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HETEROCLINIC ORBITS FOR DISCRETE HAMILTONIAN SYSTEMS

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Abstract

Recently, the existence and multiplicity results of heteroclinic orbits for a discrete pendulum equation have been investigated. In present paper, we generalize those results to a class of discrete Hamiltonian systems. Since the variational functional is identically infinite, some effective methods, provided by Rabinowitz, have to be adopted to detect critical points corresponding to heteroclinic solutions.

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1 Introduction

Difference equations have been widely used as mathematical models to describe real life situations in probability theory, matrix theory, electrical circuit analysis, combinatorial anal-

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ysis, number theory, psychology and sociology, and etc. So it is worthwhile to explore this topic. In past decades, many scholars investigate the qualitative properties of difference equations such as disconjugacy, stability, attractiveness, oscillation and boundary value problems, see for examples [1, 2, 4, 7, 12, 28]. But results on the existence and multiplicity of periodic solutions are relatively rare. In 2003, Guo and Yu [8, 9, 10] introduce the critical point theory to study discrete dynamical systems. They construct a variational framework and convert the problem of the existence of periodic solutions to the study of the existence of critical points of corresponding variational functional. Since then, the study of the existence of all kinds of solutions for difference equations attaches the attention of many researchers. For some recent results on boundary value problems, periodic solutions, homoclinic orbits, we refer the reader to [3, 5, 6, 8, 9, 10, 11, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30].

More recently, Xiao and Yu [18] investigate the existence and multiplicity of heteroclinic orbits of a discrete pendulum equation. It seems that the origin idea comes from the observing of strong similarities between the phase plane of the discrete pendulum equation and that of the classical pendulum equation. Let us recall it briefly. The phase plane portrait of the discrete pendulum equation $\triangle^2 x_{n-1} + A\sin x_n = 0,\ n \in \mathbb{Z}$ with $A = 0.1$, shown on Figure 2, can be compared with the phase plane portrait of the classical pendulum equation $x''(t) + A\sin x(t) = 0,\ t \in \mathbb{R}$ with $A = 1$, shown on Figure 1. The phase plane analysis of

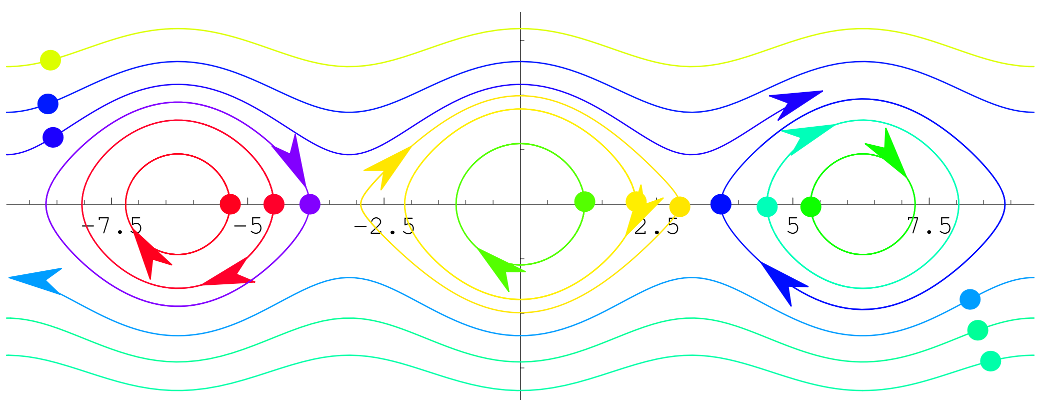

Figure 1. phase plane portrait of the classical pendulum equation

the classical pendulum equation shows the existence of two heteroclinic solutions joining $-\pi$ and π. On the other hand, the phase plane of the difference equation is similar to that of differential equation. On Figure 2, different colors are used to distinguish between different orbits. The nine ellipses represent nine periodic orbits, while two curves around the other nine ellipses are non-periodic orbits. Strong similarities observed on Figures 1 and 2 suggest the existence of heteroclinic orbits of the difference equation. Indeed, Xiao and Yu [18] prove that there exist heteroclinic orbits of the discrete pendulum equation joining equilibria. In this paper, we generalize the results to a class of discrete Hamiltonian systems.

Let us introduce some notation that will be used throughout this paper. Denote by $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ the sets of all real, integer and positive integer numbers, respectively. For $a,\ b \in \mathbb{Z}$,

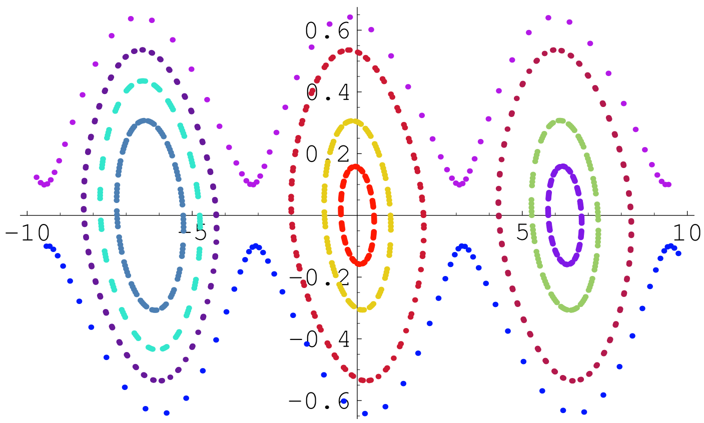

Figure 2. phase plane portrait of the discrete pendulum equation

define $Z[a] = \{a, a+1, a+2, \cdots\}$, $Z[a,\ b] = \{a, a+1, \cdots, b\}$ when $a \leq b$. For $D \subset \mathbb{R}^m, \varepsilon > 0$, denote $B_\varepsilon(D)$ as an open $\varepsilon-$neighborhood of D. For a convergent bi-infinite sequence $\{x_n\}_{n=-\infty}^{+\infty}$, denote by $x_{\pm\infty}$ the limits of the sequence as n tends to $\pm\infty$, i.e., $x_\infty := \lim\limits_{n\to\infty} x_n$ and $x_{-\infty} := \lim\limits_{n\to-\infty} x_n$.

In this paper, we investigate the following system

$$\triangle^2 x_{n-1} + V'(x_n) = 0, \qquad n \in \mathbb{Z}, \tag{1.1}$$

where $x_n \in \mathbb{R}^m$, $\triangle$ is the forward difference operator defined by $\triangle x_n = x_{n+1} - x_n$, $\triangle^2 x_n = \triangle(\triangle x_n)$, $V(z) \in C^1(\mathbb{R}^m, \mathbb{R})$, $V'(z)$ denote the gradient of V. A solution $x : \mathbb{Z} \to \mathbb{R}^m$ of (1.1) is called a heteroclinic solution (or a heteroclinic orbit) if there exist two equilibria of equation (1.1), $\xi, \eta \in \mathbb{R}^m$, $\xi \neq \eta$, such that x joins ξ and η, and the difference approaches zero as n tends to $\pm\infty$. We will study the existence and multiplicity of heteroclinic orbits of (1.1).

