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H^p_w BOUNDEDNESS OF CALDERÓN-ZYGMUND TYPE OPERATORS

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Abstract

We use atomic decomposition and molecular characterization to show that some Calderón-Zygmund operators are bounded on H^p_w-boundedness for $0 < p \le 1$, where w belongs to the Muckenhoup A_1-class.

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1 Introduction

The classical Calderón-Zygmund operators (CZO's) are a class of convolution operators defined on $\mathbb{R}^n$

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy,$$

where the Fourier transform of k is essentially bounded and k is of class C^1 outside the origin with

$$|\nabla k(x)| \le C/|x|^{n+1}. \tag{1.1}$$

It is well known that T is bounded on $L^p, 1 < p < \infty$, and is of weak type $(1,1)$. Furthermore, the results hold with (1.1) replaced by a weaker condition,

$$\int_{|x|\ge 2|y|} |k(x-y) - k(x)|\, dx \le C, \qquad |y| > 0.$$

The theory of CZO's has been studied and extended by many mathematicians in different topics, such as non-convolution type of CZO's, pseudo-differential operators, and oscillatory singular integrals (see [7]).

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In this article we are concerned about the following Calderón-Zygmund type operators. We say that $K \in K_q, 2 < q < \infty$, if there is a sequence of positive constants $\{C_j\}$ such that, for $j \in \mathbb{N}$,

$$\left(\int_{2^j|z-y|\le|x-y|<2^{j+1}|z-y|} |K(x,y)-K(x,z)|^q dx\right)^{\frac{1}{q}} \le C_j \cdot \left(2^j|z-y|\right)^{-\frac{n}{q'}} \tag{1.2}$$

and

$$\left(\int_{2^j|y-z|\le|y-x|<2^{j+1}|y-z|} |K(y,x)-K(z,x)|^q dx\right)^{\frac{1}{q}} \le C_j \cdot \left(2^j|z-y|\right)^{-\frac{n}{q'}}. \tag{1.3}$$

Here and afterwards, q' denotes the conjugate number of q satisfying $1/q+1/q'=1$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the class of all tempered distributions on $\mathbb{R}^n$. Suppose $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear operator with kernel $K(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$, which is defined initially by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{1.4}$$

The operator T is said to be a *Calderón-Zygmund type operator* if T can be extended as a bounded operator on $L^2(\mathbb{R}^n)$ and $K \in K_q$ with

$$\int_{|x-y|>2|z-y|} \Big(|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\Big)dx \le C$$

where $C>0$ is a constant independent of y and z. It is clear that (1.1) implies that (1.2) with $C_j = C2^{-j}$ when $K(x,z) = K(x-z)$ and $y=0$. This tells us that the class of Calderón-Zygmund type operators contains classical CZO's. Chang and Lee [1] obtained estimates for Calderón-Zygmund type operators acting on Hardy spaces of homogeneous groups. For the weighted case, Chang, Li and Xiao [2] showed that Calderón-Zygmund type operators are bounded on $L^p_w(\mathbb{R}^n)$.

Theorem 1.1. *[2] Let T be a Calderón-Zygmund type operator associated with kernel $K \in K_q$ and a sequence $\{C_j\}$ satisfying* (1.2)-(1.3)*. For $1 \le p < \infty$, if $\{C_j\} \in \ell^1$ and $w \in A_1 \cap RH_{q'}$, then T can be extended as a bounded operator on L^p_w for $1 < p < \infty$ and satisfies weak L^1_w boundedness.*

The purpose of this paper is to generalize the boundedness to the weighted Hardy spaces.

Theorem 1.2. *Let T be a Calderón-Zygmund type operator associated to a kernel $K \in K_q$ and a sequence $\{C_j\}$ satisfying* (1.2)-(1.3)*. For $0 < p \le 1$, suppose the following three conditions hold:*

(i) *there exists* $\varepsilon > \max\left\{\left[n(\frac{2}{p}-1)\right]r_w(r_w-1)^{-1}n^{-1}+(r_w-1)^{-1}, 1/p-1\right\}$ *such that* $\{2^{nj\varepsilon}(C_j)\} \in \ell^2$,

(ii) $T^*(x^\alpha)=0$ *with* $|\alpha| \le \left[n(\frac{2}{p}-1)\right]$, *where T^* is the adjoint operator of T,*

(iii) $w \in A_1 \cap RH_{q/(q-2)}$.

