Communications in **M**athematical **A**nalysis
Volume 11, Number 1, pp. 51–60 (2011) www.math-res-pub.org/cma
ISSN 1938-9787

A PERTURBATION OF DOUBLE DERIVATIONS ON BANACH ALGEBRAS

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(Communicated by Toka Diagana)

Abstract

In this paper, we prove the generalized Hyers – Ulam – Rassias stability of double derivations on Banach algebras.

AMS Subject Classification: 39B52, 39B82, 47B48.

Keywords: Generalized Hyers – Ulam – Rassias stability; double derivation.

1 Introduction

A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$?". Such a problem was formulated by S.M. Ulam [21] in 1940 and solved in the next year for the Cauchy functional equation by D.H. Hyers [9]. It gave rise to the *stability theory* for functional equations. For the history and various aspects of this

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theory we refer the reader to [1, 2, 7, 8, 10, 11, 14, 16, 17, 18, 19, 20]. Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}$ and let $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $\theta_1 : \mathcal{A} \to \mathcal{B}$ is called σ – derivation if

$$\theta_1(ab) = \theta_1(a)\sigma(b) + \sigma(a)\theta_1(b) \tag{1.1}$$

for all $a, b \in \mathcal{A}$.

Clearly, if $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ – derivation is an ordinary derivation. On the other hand, each homomorphism θ_1 is a $\frac{\theta_1}{2}$ – derivation. Thus, the theory of σ – derivations combines the theory of derivations and homomorphisms. If $\theta_2 : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $\theta_1 = \theta_2\sigma$ is a σ – derivation. Although, a σ – derivation is not necessarily of the form $\theta_2\sigma$, but it seems that the generalized Leibniz rule, $\theta_1(ab) = \theta_1(a)\sigma(b) + \sigma(a)\theta_1(b)$, comes from this observation.

M. Mirzavaziri and E. Omidvar Tehrani [13] took ideas from above fact, and considered two derivations θ_2, θ_3 to find a similar rule, for $\theta_1 = \theta_2\theta_3$. In this case, they saw that θ_1 satisfies

$$\theta_1(ab) = \theta_1(a)b + a\theta_1(b) + \theta_2(a)\theta_3(b) + \theta_3(a)\theta_2(b) \tag{1.2}$$

for all $a, b \in \mathcal{A}$. They said that a linear mapping $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2, θ_3) – double derivation if it satisfies (1.2). Moreover, by a θ_1 – double derivation they called a (θ_1, θ_1) – derivation and proved that if $\mathcal{A}$ is a C^* – algebra, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a $*$ - linear mapping and $\theta_2 : \mathcal{A} \to \mathcal{A}$ is a continuous θ_1 – double derivation then θ_1 is continuous.

During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; see [3, 4, 5, 6, 15] and references therein for more detailed information.

H. Khodaei and Th. M. Rassias [12] have found the general n – dimensional additive functional equation for $n \geq 2$ as follows:

$$\begin{aligned}\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r}x_{i_r}\Big)\\ + f\Big(\sum_{i=1}^{n} a_i x_i\Big) = 2^{n-1}a_1 f(x_1)\end{aligned} \tag{1.3}$$

where $a_1, \ldots, a_n \in \mathbb{Z} - \{0\}$ with $a_1 \neq \pm 1$, and investigated stability of functional equation (1.3) in random normed spaces, in non-Archimedean spaces and quasi-normed spaces.

In this paper, our main purpose is to prove the generalized Hyers – Ulam – Rassias stability of (θ_2, θ_3) – double derivations on $\mathcal{A}$ associated with the functional equation (1.3).

Throughout this paper, assume that $a_1, \ldots, a_n$ are nonzero fixed integers with $a_1 \neq \pm 1$, and that $\mathcal{A}$ is a Banach algebra.

2 Main Results

Let $l = 1,2,3$. For given mappings $f_l : \mathcal{A} \to \mathcal{A}$, we define the difference operators $D_\mu f_l : \mathcal{A}^n \to \mathcal{A}$ and $C_{f_1,f_2,f_3}(x,y) : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ by

$$D_\mu f_l(x_1,...,x_n) := \sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f_l\Big(\sum_{i=1,i\neq i_1,...,i_{n-k+1}}^{n} a_i\mu x_i - \sum_{r=1}^{n-k+1} a_{i_r}\mu x_{i_r}\Big) + f_l\Big(\sum_{i=1}^{n} a_i\mu x_i\Big) - 2^{n-1}a_1\mu f_l(x_1)$$

and

$$C_{f_1,f_2,f_3}(x,y) := f_1(xy) - f_1(x)y - xf_1(y) - f_2(x)f_3(y) - f_3(x)f_2(y)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda : \ |\lambda| = 1\}$ and all $x,y,x_i \in \mathcal{A}$ $(i = 1,2,...,n)$. We will use the following lemma in this paper.