2 Main Results

In this section, we will state and prove the main results. We assume that
(V1) $V(z) \in C^1(\mathbb{R}^m, \mathbb{R})$,
(V2) V is periodic in z_i with period T_i, $1 \leq i \leq m$.
Then V possesses a global maximum, $\overline{V}$, on $\mathbb{R}^m$. Let $\Lambda := \{z \in \mathbb{R}^m |\ V(z) = \overline{V}\}$. Assume further that
(V3) Λ consists only isolated points.

Let c be the vector space consisting of all convergent bi-infinite sequences $x = \{x_n\}_{n=-\infty}^{\infty}$, i.e.,

$$c := \{x = \{x_n\} |\ \lim_{n\to\infty} x_n \text{ and } \lim_{n\to-\infty} x_n \text{ exist},\ x_n \in \mathbb{R}^m,\ n \in \mathbb{Z}\}.$$

We define the space H by

$$H := \{x \in c | \sum_{n=-\infty}^{\infty} |\triangle x_n|^2 < \infty\}.$$

We define a bilinear product on H as follows:

$$< x,y >:= \sum_{n=-\infty}^{\infty} \triangle x_n \cdot \triangle y_n + x_0 \cdot y_0, \ \forall\, x,y \in H, \tag{2.1}$$

where $\cdot$ denote the usual inner product on $\mathbb{R}^m$. First, we place a result of matrix analysis.

Lemma 2.1. *Let $x = (x_1, x_2, \cdots, x_m)^T,\ x^k = (x_1^k, x_2^k, \cdots, x_m^k)^T \in \mathbb{R}^m,\ k = 1,2,\cdots$. Then vector sequence $\{x^k\}_{k=1}^{\infty}$ converges to x if and only if $\lim\limits_{k\to\infty} x_i^k = x_i,\ i = 1,2,\cdots,m$.*

Proposition 2.2. *The bilinear product (2.1) is an inner product on H. The space H is a Hilbert space with the norm given below*

$$\|x\| := [\sum_{n=-\infty}^{\infty} |\triangle x_n|^2 + |x_0|^2]^{\frac{1}{2}}, \ \forall\, x \in H. \tag{2.2}$$

Proof. Recall that the space l^2 of all sequences $a = \{a_k\}_{k=-\infty}^{\infty}$ such that $a_k \in \mathbb{R}^m,\ \|a\|_2 := \left[\sum_{-\infty}^{\infty} |a_k|^2\right]^{\frac{1}{2}} < \infty$, is a Hilbert space. Let $\{y^n\}_{n=1}^{\infty} \subset H$ be a Cauchy sequence in H, i.e.

$$\forall\, \varepsilon > 0,\ \exists\, N,\ \forall\, n,n' \geq N, \quad \|y^n - y^{n'}\| = \left[\sum_{k=-\infty}^{\infty} |\triangle y_k^n - \triangle y_k^{n'}|^2 + |y_0^n - y_0^{n'}|^2\right]^{\frac{1}{2}} < \varepsilon. \tag{2.3}$$

Then $\{y_0^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^m$, while $\{\triangle y^n\}_{n=1}^{\infty},\ \triangle y^n := \{\triangle y_k^n\}_{k=-\infty}^{\infty}$, is a Cauchy sequence in l^2. By completeness of l^2, there exists a limit a in l^2 of $\{\triangle y^n\}_{n=1}^{\infty}$. One can easily observe, that there exists a unique $y^0 := \{y_k^0\}_{k=-\infty}^{\infty}$ in H such that $\lim_{n\to\infty} y_0^n = y_0^0$, and $\forall\, k \in \mathbb{Z},\ \triangle y_k^0 = a_k$. By passing to the limit as n' goes to ∞, we obtain from (2.3)

$$\forall\, \varepsilon > 0,\ \exists\, N,\ \forall\, n \geq N, \quad \|y^n - y^0\| \leq \varepsilon,$$

which proves that $\{y^n\}_{n=1}^{\infty}$ converges to y^0. Consequently, H is a Hilbert space. □

We are interested in the existence of heteroclinic solutions of (1.1) which tends to the equilibria as $n \to \pm\infty$. The variation functional associated with (1.1) defined on H is

$$J(x) = \sum_{n=-\infty}^{\infty} [\frac{1}{2}|\triangle x_n|^2 - V(x_n)]. \tag{2.4}$$

Without loss of generality, we assume that $0 \in \Lambda, V(0) = 0$. Therefore $-V(z) \geq 0$ for all $z \in \mathbb{R}^m$ and $-V(z) > 0$ if $z \notin \Lambda$. Denote $\gamma = 1/3 \min_{\xi,\ \eta\in\Lambda, \xi\neq\eta} |\xi - \eta| > 0$. Given $\xi \in \Lambda \setminus \{0\}$, $\varepsilon \in (0,\gamma)$, we define $\Gamma_\varepsilon(\xi)$ to be the set of $x \in H$ satisfying

(i) $x_{-\infty} = 0$,

(ii) $x_\infty = \xi$,

(iii) $x_n \notin B_\varepsilon(\Lambda \setminus \{0,\xi\})$ for all $n \in \mathbb{Z}$.

Obviously, $\Gamma_\varepsilon(\xi)$ is not empty for all $\xi \in \Lambda$. Define

$$c_\varepsilon(\xi) := \inf_{x\in\Gamma_\varepsilon(\xi)} J(x) \qquad \text{and} \qquad \alpha_\varepsilon := \min_{x\notin B_\varepsilon(\Lambda)} [-V(x)].$$

Remark 2.3. For any $\varepsilon > 0$, we claim that $\alpha_\varepsilon > 0$. Since V is periodic in x_i and $x \notin B_\varepsilon(\Lambda)$, then α_ε can be achieved. Of course, $\alpha_\varepsilon \neq 0$. Otherwise, there exists an element y in $\mathbb{R}^m$, which does not belong to $B_\varepsilon(\Lambda)$, such that $V(y) = 0$, which implies $y \in \Lambda$. This is a contradiction.

Now we present a useful lemma without proof. One can find details in [18].