Then T can be extended as a bounded operator on $H^p_w(\mathbb{R}^n)$.

Remark 1.3. This result extends [1, Theorem 3.8] to the $H^p_w(\mathbb{R}^n)$ boundedness.

For the convenience of statement, we always assume that the letter C stands for a generic constant independent of main variables and B_j denotes either the set $\{x \in \mathbb{R}^n : 2^j|z-y| \leq |x-y| < 2^{j+1}|z-y|\}$ or the set $\{y \in \mathbb{R}^n : 2^j|x-z| \leq |y-x| < 2^{j+1}|x-z|\}$ for each $j \in \mathbb{N}$, unless a special remark is made.

2 Preliminaries

Let φ be a Schwartz function with $\int_{\mathbb{R}^n} \varphi = 1$, and set $\varphi_t(x) = t^{-n}\varphi(x/t), t > 0$. For an distribution f, we define its radial maximal function by

$$f^*(x) = \sup_{t>0} |\varphi_t * f(x)|.$$

The weighted Hardy space H^p_w consists of the distributions whose radial maximal functions are in L^p_w for $0 < p < \infty$ and $\|f\|_{H^p_w} = \|f^*\|_{L^p_w}$. Since H^p_w is L^p_w for $1 < p < \infty$, we only consider the case $0 < p \leq 1$ in this section.

Regarding the constraints of weight functions, w is required to be in the Muckenhoupt A_p-class (see [4, Chapter 4] for details about A_p). For $1 < p < \infty$, a locally integrable nonnegative function w on $\mathbb{R}^n$ is said to belong to A_p if there exists $C > 0$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C,$$

where $|B|$ denotes its Lebesgue measure. For the case $p = 1$, $w \in A_1$ if there exists $C > 0$ such that, for every cube $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(x)dx \leq C \operatorname*{ess\,inf}_{x\in B} w(x).$$

It is well-known that if $w \in A_p$ for $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. We thus use $q_w = \inf\{q > 1 : w \in A_q\}$ to denote the *critical index* of w and set the weighted measure $w(E) = \int_E w(x)dx$.

A closed relation to A_p is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)^r dx\right)^{1/r} \leq C\left(\frac{1}{|B|}\int_B w(x)dx\right),$$

we say that w satisfies the *reverse Hölder condition of order r* and write $w \in RH_r$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for $s < r$. It is known that if $w \in RH_r, r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We denote by $r_w = \sup\{r > 1 : w \in RH_r\}$ the *critical index* of w for the reverse Hölder condition. For any ball B and $\lambda > 0$, we denoted by λB the ball concentric with B whose radius is λ times as long. It is known that for $w \in A_p, 1 \le p < \infty$, w satisfies the doubling condition

$$w(\lambda B) \le C\lambda^{np} w(B) \tag{2.1}$$

and

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{w(E)}{w(B)} \quad \text{for any measurable subset } E \text{ of a ball } B. \tag{2.2}$$

We recall Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces (cf. [3, 6]). Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. Use $[\cdot]$ to denote the greatest integer function. For $s \in \mathbb{Z}$ satisfying $s \ge [n(q_w/p - 1)]$, a function $a \in L^q_w$ is called a w-(p,q,s)-atom centered at x_0 if

(i) a is supported on a ball B centered at x_0,

(ii) $\|a\|_{L^q_w} \le w(B)^{1/q-1/p}$,

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha| \le s$.

Then we can characterize weighted Hardy spaces in terms of atom decomposition. A tempered distribution f is in $H^p_w(\mathbb{R}^n)$ if and only if there exist a sequence $\{a_i\}$ of w-(p,q,s)-atoms and a sequence $\{\lambda_i\}$ of scalars with $\sum|\lambda_i|^p < \infty$ such that $f = \sum \lambda_i a_i$ in the sense of distributions. Furthermore,

$$\|f\|^p_{H^p_w} \sim \inf\left\{\sum|\lambda_i|^p : \sum \lambda_i a_i \text{ is a decomposition of } f \text{ into } w\text{-}(p,q,s)\text{-atoms}\right\}.$$

Denote by $Q^{x_0}_r$ the cube centered at x_0 with side length $2r$. We now define the molecules corresponding to the atoms mentioned above.