Lemma 2.1. *[12] A function $f : \mathcal{A} \to \mathcal{A}$ satisfies the functional equation* (1.3) *if and only if $f : \mathcal{A} \to \mathcal{A}$ is additive.*

Theorem 2.2. *Let $r > 1$, $l = 1,2,3$ and let $\theta_l : \mathcal{A} \to \mathcal{A}$ be mappings satisfying $\theta_l(rx) = r\theta_l(x)$ for all $x \in \mathcal{A}$. If there exists a function $\varphi : \mathcal{A}^8 \to [0,\infty)$ such that*

$$\lim_{j\to\infty}\frac{1}{r^j}\varphi(r^jx,r^jy,r^jz,r^jt,r^ja,r^jb,r^jc,r^jd) = 0, \tag{2.1}$$

$$\begin{aligned}\|\theta_1(\mu x+\mu y+zt) &- \mu\theta_1(x) - \mu\theta_1(y) - \theta_1(z)t - z\theta_1(t) - \theta_2(z)\theta_3(t) - \theta_3(z)\theta_2(t)\\ &- \theta_2(\mu a+\mu b) - \mu\theta_2(a) - \mu\theta_2(b) - \theta_3(\mu c+\mu d) - \mu\theta_3(c) - \mu\theta_3(d)\|\\ &\leq \varphi(x,y,z,t,a,b,c,d) \qquad (2.2)\end{aligned}$$

for all $\mu \in \mathbb{C}$ and all $x,y,z,t,a,b,c,d \in \mathcal{A}$. Then θ_1 is a (θ_2,θ_3) – double derivation on $\mathcal{A}$.

Proof. $\theta_l(0) = 0$ since $\theta_l(0) = r\theta_l(0)$. Put $z = t = a = b = c = d = 0$ in (2.2). Then

$$\begin{aligned}\|\theta_1(\mu x+\mu y) - \mu\theta_1(x) - \mu\theta_1(y)\| &= \frac{1}{r^j}\|\theta_1(\mu r^jx+\mu r^jy) - \mu\theta_1(r^jx) - \mu\theta_1(r^jy)\|\\ &\leq \frac{1}{r^j}\varphi(r^jx,r^jy,0,0,0,0,0,0)\end{aligned}$$

for all $x,y \in \mathcal{A}$ and all $\mu \in \mathbb{C}$. The right hand side tends to zero as $j \to \infty$. So

$$\theta_1(\mu x+\mu y) = \mu\theta_1(x) + \mu\theta_1(y)$$

for all $x,y \in \mathcal{A}$ and all $\mu \in \mathbb{C}$. Similarly, one can show that

$$\theta_2(\mu x+\mu y) = \mu\theta_2(x) + \mu\theta_2(y),$$

$$\theta_3(\mu x+\mu y) = \mu\theta_3(x) + \mu\theta_3(y)$$

for all $x,y \in \mathcal{A}$ and all $\mu \in \mathbb{C}$. Let $\mu = 1$ and $x = y = a = b = c = d = 0$ in (2.2), we get

$$\|\theta_1(zt) - \theta_1(z)t - z\theta_1(t) - \theta_2(z)\theta_3(t) - \theta_3(z)\theta_2(t)\| = \frac{1}{r^{2j}}\|\theta_1(r^j z r^j t) - \theta_1(r^j z)r^j t - r^j z\theta_1(r^j t) - \theta_2(r^j z)\theta_3(r^j t) - \theta_3(r^j z)\theta_2(r^j t)\| \le \frac{1}{r^{2j}}\varphi(0,0,r^j z, r^j t, 0,0,0,0) \le \frac{1}{r^j}\varphi(0,0,r^j z, r^j t,0,0,0,0)$$

for all $z,t \in \mathcal{A}$. The right hand side tends to zero as $j \to \infty$. Then

$$\theta_1(zt) = \theta_1(z)t + z\theta_1(t) + \theta_2(z)\theta_3(t) + \theta_3(z)\theta_2(t)$$

for all $z,t \in \mathcal{A}$. □

Now, we investigate the generalized Hyers – Ulam – Rassias stability of (θ_2,θ_3) – double derivations on Banach algebras for functional equation (1.3).