Proposition 2.4. *Let $n \leq l \in \mathbb{Z}$. If $x \in H$ such that $x_i \notin B_\varepsilon(\Lambda)$ for all $i \in Z[n,l]$, then*

$$J(x) \geq \sqrt{2\alpha_\varepsilon}|x_{l+1} - x_n|.$$

Lemma 2.5. *If $x \in H, J(x) < \infty$, then there exist $\zeta,\ \eta \in \Lambda$ such that $x_{-\infty} = \zeta,\ x_\infty = \eta$.*

Proof. Since $\{x_n\}_{n=-\infty}^{\infty}$ is a convergent bi-infinite sequence, the existence of ζ and η is obvious. We claim that $\zeta,\eta \in \Lambda$. We only prove that $\zeta = \lim_{n\to-\infty} x_n \in \Lambda$, since the proof that $\eta = \lim_{n\to\infty} x_n \in \Lambda$ follows similarly. Suppose that there exists $\delta > 0$ such that $x_n \notin B_\delta(\Lambda)$ for all n near $-\infty$. By remark 2.3, $\alpha_\delta > 0$. Then

$$J(x) \geq \sum_{i=-\infty}^{n} [-V(x_i)] \geq \sum_{i=-\infty}^{n} \alpha_\delta$$

for any $n \in \mathbb{Z}$, which shows $J(x) = \infty$, contrary to the hypothesis. Hence $\zeta \in \Lambda$. ☐

The main idea of this paper is to find heteroclinic orbits through minimization arguments. As we can see, for any $\varepsilon > 0$, J is identically infinite on the set $H \setminus \bigcup_{\xi\in\Lambda} \Gamma_\varepsilon(\xi)$. So the critical point theory, which determines the existence of critical points by values of J on the whole space, can not be used directly to find minima. However, we can restrict J to a subset $\Gamma_\varepsilon(\xi)$ of H and get a minimum $c_\varepsilon(\xi)$ on $\Gamma_\varepsilon(\xi)$. The minimum $c_\varepsilon(\xi)$ achieves at some $x_{\varepsilon,\xi} \in \Gamma_\varepsilon(\xi)$. For a fixed ε, we will prove that there are only finite many ξ's candidate for the minimum $c_\varepsilon = \min_{\xi\in\Lambda} c_\varepsilon(\xi)$. Thus, there exists at least an element of Λ, denoted by $\xi(\varepsilon)$, such that $c_\varepsilon = c_\varepsilon(\xi(\varepsilon))$. Choosing a decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_n \to 0$ as $n \to \infty$, we will prove that the sequence $\{\xi(\varepsilon_n)\}_{n=1}^{\infty}$ is independent of ε_n, denoted by ξ_0. Thus, $c = \min_\varepsilon c_\varepsilon$ achieves at some $x \in \Gamma_\varepsilon(\xi_0)$. Finally, we will prove that x is a desired heteroclinic orbit.

Lemma 2.6. *Consider $\xi \in \Lambda \setminus \{0\}$ and let $\{x^k\}_{k=1}^{\infty}$ be a minimizing sequence for (2.4) restricted to $\Gamma_\varepsilon(\xi)$ such that for any $n \in \mathbb{N}$, $x^k \to x$ uniformly for $i \in Z[-n,n]$. If $x \in H$ and $J(x) < \infty$, then $x \in \Gamma_\varepsilon(\xi)$.*

Proof. Fix $\varepsilon \in (0,\gamma)$. By Lemma 2.5, there exist $\zeta,\eta \in \Lambda$ such that $x_{-\infty} = \zeta,\ x_\infty = \eta$. Since $x^k \to x$ uniformly for $i \in Z[-n,n]$ and $x^k \in \Gamma_\varepsilon(\xi)$, we will prove this lemma in four partions.

Claim 1: $x_n \notin B_\varepsilon(\Lambda \setminus \{0,\xi\})$ for all $n \in \mathbb{Z}$.
If there exist n_0 and $\theta \in \Lambda \setminus \{0,\xi\}$ such that $x_{n_0} \in B_\varepsilon(\theta)$, then $\delta = |x_{n_0} - \theta| < \varepsilon$. Since $x^k \to x$ uniformly for $i \in Z[-n_0, n_0]$, for sufficiently large k, we have $|x^k_{n_0} - x_{n_0}| < \varepsilon - \delta$. Thus $|x^k_{n_0} - \theta| \leq |x^k_{n_0} - x_{n_0}| + |\theta - x_{n_0}| < \varepsilon$. This is a contradiction.

Claim 2: $x_{\pm\infty} \in \{0,\xi\}$.
If $x_{-\infty} = \zeta \in \Lambda \setminus \{0,\xi\}$, then for any $0 < \varepsilon_1 < \varepsilon$, $\exists N_1 \in \mathbb{N}, x_{-n} \in B_{\varepsilon_1/2}(\zeta)$, $\forall\ n \geq N_1$. Since $x^k \to x$ uniformly for $n \in Z[-N_1, N_1]$, there exists $N_2 \in \mathbb{N}$, such that for any $k > N_2$, $|x^k_{-N_1} - x_{-N_1}| < \varepsilon_1/2$. Consequently, for these ε_1, N_1, N_2, we have $|x^k_{-N_1} - \zeta| \leq |x^k_{-N_1} - x_{-N_1}| + |x_{-N_1} - \zeta| < \varepsilon_1$. Thus $x^k_{-N_1} \in B_{\varepsilon_1}(\zeta)$, which contradicts with the definition of $\Gamma_\varepsilon(\xi)$. Thus $\zeta \in \{0,\xi\}$. The same argument guarantees $\eta \in \{0,\xi\}$.

Claim 3: $x_{-\infty} = 0$.
For each $k \in \mathbb{N}$, since $x^k \in \Gamma_\varepsilon(\xi)$, there exists $n(k) \in \mathbb{Z}$ such that $x^k_{n(k)} \notin B_\varepsilon(0)$ and $x^k_n \in B_\varepsilon(0)$ for all $n < n(k)$. For $x \in H$, putting $x_n(k) := x_{n-n(k)}$, we have $J(x(k)) = J(x)$ where $x(k) = \{x_n(k)\}$. Therefore it can be assumed that $n(k) = 0$ for all $k \in \mathbb{N}$. Consequently $x^k_n \in B_\varepsilon(0)$ and $x_n \in \overline{B}_\varepsilon(0)$, $\forall\ n < 0$. Thus $\zeta \in \overline{B}_\varepsilon(0) \cap \{0,\xi\} = \{0\}$, i.e., $\zeta = 0$.

Claim 4: $x_\infty = \xi$.
Suppose, to the contrary, that $x_\infty = 0$. We will show that, under this assumption, there will be always a contradiction with the minimizing sequence $\{x^k\}_{k=1}^\infty$. Since $x^k_0 \notin B_\varepsilon(0)$, we go on our discussion in two cases.