Definition 2.1. For $0 < p \le 1 \le q \le \infty$ and $p \ne q$, let $w \in A_q$ with critical index q_w and r_w. Set $s \ge [n(q_w/p - 1)]$, $\varepsilon > \max\{sr_w(r_w-1)^{-1}n^{-1} + (r_w-1)^{-1}, 1/p - 1\}$, $a = 1 - 1/p + \varepsilon$, and $b = 1 - 1/q + \varepsilon$. A (p,q,s,ε)*-molecule centered at* x_0 *with respect to* w (or w-(p,q,s,ε)-*molecule centered at* x_0) is a function $M \in L^q_w(\mathbb{R}^n)$ satisfying

(i) $M(x) \cdot w(Q^{x_0}_{|x-x_0|})^b \in L^q_w(\mathbb{R}^n)$,

(ii) $\|M\|^{a/b}_{L^q_w} \cdot \left\|M(x) \cdot w(Q^{x_0}_{|x-x_0|})^b\right\|^{1-a/b}_{L^q_w} \equiv \mathfrak{N}_w(M) < \infty$,

(iii) $\int_{\mathbb{R}^n} M(x)x^\alpha\,dx = 0 \quad |\alpha| \le s$.

The above $\mathfrak{N}_w(M)$ is called the molecular norm of M with respect to w (or w-molecular norm of M).

We have the following molecular characterization of weighted Hardy spaces.

Theorem 2.2. [6] *Let* (p,q,s,ε) *be the quadruple in the definition of* w*-molecule, and let* $w \in A_q$. *Every* (p,q,s,ε)*-molecule* M *centered at any point with respect to* w *is in* $H^p_w(\mathbb{R}^n)$ *and* $\|M\|_{H^p_w} \le C\mathfrak{N}(M)$, *where the constant* C *is independent of* M.

If a linear mapping is bounded on L^2_w, then to prove its H^p_w-boundedness is sufficient to show that this operator maps the w-atoms into w-molecules with uniform w-molecular norms.

Theorem 2.3. [5] *Let $0 < p \leq 1$ and $w \in A_2$. For a linear operator T bounded on $L^2_w(\mathbb{R}^n)$, T can be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $H^p_w(\mathbb{R}^n)$ if and only if there exists an absolute constant C such that*

$$\|Ta\|_{H^p_w} \leq C \qquad \text{for any } w\text{-}(p,2,[n(\frac{2}{p}-1)])\text{-atom } a.$$

3 Proof of Theorem 1.2

We now are ready to prove the main theorem.

Proof of Theorem 1.2. By Theorems 2.1 and 2.3, it suffices to show that Tf is a w-$(p,2,s,\varepsilon)$-molecule satisfying

$$\mathfrak{N}_w(Tf) = \|Ta\|^{a/b}_{L^2_w(\mathbb{R}^n)} \|Tf(x)\cdot w(Q^{x_0}_{|x-x_0|})^b\|^{1-\frac{a}{b}}_{L^2_w(\mathbb{R}^n)} \leq C$$

for any w-$(p,2,s)$-atom f and $s := \left[n(\frac{2}{p}-1)\right]$. Given w-$(p,2,s)$-atom f supported on a ball B centered at x_0, let $a = 1-\frac{1}{p}+\varepsilon$ and $b = \frac{1}{2}+\varepsilon$. Then Theorem 1.1 and (2.1) yield

$$\int_{6B} |Tf(x)|^2 w(Q^{x_0}_{|x-x_0|})^{2b} w(x)dx \leq Cw(B)^{2b+1-\frac{2}{p}} = Cw(B)^{2a}. \tag{3.1}$$

On the other hand, set $2\ell = q$. Hölder's inequality and (2.1) imply

$$\begin{aligned}&\int_{|x-y|>2|x_0-y|} w(Q^{x_0}_{|x-x_0|})^{2b}|K(x,y)-K(x,x_0)|^2 w(x)dx\\ &\leq C\sum_{j=1}^{\infty} w(B_j)^{2b}\left(\int_{B_j}|K(x,y)-K(x,x_0)|^{2\ell}dx\right)^{\frac{1}{\ell}}\left(\int_{B_j} w(x)^{\ell'}dx\right)^{\frac{1}{\ell'}}.\end{aligned}$$