Theorem 2.3. *Let $l = 1,2,3$. If $f_l : \mathcal{A} \to \mathcal{A}$ with $f_l(0) = 0$ are mappings for which there exists a function $\varphi : \mathcal{A}^{n+2} \to [0,\infty)$ such that*

$$\tilde{\varphi}(x) := \sum_{j=0}^{\infty} \frac{1}{|a_1|^j}\varphi(a_1{}^j x, 0, \dots, 0,0,0) < \infty, \tag{2.3}$$

$$\lim_{j\to\infty} \frac{1}{|a_1|^j}\varphi(a_1{}^j x_1, a_1{}^j x_2, \dots, a_1{}^j x_n, a_1{}^j a, a_1{}^j b) = 0, \tag{2.4}$$

$$\max_l\{\|D_\mu f_l(x_1,x_2,\dots,x_n) - C_{f_1,f_2,f_3}(a,b)\|\} \le \varphi(x_1,x_2,\dots,x_n,a,b) \tag{2.5}$$

for all $a,b,x_i \in \mathcal{A}$ $(i = 1,2,\dots,n)$ and all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$. Then there exist unique $\mathbb{C}$– linear mappings $\theta_l : \mathcal{A} \to \mathcal{A}$ such that

$$\|f_l(x) - \theta_l(x)\| \le \frac{1}{2^{n-1}|a_1|}\tilde{\varphi}(x) \tag{2.6}$$

for all $x \in \mathcal{A}$. Moreover, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2,θ_3) – double derivation on $\mathcal{A}$.

Proof. It follows from the inequality (2.5) that

$$\|D_\mu f_1(x_1,x_2,\dots,x_n) - C_{f_1,f_2,f_3}(a,b)\| \le \varphi(x_1,x_2,\dots,x_n,a,b), \tag{2.7}$$

$$\|D_\mu f_2(x_1,x_2,\dots,x_n) - C_{f_1,f_2,f_3}(a,b)\| \le \varphi(x_1,x_2,\dots,x_n,a,b), \tag{2.8}$$

$$\|D_\mu f_3(x_1,x_2,\dots,x_n) - C_{f_1,f_2,f_3}(a,b)\| \le \varphi(x_1,x_2,\dots,x_n,a,b) \tag{2.9}$$

for all $a,b,x_i \in \mathcal{A}$ $(i = 1,2,\dots,n)$ and all $\mu \in \mathbb{T}^1$. Let $\mu = 1$. We use the relation

$$1 + \sum_{i=1}^{n-1}\binom{n-1}{i} = \sum_{i=0}^{n-1}\binom{n-1}{i} = 2^{n-1} \tag{2.10}$$

and put $x_1 = x$ and $a = b = x_i = 0$ $(i = 2,\dots,n)$ in (2.7). Then we obtain

$$\|2^{n-1}f_1(a_1 x) - 2^{n-1}a_1 f_1(x)\| \le \varphi(x,0,\dots,0,0,0) \tag{2.11}$$

for all $x \in \mathcal{A}$. So

$$\|f_1(x) - \frac{1}{a_1} f_1(a_1 x)\| \leq \frac{1}{2^{n-1}|a_1|} \varphi(x, 0, ..., 0, 0, 0) \tag{2.12}$$

for all $x \in \mathcal{A}$. Replacing x by $a_1 x$ in (2.12) and dividing by a_1 and summing the resulting inequality with (2.12), we get

$$\|f_1(x) - \frac{1}{a_1^2} f_1(a_1^2 x)\| \leq \frac{1}{2^{n-1}|a_1|} (\varphi(x, 0, ..., 0, 0, 0) + \frac{\varphi(a_1 x, 0, ..., 0, 0, 0)}{|a_1|}) \tag{2.13}$$

for all $x \in \mathcal{A}$. Hence

$$\|\frac{1}{a_1^l} f_1(a_1^l x) - \frac{1}{a_1^m} f_1(a_1^m x)\| \leq \frac{1}{2^{n-1}\ |a_1|} \sum_{j=k}^{m-1} \frac{1}{|a_1|^j} \varphi(a_1^j x, 0, ..., 0, 0, 0) \tag{2.14}$$