Case 1 If there exists a subsequence of $\{x^k\}_{k=1}^\infty$ (denoted again by $\{x^k\}_{k=1}^\infty$) such that $x^k_0 \in B_\varepsilon(\xi)$, then $|x^k_0 - x^k_{-1}| \geq \gamma/3$. Since $x_\infty = 0$, there exists $n_0 > 0$ such that $x_{n_0} \in B_{\varepsilon/6}(0)$. On one hand, since $x^k \to x$ uniformly for $i \in Z[-n_0, n_0]$, for large k, we have $x^k_{n_0} \in B_{\varepsilon/3}(0)$. On the other hand,

$$\begin{aligned} J(x^k) &= \sum_{n=-\infty}^{\infty} [\frac{1}{2}|\triangle x^k_n|^2 - V(x^k_n)] \\ &\geq \frac{1}{2}|x_0 - x_{-1}|^2 + \sum_{n=n_0}^{\infty} [\frac{1}{2}|\triangle x^k_n|^2 - V(x^k_n)] \\ &\geq \frac{1}{18}\gamma^2 + \sum_{n=n_0}^{\infty} [\frac{1}{2}|\triangle x^k_n|^2 - V(x^k_n)]. \end{aligned} \tag{2.5}$$

Define

$$y^k_n := \begin{cases} 0 & n \leq n_0 - 1 \\ x^k_n & n \geq n_0. \end{cases}$$

Then $y^k := \{y_n^k\} \in \Gamma_\varepsilon(\xi)$ and

$$\begin{aligned} J(y^k) &= \sum_{n=n_0-1}^{\infty}[\frac{1}{2}|\triangle y_n^k|^2 - V(y_n^k)] \\ &= \frac{1}{2}|x_{n_0}^k|^2 + \sum_{n=n_0}^{\infty}[\frac{1}{2}|\triangle x_n^k|^2 - V(x_n^k)] \\ &< \frac{1}{18}\varepsilon^2 + \sum_{n=n_0}^{\infty}[\frac{1}{2}|\triangle x_n^k|^2 - V(x_n^k)]. \end{aligned} \tag{2.6}$$

(2.5) and (2.6) imply that $c_\varepsilon(\xi) = \lim_{k\to\infty} J(x^k) > \lim_{k\to\infty} J(y^k) + \gamma^2/18 - \varepsilon^2/18 \geq c_\varepsilon(\xi) + (\gamma^2 - \varepsilon^2)/18$, which is a contradiction since $\gamma > \varepsilon$.

Case 2 Otherwise, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $x_0^k \notin B_\varepsilon(\Lambda)$. Since $x_{-\infty}^k = 0$, two possibilities should be considered:

Subcase I There exists a subsequence of $\{x^k\}_{k=1}^\infty$, denoted again by $\{x^k\}_{k=1}^\infty$, such that $x_{-1}^k \in B_{\varepsilon/2}(0)$. Then we have $|x_0^k - x_{-1}^k| \geq \varepsilon/2$. By using a similar argument as case I, we get a contradiction.

Subcase II There exists $M \geq K$ such that $x_{-1}^k \notin B_{\varepsilon/2}(0)$ for all $k \geq M$. Denote $n(k) = \{n_1 < 0|\ x_{n_1-1}^k \in B_{\varepsilon/2}(0),\ x_n^k \notin B_{\varepsilon/2}(0)$, for all $0 > n \geq n_1\}$. Then

$$J(x^k) \geq \sqrt{2\alpha_{\varepsilon/2}}\varepsilon/2 + \sum_{n=n(k)}^{\infty}[\frac{1}{2}|\triangle x_n^k|^2 - V(x_n^k)].$$

Set $\rho^2 = \sqrt{2\alpha_{\varepsilon/2}}\varepsilon/2$. Since $x_\infty = 0$, there exists $n_2 > 0$ such that $x_{n_2} \in B_{\rho/2}(0)$. Since $x^k \to x$ uniformly for $i \in Z[-n_2, n_2]$, for $k \geq M$ large enough, we have $x_{n_2}^k \in B_\rho(0)$. Define

$$z_n^k := \begin{cases} 0 & n < n_2 \\ x_n^k & n \geq n_2. \end{cases}$$

Then $z^k := \{z_n^k\} \in \Gamma_\varepsilon(\xi)$ and

$$\begin{aligned} J(z^k) &= \sum_{n=n_2-1}^{\infty}[\frac{1}{2}|\triangle z_n^k|^2 - V(z_n^k)] \\ &= \frac{1}{2}|x_{n_2}^k|^2 + \sum_{n=n_2}^{\infty}[\frac{1}{2}|\triangle x_n^k|^2 - V(x_n^k)] \\ &\leq \frac{1}{2}\rho^2 + J(x^k) - \rho^2 \\ &< J(x^k) - \frac{1}{2}\rho^2. \end{aligned}$$

Thus $c_\varepsilon(\xi) = \lim_{k\to\infty} J(x^k) > \lim_{k\to\infty} J(z^k) + \rho^2/2 \geq c_\varepsilon(\xi) + \rho^2/2$. Again, it is a contradiction. This completes the proof. □

Lemma 2.7. *For any* $\varepsilon \in (0,\gamma)$, $\xi \in \Lambda \setminus \{0\}$, *there exists* $x^0 = x(\varepsilon,\xi) \in \Gamma_\varepsilon(\xi)$ *such that* $J(x(\varepsilon,\xi)) = c_\varepsilon(\xi)$, *i.e.* $x(\varepsilon,\xi)$ *minimizes* $J|_{\Gamma_\varepsilon(\xi)}$.

Proof. Let $\{x^k\}_{k=1}^\infty$ be a minimizing sequence for (2.4) restricted to $\Gamma_\varepsilon(\xi)$. There exists a positive number $M > 0$ such that $M \geq J(x^k) \geq 1/2 \sum\limits_{n=-\infty}^{\infty} |\triangle x_n^k|^2$. We claim that $\{x_0^k\}_{k=1}^\infty$ is a bounded sequence. Suppose, to the contrary, that for any $j \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ such that $|x_0^{k_j}| \geq j$. Then $\lim_{j\to\infty} |x_0^{k_j}| = \infty$, which implies the existence of $j_0 \in \mathbb{N}$ such that $x_0^{k_j} \notin B_\varepsilon(\xi)$ when $j \geq j_0$. We consider the sequence $\{x_1^{k_j}\}_{j=j_0}^\infty$. If there is a subsequence of $\{x_1^{k_j}\}_{j=j_0}^\infty$ (denoted again by $\{x_1^{k_j}\}_{j=j_0}^\infty$) such that $x_1^{k_j} \in \overline{B}_\varepsilon(\xi)$, then

$$J(x^{k_j}) \geq |x_0^{k_j} - \xi - \varepsilon|^2/2 \quad \text{for all } j > j_0.$$

Letting $j \to \infty$, we have $J(x^{k_j}) \to \infty$, which contradicts with the assumptions. Otherwise, there exists $J > 0$ such that $x_1^{k_j} \notin \overline{B}_\varepsilon(\xi)$ for all $j \geq J$. Denote $n_j := \{n > 0 |\ x_l^{k_j} \notin B_\varepsilon(\xi)$ for all $l \in Z[0,n]$ and $x_{n+1}^{k_j} \in B_\varepsilon(\xi)\}$. Then

$$J(x^{k_j}) \geq \sqrt{2\alpha_\varepsilon}|x_0^{k_j} - x_{n_j}^{k_j}| + \frac{1}{2}|x_{n_j+1}^{k_j} - x_{n_j}^{k_j}|^2 \text{ for all } j > j_0. \tag{2.7}$$

Letting $j \to \infty$, we have $|x_0^{k_j} - x_{n_j+1}^{k_j}| \to \infty$. But $|x_0^{k_j} - x_{n_j+1}^{k_j}| \to \infty$ if and only if $|x_0^{k_j} - x_{n_j}^{k_j}| + |x_{n_j}^{k_j} - x_{n_j+1}^{k_j}| \to \infty$ if and only if $\sqrt{2\alpha_\varepsilon}|x_0^{k_j} - x_{n_j}^{k_j}| + |x_{n_j+1}^{k_j} - x_{n_j}^{k_j}|^2/2 \to \infty$, which contradicts again with the assumptions. Thus $\{x_0^k\}_{k=1}^\infty$ is a bounded sequence. From the definition of norm on H, we can learn that $\{x^k\}_{k=1}^\infty$ is a bounded sequence in H. Therefore passing to a subsequence if necessary, there exists $x^0 \in H$ such that x^k converges weakly to x^0 in H .

