equation: $\mathbb{T} = \{0\} \cup \mathbb{N}^2$, $y^\Delta(t) = \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}}$

$$\Delta_N^2 y(n) = f(t, y((\sqrt{t}+1)^2)).$$

It is clear, that we consider also different types of time scales and different dynamic problems, not only for the existence of solutions, but also for all type of problems investigated in Banach spaces (periodicity, asymptotic properties etc.).

All the above problems can be solved by finding a fixed point of the operator

$$Ty(t) = \int_0^t G(t,s) f(s, y^\sigma(s)) \Delta(s),$$

where $G$ is a Green function for the considered problem (3.1). The integral sign "$\int$" is considered in the sense which agree with a type of solutions (cf. [19] for the case of differential equations or [21] for the case of dynamic equations). Thus any of the defined integrals can be useful for solving the above problem. It depends only on the desired type of solutions. For details of the presented method let us refer to [23] (the necessity of the results proved in this paper will be then clear). The statements of exact results and the proofs are too big to be placed here and do not fully harmonize with the results presented above, then they will be published elsewhere.

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