for all $x \in \mathcal{A}$. for all nonnegative integers m and k with $m > k$ and for all $x \in \mathcal{A}$. It follows from (2.3) and (2.14) that the sequence $\{\frac{1}{a_1^m} f_1(a_1^m x)\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{\frac{1}{a_1^m} f_1(a_1^m x)\}$ converges. Therefore, one can define the function $\theta_1 : \mathcal{A} \to \mathcal{A}$ by

$$\theta_1(x) := \lim_{m\to\infty} \frac{1}{a_1^m} f_1(a_1^m x)$$

for all $x \in \mathcal{A}$. In the inequality (2.7), assume that $a = b = 0$ and $\mu = 1$. Then By (2.4),

$$\begin{aligned} \|D_1\theta_1(x_1, ..., x_n)\| &= \lim_{m\to\infty} \frac{1}{|a_1|^m} \|D_1 f_1(a_1^m x_1, ..., a_1^m x_n)\| \\ &\leq \lim_{m\to\infty} \frac{1}{|a_1|^m} \varphi(a_1^m x_1, ..., a_1^m x_n, 0, 0) = 0 \end{aligned}$$

for all $x_1, ..., x_n \in \mathcal{A}$. So $D_1\theta_1(x_1, ..., x_n) = 0$. By Lemma 2.1, the function $\theta_1 : \mathcal{A} \to \mathcal{A}$ is additive. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (2.14), we get the inequality (2.6) for $l = 1$. Now, let $\theta_1' : \mathcal{A} \to \mathcal{A}$ be another additive function satisfying (1.3) and (2.6). So

$$\begin{aligned} \|\theta_1(x) - \theta_1'(x)\| &= \frac{1}{|a_1|^m} \|\theta_1(a_1^m x) - \theta_1'(a_1^m x)\| \leq \frac{1}{|a_1|^m} (\|\theta_1(a_1^m x) - f_1(a_1^m x)\| \\ &+ \|\theta_1'(a^m x) - f_1(a^m x)\|) \leq \frac{2}{|a_1|^m 2^{(n-1)} |a_1|} \widetilde{\varphi}(a_1^m x) \end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in \mathcal{A}$. So we can conclude that $\theta_1(x) = \theta_1'(x)$ for all $x \in \mathcal{A}$. This proves the uniqueness of θ_1.

For $l = 2$ and $l = 3$, a similar argument shows that there exist unique additive mappings $\theta_2, \theta_3 : \mathcal{A} \to \mathcal{A}$ satisfying (2.6). The additive mappings $\theta_2, \theta_3 : \mathcal{A} \to \mathcal{A}$ are defined by

$$\theta_2(x) := \lim_{m\to\infty} \frac{1}{a_1^m} f_2(a_1^m x) \tag{2.15}$$

and

$$\theta_3(x) := \lim_{m\to\infty} \frac{1}{a_1^m} f_3(a_1^m x) \tag{2.16}$$

for all $x \in \mathcal{A}$. Since θ_1 is additive, we have $a_1\theta_1(x) = \theta_1(a_1x) = \lim_{m\to\infty} \frac{1}{{a_1}^m} f_1({a_1}^{m+1}x)$ for all $x \in \mathcal{A}$. Thus $\theta_1(x) = \lim_{m\to\infty} \frac{1}{{a_1}^{m+1}} f_1({a_1}^{m+1}x)$ for all $x \in \mathcal{A}$. Let $\mu \in \mathbb{T}^1$. Set $x_1 = x$ and $a = b = x_i = 0 \ \ (i = 2,...,n)$ in (2.7). Then by the relation (2.10), we get

$$\|2^{n-1} f_1(a_1\mu x) - 2^{n-1} a_1\mu f_1(x)\| \le \varphi(x,0,...,0,0,0) \tag{2.17}$$

for all $x \in \mathcal{A}$. So that

$$\|a_1^{-(m+1)}(2^{n-1} f_1(a_1^{m+1}\mu x) - 2^{n-1} a_1\mu f_1(a_1^m x))\| \le |a_1|^{-(m+1)}\varphi(a_1^m x,0,...,0,0,0),$$