We claim that $J(x^0) < \infty$. Indeed, consider $-\infty < s < t < \infty$ and define for $x \in H$

$$J(s,t,x) = \sum_{n=s}^{t} [\frac{1}{2}|\triangle x_n|^2 + V(x_n)].$$

Since $x^k \to x^0$ weakly in H, $x_n^k \to x_n^0$ for any $n \in \mathbb{Z}$. Then $\{x_n^k\}_{n=s}^t$ converges uniformly to $\{x_n^0\}_{n=s}^t$. Clearly, $J(s,t,x)$ is lower continuous, and it must be lower semi-continuous. Combining $M \geq J(x^k) \geq J(s,t,x^k)$ with the lower semi-continuous property of $J(s,t,x)$, we have

$$J(s,t,x^0) \leq \liminf_{k\to\infty} J(s,t,x^k) \leq c_\varepsilon(\xi) = \liminf_{k\to\infty} J(x^k) \leq M. \tag{2.8}$$

Since $x^0 \in H$ and s,t are arbitrary, (2.8) implies $J(x^0) \leq \inf\limits_{x\in\Gamma_\varepsilon(\xi)} J(x)$. Lemma 2.6 implies $x^0 \in \Gamma_\varepsilon(\xi)$. Thus $J(x^0) = c_\varepsilon(\xi)$. □

Put

$$c_\varepsilon = \inf_{\xi\in\Lambda\setminus\{0\}} c_\varepsilon(\xi).$$

We will show that for a fixed $\varepsilon > 0$ there exist $\xi_0 \in \Lambda \setminus \{0\}$ and $q \in \Gamma_\varepsilon(\xi_0)$ such that $c_\varepsilon(\xi_0) = c_\varepsilon$, $J(q) = \inf\{J(x) : x \in \bigcup_{\xi\in\Lambda\setminus\{0\}} \Gamma_\varepsilon(\xi)\}$. That is, J achieves its minimum on the set $\bigcup_{\xi\in\Lambda\setminus\{0\}} \Gamma_\varepsilon(\xi)$.

Lemma 2.8. *The set* $\Upsilon_\varepsilon := \{\xi \in \Lambda \setminus \{0\} | \; c_\varepsilon(\xi) = c_\varepsilon\}$ *is finite.*

Proof. Consider $\xi \in \Lambda \setminus \{0\}$. For any $x \in \Gamma_\varepsilon(\xi)$, we have $x_{-\infty} = 0,\ x_\infty = \xi,\ x_n \notin B_\varepsilon(\Lambda \setminus \{0, \xi\})$. Put $k_1 := \max\{n | \; x_n \in B_\varepsilon(0)\},\ k_2 := \min\{n | \; n > k_1,\ x_n \in B_\varepsilon(\xi)\}$. If $k_2 > k_1 + 2$, then by Proposition 2.4

$$J(x) \geq \sum_{n=k_1}^{k_2-1} \frac{1}{2}|\triangle x_n|^2 \geq \sqrt{2\alpha_\varepsilon}|x_{k_2-1} - x_{k_1+1}| + \frac{1}{2}|\triangle x_{k_1}|^2 + \frac{1}{2}|\triangle x_{k_2-1}|^2.$$

Notice that $\xi \to \infty$ if and only if $|x_{k_2-1} - x_{k_1+1}| + |\triangle x_{k_1}| + |\triangle x_{k_2-1}| \to \infty$ which is equivalent to $\sqrt{2\alpha_\varepsilon}|x_{k_2-1} - x_{k_1+1}| + |\triangle x_{k_1}|^2/2 + |\triangle x_{k_2-1}|^2/2 \to \infty$. Thus $J(x) \to \infty$ as $\xi \to \infty$. For the case $k_1 \leq k_2 \leq k_1 + 2$, using a similar analysis as above, we get the same result. Choose $\xi_1 \in \Lambda \setminus \{0\}$. Obviously, $c_\varepsilon(\xi_1) \geq c_\varepsilon$. There exists $K_1 > 0$ such that for any $\xi \in \Lambda,\ |\xi| > K_1$, we have $\inf_{x \in \Gamma_\varepsilon(\xi)} J(x) > c_\varepsilon(\xi_1)$. Thus, by assumption (V3), there are only finite many $c_\varepsilon(\xi)$ candidate for c_ε. □

For a fixed ε, Lemma 2.8 implies that there exists $\zeta(\varepsilon) \in \Upsilon_\varepsilon$ such that $c_\varepsilon = c_\varepsilon(\zeta(\varepsilon))$. The existence of $x(\varepsilon, \zeta(\varepsilon))$ such that $c_\varepsilon = J(x(\varepsilon, \zeta(\varepsilon)))$ follows by Lemma 2.7. Now by choosing a decreasing sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that $\varepsilon_j \to 0$ as $j \to \infty$, we claim that for sufficiently large j, $\zeta(\varepsilon_j) \in \Upsilon_{\varepsilon_j}$ is independent of ε_j, i.e., we have the following:

Lemma 2.9. *Suppose that* ε_j *is a decreasing sequence of positive numbers such that* $\varepsilon_j \to 0$ *as* $j \to \infty$*. Then there exists a set* $\Upsilon \subset \Lambda \setminus \{0\}$ *such that* $\Upsilon_{\varepsilon_j} = \Upsilon$ *for sufficiently large* j*.*

Proof. Consider $\{c_{\varepsilon_j}\}_{j=1}^\infty$. For any $x \in \Gamma_{\varepsilon_j}(\eta)$, $x_n \notin B_{\varepsilon_j}(\Lambda \setminus \{0, \eta\})$, and also $x_n \notin B_{\varepsilon_{j+1}}(\Lambda \setminus \{0, \eta\})$ for all $n \in \mathbb{Z}$. Thus $x \in \Gamma_{\varepsilon_{j+1}}(\eta)$. Now consider a monotone sequence $\Gamma_{\varepsilon_1}(\eta) \subset \Gamma_{\varepsilon_2}(\eta) \subset \ldots \Gamma_{\varepsilon_j}(\eta) \subset \Gamma_{\varepsilon_{j+1}}(\eta) \subset \ldots$. By definition of $c_{\varepsilon_j}(\eta)$, we have

$$c_{\varepsilon_j}(\eta) = \inf_{x \in \Gamma_{\varepsilon_j}(\eta)} J(x) \geq \inf_{x \in \Gamma_{\varepsilon_{j+1}}(\eta)} J(x) = c_{\varepsilon_{j+1}}(\eta). \tag{2.9}$$