Since $w \in RH_{q/(q-2)}$, we obtain

$$\left(\int_{B_j} w(x)^{\ell'}dx\right)^{\frac{1}{\ell'}} \leq C|2^{j+1}B|^{-2/q} w(2^{j+1}B)$$

and $K \in K_q$ shows

$$\left(\int_{B_j}|K(x,y)-K(x,x_0)|^{2\ell}dx\right)^{\frac{1}{\ell}} \leq C(C_j)^2 w(B_j)^{-2+2/q}.$$

The condition (i) of Theorem 1.2 and (2.2) yield

$$\begin{aligned}&\int_{|x-y|>2|x_0-y|} w(Q^{x_0}_{|x-x_0|})^{2b}|K(x,y)-K(x,x_0)|^2 w(x)dx\\ &\leq C\sum_{j=1}^{\infty} w(2^{j+1}B)^{2b}(C_j)^2\frac{w(2^{j+1}B)}{|2^{j+1}B|^2}\\ &\leq C\frac{w(B)^{2+2\varepsilon}}{|B|^2}\sum_{j=1}^{\infty}(C_j)^2 2^{2n\varepsilon j} \leq C\frac{w(B)^{2+2\varepsilon}}{|B|^2}.\end{aligned}$$

By Hölder's inequality and $w \in A_1 \subset A_2$,

$$\int_B |f(y)|\,dy \le \|f\|_{L^2_w} \left(\int_B w^{-1}(x)\,dx \right)^{\frac{1}{2}} \le C\|f\|_{L^2_w} \left(\frac{|B|}{w(B)} \right)^{\frac{1}{2}} |B|^{\frac{1}{2}}.$$

Hence, we may establish the following estimates:

$$\begin{aligned}
&\int_{(6B)^c} w(Q^{x_0}_{|x-x_0|})^{2b} |Tf(x)|^2 w(x)dx \\
&\le \int_{(6B)^c} w(Q^{x_0}_{|x-x_0|})^{2b} \left(\int_B |f(y)||K(x,y) - K(x,x_0)|dy \right)^2 w(x)dx \\
&\le \left\{ \int_B |f(y)| \left(\int_{|x-y|>2|x'-y|} w(Q^{x_0}_{|x-x_0|})^{2b} |K(x,y) - K(x,x_0)|^2 w(x)dx \right)^{\frac{1}{2}} dy \right\}^2 \\
&\le \frac{w(B)^{2+2\varepsilon}}{|B|^2} \left(\int_B |f(y)|dy \right)^2 \le Cw(B)^{2a}.
\end{aligned}$$

Combining (3.1) and the above estimate,

$$\begin{aligned}
\mathfrak{N}_w(Tf) &= \|Ta\|^{a/b}_{L^2_w(\mathbb{R}^n)} \|Tf(x) \cdot w(Q^{x_0}_{|x-x_0|})^b\|^{1-\frac{a}{b}}_{L^2_w(\mathbb{R}^n)} \\
&\le C\|a\|^{a/b}_{L^2_w(\mathbb{R}^n)} w(B)^{a(1-\frac{a}{b})} \\
&\le C.
\end{aligned}$$

It follows from the assumption $T^*(x^\alpha) = 0$ for all $|\alpha| \le [n(\frac{2}{p} - 1)]$ that Tf satisfies the required moment condition. Hence the proof is completed. □

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References

[1] D.-C. Chang and M.-Y. Lee, Generalized Calderón-Zygmund operators on homogeneous groups and applications. *Appl. Anal.* **87** (2008), pp 531-554.

[2] D.-C. Chang, J.-F. Li, and J. Xiao, Weighted scale estimates for Calderón-Zygmund type operators. *Contemp. Math.* **445** (2007), pp 61-70.

[3] J. Garcia-Cuerva, Weighted H^p spaces. *Dissertations Math.* **162** (1979), pp 1-63.

[4] J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[5] Y. S. Han, M.-Y. Lee and C.-C. Lin, Atomic decomposition and boundedness of operators on weighted Hardy spaces. *Canad. Math. Bull.*, to appear.

[6] M.-Y. Lee and C.-C. Lin, The molecular characterization of weighted Hardy spaces. *J. Funct. Anal.* **188** (2002), pp 442-460.

[7] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.