that is,

$$\|a_1^{-(m+1)} f_1(a_1^{m+1}\mu x) - a_1^{-m}\mu f_1(a_1^m x))\| \le \frac{|a_1|^{-m}\varphi(a_1^m x,0,...,0,0,0)}{|a_1|2^{n-1}} \tag{2.18}$$

for all $x \in \mathcal{A}$. Since the right hand side tends to zero as $m \to \infty$, we have

$$\theta_1(\mu x) = \lim_{m\to\infty} \frac{1}{{a_1}^{m+1}} f_1(\mu {a_1}^{m+1}x) = \lim_{m\to\infty} \frac{\mu f_1({a_1}^m x)}{{a_1}^m} = \mu\theta_1(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Obviously, $\theta_1(0x) = 0 = 0\theta_1(x)$.
Next, let $\lambda = \beta_1 + i\beta_2 \in \mathbb{C}$, where $\beta_1, \beta_2 \in \mathbb{R}$. Let $\alpha_1 = \beta_1 - [\beta_1], \alpha_2 = \beta_2 - [\beta_2]$, in which $[r]$ denotes the greatest integer less than or equal to the number r. Then $0 \le \alpha_i \le 1, \ (1 \le i \le 2)$ and one can represent α_i as $\alpha_i = \frac{\mu_{i,1}+\mu_{i,2}}{2}$ in which $\mu_{i,j} \in \mathbb{T}^1, \ \ (1 \le i,j \le 2)$. Since θ_1 is additive we infer that

$$\begin{aligned}\theta_1(\lambda x) &= \theta_1(\beta_1 x) + i\theta_1(\beta_2 x) = [\beta_1]\theta_1(x) + \theta_1(\alpha_1 x) + i([\beta_2]\theta_1(x) + \theta_1(\alpha_2 x))\\ &= ([\beta_1]\theta_1(x) + \frac{1}{2}\theta_1(\mu_{1,1}x + \mu_{1,2}x)) + i([\beta_2]\theta_1(x) + \frac{1}{2}\theta_1(\mu_{2,1}x + \mu_{2,2}x))\\ &= \beta_1\theta_1(x) + i\beta_2\theta_1(x) = \lambda\theta_1(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a $\mathbb{C}$ – linear mapping. A similar argument shows that θ_2, θ_3 are $\mathbb{C}$– linear.
Setting $x_1 = x_2 = ... = x_n = 0$ in the inequality (2.7), we get

$$\begin{aligned}|a_1|^{-2m}\|C_{f_1,f_2,f_3}({a_1}^m a, {a_1}^m b)\| &= |a_1|^{-2m}\|f_1({a_1}^{2m}ab) - f_1({a_1}^m a){a_1}^m b - {a_1}^m a f_1({a_1}^m b)\\ &\quad - f_2({a_1}^m a) f_3({a_1}^m b) - f_3({a_1}^m a) f_2({a_1}^m b)\|\\ &\le |a_1|^{-2m}\varphi(0,...,0,{a_1}^m a,{a_1}^m b) \le |a_1|^{-m}\varphi(0,...,0,{a_1}^m a,{a_1}^m b),\end{aligned}$$

which tends to zero as $m \to \infty$ for all $a, b \in \mathcal{A}$ by (2.4). Hence

$$\theta_1(ab) = \theta_1(a)b + a\theta_1(b) + \theta_2(a)\theta_3(b) + \theta_3(a)\theta_2(b)$$

for all $a, b \in \mathcal{A}$. So the $\mathbb{C}$– linear mapping $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2, θ_3) – double derivation on $\mathcal{A}$. □

Corollary 2.4. *Let $l = 1,2,3$. Let $f_l : \mathcal{A} \to \mathcal{A}$ be mappings with $f_l(0) = 0$ for which there exist constants $\varepsilon \geq 0$ and $p < 1$ such that*

$$\max_l \{\|D_\mu f_l(x_1,x_2,...,x_n) - C_{f_1,f_2,f_3}(a,b)\|\} \leq \varepsilon(\|a\|^p + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p)$$

for all $a,b,x_i \in \mathcal{A}$ $(i = 1,2,...,n)$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$ – linear mappings $\theta_l : \mathcal{A} \to \mathcal{A}$ such that

$$\|f_l(x) - \theta_l(x)\| \leq \frac{\varepsilon\|x\|^p}{2^{n-1}|a_1|(1 - |a_1|^{p-1})},$$

for all $x \in \mathcal{A}$. Moreover, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2,θ_3) – double derivation on $\mathcal{A}$.