Then $\{c_{\varepsilon_j}\}_{j=1}^\infty$ forms a new monotonically decreasing real number sequence which is bounded both from above and from below. Using a similar argument as in Lemma 2.8, we know that $\{\zeta(\varepsilon_j)\}_{j=1}^\infty$ is bounded. Thus there is a convergent subsequence of $\{\zeta(\varepsilon_j)\}_{j=1}^\infty$. Since Λ consists of isolated points, the convergent subsequence must be a constant sequence, denoted by $\{\zeta\}_1^\infty$. Denote by Υ the set of elements of constant sequences. Then Υ is independent of ε_j. □

Theorem 2.10. *For* j *sufficiently large,* $x(\varepsilon_j, \zeta)$ *is a heteroclinic solution joining* 0 *and* ζ*.*

Proof. Denote $x(j) = x(\varepsilon_j, \zeta)$. By the definition of $\Gamma_\varepsilon(\zeta)$ and H, it is sufficient to show that for large j, $x_n(j) \notin \partial B_{\varepsilon_j}(\Lambda \setminus \{0, \zeta\})$ for all $n \in \mathbb{Z}$. If not, there exist a sequence $\{\eta_k\}_{k=1}^\infty \subset \Lambda \setminus \{0, \zeta\}$ and a sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{Z}$ such that

$$x_{n_k}(k) \in \partial B_{\varepsilon_k}(\eta_k) \text{ and } x_n(k) \notin \partial B_{\varepsilon_k}(\eta_k),\ \forall\ n < n_k.$$

Using a similar argument as in Lemma 2.8, we have $\{\eta_k\}_{k=1}^\infty$ is bounded. Passing to a subsequence, if necessary, $\{\eta_k\}_{k=1}^\infty$ must be a constant sequence, denoted by $\{\eta\}_1^\infty$. There are two cases to be considered:

Case 1 There is an increasing sequence of integers $k' \to \infty$ such that $x_n(k') \notin \overline{B}_{\varepsilon_j}(\zeta)$ for all $n < n_{k'}$.
Case 2 For every $j \in \mathbb{N}$, there is $l_k < n_k$ such that $x_{l_k}(k) \in \partial B_{\varepsilon_k}(\zeta)$.

If Case 1 occurs, we define

$$y_n(k') = \begin{cases} x_n(k') & n \le n'_k \\ \eta & n \ge n'_k + 1. \end{cases}$$

Then $y(k') := \{y_n(k')\} \in \Gamma_{\varepsilon_j}(\eta)$ and

$$\begin{aligned} J(x(k')) - J(y(k')) &= \sum_{n=n_{k'}}^{\infty} [\frac{1}{2}|\triangle x_n(k')|^2 - V(x_n(k'))] - \frac{1}{2}|\triangle y_{n_{k'}}(k')|^2 + V(y_{n_{k'}}(k')) \\ &= \sum_{n=n_{k'}}^{\infty} [\frac{1}{2}|\triangle x_n(k')|^2 - V(x_n(k'))] - \frac{1}{2}\varepsilon_j^2 + V(x_{n_{k'}}(k')). \end{aligned}$$

If there exists $n_0 > n_{k'}$ such that $x_{n_0} \notin B_\gamma(\Lambda)$, then

$$\begin{aligned} J(x(k')) - J(y(k')) &\ge V(x_{n_0}(k')) - \frac{1}{2}\varepsilon_j^2 + V(x_{n_{k'}}(k')) \\ &\ge \alpha_\gamma - \frac{1}{2}\varepsilon_j^2 + V(x_{n_{k'}}(k')). \end{aligned} \tag{2.10}$$

Letting $j \to \infty$, the second and third item of (2.10) approach 0. Hence for large j, $c_{\varepsilon_j} = J(x(k')) > J(y(k'))$. This is a contradiction. Otherwise, there exist two adjacent points such that the distance between them is larger than γ. Then

$$J(x(k')) - J(y(k')) \ge \sum_{n=n_k}^{\infty} \frac{1}{2}|\triangle x_n(k')|^2 - \frac{1}{2}\varepsilon_j^2 + V(x_{n_{k'}}(k')) > \gamma^2/2 - \frac{1}{2}\varepsilon_j^2 + V(x_{n_{k'}}(k')).$$

We get a contradiction by using an analogous discussion as above.
If Case 2 occurs, define

$$z_n(k) := \begin{cases} x_n(k) & n \le l_k \\ \zeta & n \ge l_k + 1. \end{cases}$$

Then $z(k) := \{z_n(k)\} \in \Gamma_{\varepsilon_j}(\zeta)$ and

$$\begin{aligned} J(x(k)) - J(z(k)) &= \sum_{n=l_k}^{\infty} [\frac{1}{2}|\triangle x_n(k)|^2 - V(x_n(k))] - \frac{1}{2}|\triangle z_{l_k}(k)|^2 + V(z_{l_k}(k)) \\ &= \sum_{n=l_k}^{\infty} [\frac{1}{2}|\triangle x_n(k)|^2 - V(x_n(k))] - \frac{1}{2}{\varepsilon_j}^2 + V(x_{l_k}(k)). \end{aligned}$$

Using a similar argument as in case 1, we get a contradiction and finish our proof. □

Theorem 2.11. *Suppose that V satisfies (V1-V3). For each $\beta \in \Lambda$, there exist two heteroclinic solutions connecting β to $\Lambda \setminus \{\beta\}$, one of which originates at β and one of which terminates at β.*

Proof. Without loss of generality, we only need to check heteroclinic orbits connecting 0 to $\Lambda\setminus\{0\}$. In Theorem 2.10, we have proved that there exists a heteroclinic solution, denote by $\{x_n\}_{n=-\infty}^{\infty}$, connecting 0 and ζ, which originates at 0. Then $\{x_{-n}\}_{n=-\infty}^{\infty}$ is also a heterclinic orbit joining ζ and 0, which terminates at 0. This completes the proof. □

Next the multiplicity of heteroclinic orbits will be studied in the simplest possible setting. Suppose that V satisfies
(V4) Λ/T^m is a singleton.
By (V4), Λ consists only of the translates of a single point. Without loss of generality, we can take the single point to be 0. Denote Ξ as the set of ξ such that for $\varepsilon\in(0,\gamma)$, $c_\varepsilon(\xi)$ corresponds to a connecting orbits of (1.1) joining 0 and ξ. Let Ψ denote the set of finite linear combinations over $\mathbb{Z}$ of elements of Ξ.

Lemma 2.12. $\Psi=\Lambda$.

Proof. If not, denote $\Omega=\Lambda\setminus\Psi\neq\emptyset$. Using a similar argument as in the proof of Theorem 2.10, we shows that for each $\varepsilon\in(0,\gamma)$, there exists $\xi_\varepsilon\in\Omega$ such that

$$c_\varepsilon(\xi_\varepsilon)=\inf_{\zeta\in\Omega}c_\varepsilon(\zeta).$$

And there exists $x(\varepsilon)=x(\varepsilon,\xi_\varepsilon)\in\Gamma_\varepsilon(\xi_\varepsilon)$ such that $J(x(\varepsilon))=c_\varepsilon(\xi_\varepsilon)$. We claim that for ε sufficiently small,

$$x_n(\varepsilon)\notin\partial B_\varepsilon(\Lambda\setminus\{0,\xi_\varepsilon\})\qquad\text{for all } n\in\mathbb{Z}. \tag{2.11}$$

And therefore $x(\varepsilon)$ is a heteroclinic orbit of (1.1) joining 0 and ξ_ε. Then $\xi_\varepsilon\in\Xi$, which is contrary to the choice of $\xi_\varepsilon\in\Omega$. Thus $\Psi=\Lambda$.

Assume to the contrary of (2.11), that there exist $\eta_\varepsilon\in\Lambda\setminus\{0,\xi_\varepsilon\}$ and $n_\varepsilon\in\mathbb{Z}$ such that $x_{n_\varepsilon}(\varepsilon)\in\partial B_\varepsilon(\eta_\varepsilon)$. Two cases need to be considered:
(a) $\eta_\varepsilon\in\Omega$,
(b) $\xi_\varepsilon-\eta_\varepsilon\in\Omega$.
No other case will happen. Indeed, if $\eta_\varepsilon\in\Lambda,\xi_\varepsilon-\eta_\varepsilon\in\Lambda$, then $\xi_\varepsilon=\xi_\varepsilon-\eta_\varepsilon+\eta_\varepsilon\in\Lambda$, a contradiction.

If case (a) happens, two further possibilities arise:
(i) $x_n(\varepsilon)\notin\overline{B}_\varepsilon(\xi_\varepsilon)$ for $n<n_\varepsilon$,
(ii) there exists $n'_\varepsilon<n_\varepsilon$ such that $x_{n'_\varepsilon}(\varepsilon)\in\partial B_\varepsilon(\xi_\varepsilon)$.

For case (a) (i) occurs, define

$$y_n(\varepsilon)=\begin{cases} x_n(\varepsilon) & n\le n_\varepsilon\\ \eta_\varepsilon & n\ge n_\varepsilon+1.\end{cases}$$

Then $y\in\Gamma_\varepsilon(\eta_\varepsilon)$ and

$$\begin{aligned} J(y(\varepsilon))-J(x(\varepsilon)) &= \frac{1}{2}|\triangle y_{n_\varepsilon}(\varepsilon)|^2-\frac{1}{2}|\triangle x_{n_\varepsilon}(\varepsilon)|^2-\sum_{n=n_\varepsilon+1}^{\infty}[\frac{1}{2}|\triangle x_n(\varepsilon)|^2-V(x_n(\varepsilon))]\\ &=\frac{1}{2}\varepsilon^2-\frac{1}{2}|\triangle x_{n_\varepsilon}(\varepsilon)|^2-\sum_{n=n_\varepsilon+1}^{\infty}[\frac{1}{2}|\triangle x_n(\varepsilon)|^2-V(x_n(\varepsilon))].\end{aligned}$$

Using a similar argument as in Theorem 2.10 case I, we conclude $J(y(\varepsilon)) < J(x(\varepsilon))$ for small ε. Consequently, $c_\varepsilon(\eta_\varepsilon) < c_\varepsilon(\xi_\varepsilon)$, which is a contradiction. If case (a) (ii) occurs, using a similar argument as in case (a) (i), we shows this case is impossible. Next, if case (b) occurs, two further possibilities will also be met:
(iii) $x_n(\varepsilon) \notin \overline{B}_\varepsilon(0)$ for all $n \geq n_\varepsilon$,
(iv) there exists $n'_\varepsilon > n_\varepsilon$ such that $x_{n'_\varepsilon}(\varepsilon) \in \partial B_\varepsilon(0)$.

For case (b) (iii) occurs, define

$$z_n(\varepsilon) = \begin{cases} 0 & n \leq n_\varepsilon - 1 \\ x_n(\varepsilon) - \eta_\varepsilon & n \geq n_\varepsilon. \end{cases}$$

Then $z := \{z_n(\varepsilon)\} \in \Gamma_\varepsilon(\xi_\varepsilon - \eta_\varepsilon)$ and

$$\begin{aligned} J(z(\varepsilon)) - J(x(\varepsilon)) &= \frac{1}{2}|\triangle z_{n_\varepsilon-1}(\varepsilon)|^2 - \sum_{n=-\infty}^{n_\varepsilon-2}[\frac{1}{2}|\triangle x_n(\varepsilon)|^2 - V(x_n(\varepsilon))] - \frac{1}{2}|\triangle x_{n_\varepsilon-1}(\varepsilon)|^2 + V(x_{n_\varepsilon-1}(\varepsilon)) \\ &= \frac{1}{2}\varepsilon^2 - \frac{1}{2}|\triangle x_{n_\varepsilon-1}(\varepsilon)|^2 + V(x_{n_\varepsilon-1}(\varepsilon)) - \sum_{n=-\infty}^{n_\varepsilon-2}[\frac{1}{2}|\triangle x_n(\varepsilon)|^2 - V(x_n(\varepsilon))]. \end{aligned}$$

Using a similar argument as in case (a) (i), we get $c_\varepsilon(\xi_\varepsilon - \eta_\varepsilon) < c_\varepsilon(\xi_\varepsilon)$, a contradiction. A simple comparison argument shows that if case (b) (iv) occurs, $x(\varepsilon)$ would not minimize J on $\Gamma_\varepsilon(\xi_\varepsilon)$, which finishes our proof. □

Theorem 2.13. *If V satisfies (V1), (V2) and (V4), for any $\beta \in \Lambda$, (1.1) has at least $4m$ heteroclinic orbits joining β to $\Lambda \setminus \{\beta\}$, $2m$ of which originate at β and $2m$ of which terminate at β.*

Proof. Without loss of generality, we can assume $\beta = 0$. Since $\Psi = \Lambda$, there are at least m distinct heteroclinic orbits of (1.1) emanating from 0. If $\{x_n\}_{n=-\infty}^{\infty}$ is a heteroclinic orbit joining 0 to $\zeta \in \Lambda \setminus \{0\}$, then $\{x_{-n}\}_{n=-\infty}^{\infty}$ is also a heteroclinic orbit joining ζ to 0. And $\{x_n - \zeta\}_{n=-\infty}^{\infty}$, $\{x_{-n} - \zeta\}_{n=-\infty}^{\infty}$ are also two heteroclinic solutions joining ζ and 0. Among these four heteroclinic orbits, $\{x_n\}_{n=-\infty}^{\infty}$ and $\{x_{-n} - \zeta\}_{n=-\infty}^{\infty}$ emanate from 0, $\{x_{-n}\}_{n=-\infty}^{\infty}$ and $\{x_n - \zeta\}_{n=-\infty}^{\infty}$ terminate at 0. This completes our proof. □

Now we give a example to illustrate our results. Let $z = (x,y)^T$ be an element of $\mathbb{R}^2$. Consider difference equations on plane. We choose potential function V as below:

$$V(z) = \sin x \cos y.$$

Thus $V \in C^1(\mathbb{R}^2, \mathbb{R})$ and V is 2π periodic in x and y. The gradient of V gives by $V'(z) = (\cos x \cos y,\ -\sin x \sin y)^T$. One can easily compute that $\Lambda = \{(x,y) |\ x = 2k\pi + \pi/2,\ y = 2l\pi,\ k,\ l \in \mathbb{Z}\}$.