Proof. Define $\varphi(x_1,x_2,...,x_n,a,b) := \varepsilon(\|a\|^p + \|b\|^p + \sum_{i=1}^{n} \|x_i\|^p)$ for all $a,b,x_i \in \mathcal{A}$ $(i = 1,...,n)$, and apply Theorem 2.3. □

Theorem 2.5. *Let $l = 1,2,3$. Let $r,s,r_1,r_2,..,r_n$ and ε be non-negative real numbers such that $r+s < 2$. If $f_l : \mathcal{A} \to \mathcal{A}$ are mappings satisfying*

$$\max_l \{\|D_\mu f_l(x_1,x_2,...,x_n)\|\} \leq \varepsilon \prod_{i=1}^{n} \|x_i\|^{r_i}, \tag{2.19}$$

$$\|C_{f_1,f_2,f_3}(a,b)\| \leq \varepsilon\|a\|^r\|b\|^s \tag{2.20}$$

for all $\mu \in \mathbb{T}^1$ and all $a,b,x_1,...,x_n \in \mathcal{A}$, then the mappings $f_l : \mathcal{A} \to \mathcal{A}$ are $\mathbb{C}$–linear. Moreover, $f_1 : \mathcal{A} \to \mathcal{A}$ is a (f_2,f_3) – double derivation. (We put $\|.\|^0 = 1$).

Proof. It follows from the inequality (2.19) that

$$\|D_\mu f_1(x_1,x_2,...,x_n)\| \leq \varepsilon \prod_{i=1}^{n} \|x_i\|^{r_i}, \tag{2.21}$$

$$\|D_\mu f_2(x_1,x_2,...,x_n)\| \leq \varepsilon \prod_{i=1}^{n} \|x_i\|^{r_i}, \tag{2.22}$$

$$\|D_\mu f_3(x_1,x_2,...,x_n)\| \leq \varepsilon \prod_{i=1}^{n} \|x_i\|^{r_i} \tag{2.23}$$

for all $x_i \in \mathcal{A}$ $(i = 1,2,...,n)$. Letting $x_i = 0$ $(i = 1,...,n)$ in (2.21), we get that

$$D_\mu f_1(0,0,...,0) = 0$$

that is,

$$\sum_{k=2}^{n} (\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n}) f_1(0) + f_1(0) = 2^{n-1} a_1 f_1(0)$$

that is,

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n} f_1(0)+\sum_{i_1=2}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n} f_1(0)+\ldots+\sum_{i_1=2}^{n} f_1(0) + f_1(0) = 2^{n-1}a_1 f_1(0)$$

whence,

$$\left(\binom{n-1}{n-1}+\binom{n-1}{n-2}+\ldots+\binom{n-1}{1}+1\right)f_1(0) = 2^{n-1}a_1 f_1(0). \tag{2.24}$$

It follows from (2.10) and (2.22) that $2^{n-1}(a_1-1)f_1(0)=0$. Since $a_1 \neq \pm 1$, so $f_1(0)=0$. By Lemma 2.1 and Theorem 2.3, the mapping $f_1 : \mathcal{A} \to \mathcal{A}$ is $\mathbb{C}$ – linear. Similarly, $f_2(0) = f_3(0) = 0$ and the mappings f_2, f_3 are $\mathbb{C}$ – linear.
It follows from (2.20) that

$$\|C_{f_1,f_2,f_3}(a,b)\| = \frac{1}{2^{2j}}\|C_{f_1,f_2,f_3}(2^j a, 2^j b)\| \leq \left(\frac{2^{(r+s)}}{2^2}\right)^j \varepsilon \|a\|^r \|b\|^s$$

for all $a,b \in \mathcal{A}$. Since the right hand side tends to zero as $j \to \infty$, we have

$$C_{f_1,f_2,f_3}(a,b) = 0$$

for all $a,b \in \mathcal{A}$. Hence f_1 is a (f_2, f_3) – double derivation on $\mathcal{A}$. □

Remark 2.6. We can obtain similar result to Theorem 2.5 for $r+s>2$.