Consider the following difference equations:

$$\begin{cases} \triangle^2 x_{n-1} + \cos x_n \cos y_n = 0 \\ \triangle^2 y_{n-1} + \sin x_n \sin y_n = 0. \end{cases} \tag{2.12}$$

Since V satisfies (V1-V3), Theorem 2.11 implies that there exist two heteroclinic orbits of (2.12). Further, $\Lambda/T^2 = \{(\pi/2, 0)\}$ is a singleton. Theorem 2.13 can be applied. Thus, there exist 8 heteroclinic orbits of (2.12), four of which originate at $(\pi/2, 0)$, four of which terminate at $(\pi/2, 0)$.

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References

[1] C. D. Ahlbrandt, Equivalence of discrete Euler equations and discrete Hamiltonian systems. *J. Math. Anal. Appl.* **180** (1993), pp 498-517.

[2] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions. *J. Math. Anal. Appl.* **199** (1996), pp 804-826.

[3] P. Chen and H. Fang, Existence of periodic and subharmonic solution for second-order p-Laplacian difference equations. *Adva. Diff. Equa.* **2007** (2007), pp 1-9.

[4] S. Z. Chen, Disconjugacy, disfocality, and oscillation of second order difference equations. *J. Diff. Equat.* **107** (1994), pp 383-394.

[5] X. Q. Deng, Periodic Solutions for Subquadratic Discrete Hamiltonian Systems. *Adva. Diff. Equa.* **2007** (2007), pp 1-16.

[6] X. Q. Deng and G. Cheng, Homoclinic Orbits for Second Order Discrete Hamiltonian Systems with Potential Changing Sign. *Acta Appl. Math.* **103** (2008), pp 301-314.

[7] L. H. Erbe and P. X. Yan, Disconjugacy for linear Hamiltonian difference systems. *J. Math. Anal. Appl.* **167** (1992), pp 355-367.

[8] Z. M. Guo and J. S. Yu, Existence of periodic and subharmonic solutions for second-order superlinear difference equations. *Sci.China (Ser A)* **46** (2003), pp 506-515.

[9] Z. M. Guo and J. S. Yu, Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems.*Nonlinear Anal.* **55** (2003), pp 969-983.

[10] Z. M. Guo and J. S. Yu, The existence of periodic and subharmonic solutions for solutions of subquadratic second order difference equations. *J. London Math. Soc.* **68** (2003), pp 419-430.

[11] Z. M. Guo and J. S. Yu, Multiplicity results for periodic solutions to second-order difference euqations. *J. Dyna. Diff. Equa.* **18** (2006), pp 943-960.

[12] P. Hartman, Difference equations: disconjugacy, principal solutions, Greens functions complete monotonicity. *Trans. Am. Math. Soc.* **246** (1978), pp 1-30.

[13] Y. H. Long, Applications of Clark duality to periodic solutions with minimal period for discrete Hamiltonian systems. *J. Math. Anal. Appl.* **342** (2008), pp 726-741.

[14] M. J. Ma and Z. M. Guo, Homoclinic orbits for second order self-adjoint difference equations. *J. Math. Anal. Appl.* **323** (2006), pp 513-521.

[15] M. J. Ma and Z. M. Guo, Homoclinic orbits and subharmonics for nonlinear second order difference equations. *Nonlinear Anal.* **67** (2007), pp 1737-1745.

[16] R. H. Rabinowitz, Periodic and heteroclinic orbits for a periodic Hamiltonian system. *Nonlinear Anal.* **6** (1989), pp 331-346.

[17] H. P. Shi, Homoclinic orbits of nonlinear functional difference equations. *Acta. Appl. Math.* **106** (2009), pp 135-147.

[18] H. F. Xiao and J. S. Yu, Existence of Heteroclinic Orbits for Discrete Pendulum Equation. *J. Diff. Equat. Appl.* Accept.

[19] Y. F. Xue and C. L. Tang, Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system. *Nonlinear Anal.* **67** (2007), pp 2072-2080.

[20] Y. F. Xue and C. L. Tang, Multiple periodic solutions for superquadratic second-order discrete Hamiltonian systems. *Appl. Math. Comp.* **196** (2008), pp 494-500.

[21] J. S. Yu, H. H. Bin and Z. M. Guo,Multiple periodic solutions for discrete Hamiltonian systems. *Nonlinear Anal.* **66** (2007), pp 1498-1512.

[22] J. S. Yu, X. Q. Deng and Z. M. Guo, Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential. *J. Math. Anal. Appl.* **324** (2006), pp 1140-1151.

[23] J. S. Yu and Z. M. Guo, Boundary value problems of discrete generalized Emden-Foeler equation. *Sci. China Ser. A* **49** (2006), pp 1303-1314.

[24] J. S. Yu and Z. M. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation. *SJ. Diff. Equat.* **231** (2006), pp 18-31.

[25] J. S. Yu, Y. H. Long and Z. M. Guo, Subharmonic solutions with prescribed minimal period of a class of a discrete forced pendulum equation. *J. Dyna. diff. Equat.* **16** (2004), pp 575-586.

[26] J. S. Yu, H. P. Shi and Z. M. Guo, Homoclinic orbits for nonlinear difference equations containing both advance and retardation. *J. Math. Anal. Appl.* **352** (2009), pp 799-806.

[27] J. S. Yu, B. S. Zhu and Z. M. Guo, Positive solutions for multiparameter semipositone discrete boundary value problems via variational method. *Adv. Difference Equ.* **2008** (2008), pp 1-15.

[28] B. G. Zhang and G. D. Chen, Oscillation of certain second order nonlinear difference equations. *J. Math. Anal. Appl.* **199** (1996), pp 827-841.

[29] B. Zheng, Multiple periodic solutions to nonlinear discrete Hamiltonian systems, *Adva. Diff. Equa.* **2007** (2007), pp 1-13.

[30] Z. Zhou, J. S. Yu and Z. M. Guo, The existence of periodic and subharmonic solutions to subquadratic discrete Hamiltonian systems. *The Australian & Zealand Industrial and Applied Mathematics Journal* **47** (2005), pp 89-102.