Theorem 2.7. *Let $l = 1,2,3$. Suppose that $f_l : \mathcal{A} \to \mathcal{A}$ with $f_2(0) = f_3(0) = 0$ are mappings satisfying (2.5). If there exists a function $\varphi : \mathcal{A}^{n+2} \to [0,\infty)$ such that*

$$\tilde{\varphi}(x) := \sum_{j=1}^{\infty} |a_1|^j \varphi\left(\frac{x}{{a_1}^j}, 0, \ldots, 0, 0, 0\right) < \infty, \tag{2.25}$$

$$\lim_{j\to\infty} |a_1|^j \varphi\left(\frac{x_1}{{a_1}^j}, \ldots, \frac{x_n}{{a_1}^j}, \frac{a}{{a_1}^j}, \frac{b}{{a_1}^j}\right) = 0, \tag{2.26}$$

for all $x_1, \ldots, x_n, a, b \in \mathcal{A}$, then there exist unique $\mathbb{C}$ – linear mappings $\theta_l : \mathcal{A} \to \mathcal{A}$ such that

$$\|f_l(x) - \theta_l(x)\| \leq \frac{1}{2^{n-1}}\tilde{\varphi}\left(\frac{x}{a_1}\right) \tag{2.27}$$

for all $x \in \mathcal{A}$. Moreover, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2, θ_3) – double derivation on $\mathcal{A}$.

Proof. Letting $a = b = x_i = 0$ $(i = 1, \ldots, n)$ in (2.26), we get $\lim_{j\to\infty}|a_1|^j\varphi(0,\ldots,0,0,0) = 0$. Hence, $\varphi(0,\ldots,0,0,0) = 0$. Now, put $a = b = x_i = 0$ $(i = 1,\ldots,n)$ in (2.7). Since $g(0) = h(0) = 0$, we get $D_\mu f(0,\ldots,0,0,0) = 0$. Therefore, by Theorem 2.5 we obtain $f(0) = 0$. It follows from (2.11) that

$$\|f_1(x) - a_1 f_1\left(\frac{x}{a_1}\right)\| \leq \frac{1}{2^{n-1}}\varphi\left(\frac{x}{a_1}, 0, \ldots, 0, 0, 0\right)$$

for all $x \in \mathcal{A}$. Hence

$$\|a_1^l f_1(\frac{x}{a_1^l}) - a_1^m f_1(\frac{x}{a_1^m})\| \leq \frac{1}{2^{n-1}} \sum_{j=k}^{m-1} |a_1|^j \varphi(\frac{x}{a_1^{j+1}}, 0, ..., 0, 0, 0) \qquad (2.28)$$

for all nonnegative integers m and k with $m > k$ and for all $x \in \mathcal{A}$. It follows from (2.28) that the sequence $\{a_1{}^m f_1(\frac{x}{a_1{}^m})\}$ is a Cauchy sequence for all $x \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\{a_1{}^m f_1(\frac{x}{a_1{}^m})\}$ converges. So one can define the function $\theta_1 : \mathcal{A} \to \mathcal{A}$ by

$$\theta_1(x) := \lim_{m\to\infty} a_1^m f_1(\frac{x}{a_1^m})$$

for all $x \in \mathcal{A}$.
The rest of the proof is similar to the proof of Theorem 2.3 and we omit it. □

Corollary 2.8. *Let $l = 1,2,3$. Suppose $f_l : \mathcal{A} \to \mathcal{A}$ are mappings with $f_l(0) = 0$ for which there exist constants $\varepsilon \geq 0$ and $p > 1$ such that*

$$\max_l \{\|D_\mu f_l(x_1, x_2, ..., x_n) - C_{f_1,f_2,f_3}(a,b)\|\}$$
$$\leq \varepsilon(\|a\|^p + \|b\|^p + (\sum_{i=1}^{n} \|x_i\|^p)$$

for all $a, b, x_i \in \mathcal{A}$ $(i = 1,2,...,n)$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$ – linear mappings $\theta_l : \mathcal{A} \to \mathcal{A}$ such that

$$\|f_l(x) - \theta_l(x)\| \leq \frac{\varepsilon\|x\|^p}{2^{n-1}|a_1|(|a_1|^{1-p} - 1)},$$

for all $x \in \mathcal{A}$. Moreover, $\theta_1 : \mathcal{A} \to \mathcal{A}$ is a (θ_2, θ_3) – double derivation on $\mathcal{A}$.

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