Communications in Mathematical Analysis
Special Volume in Honor of Prof. Peter Lax
Volume 8, pp. 16–21 (2010)
ISSN 1938-9787
www.commun-math-anal.org

A Characterization of Inner Product Spaces Concerning an Euler–Lagrange Identity

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(Communicated by S. S. Dragomir)

Abstract

In this paper we present a new criterion on characterization of real inner product spaces concerning the Euler–Lagrange type identity

$$\|r_2x_1 - r_1x_2\|^2 + \|r_1x_1 + r_2x_2\|^2 = (r_1^2 + r_2^2)(\|x_1\|^2 + \|x_2\|^2)\,.$$

AMS Subject Classification: 46C015, 46B20, 46C05.

Keywords: inner product space; Day's condition; normed space; characterization of inner product spaces; operator.

1 Introduction

In 1932, the notion of (complete) normed linear space was introduced by S. Banach [6]. Then P. Jordan and J. von Neumann [12] showed that a normed linear space V is an inner product space if and only if the parallelogram equality $\|x-y\|^2 + \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$

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holds for all x and y. Later M.M. Day [9] showed that a normed linear space V is an inner product space if we require only that the parallelogram equality holds for x and y on the unit sphere. In other words, M. M. Day showed that the parallelogram equality may be replaced by the condition $R_2 = 4$ ($\|x\| = 1$, $\|y\| = 1$), where $R_2 = \|x-y\|^2 + \|x+y\|^2$. Over the years, interesting characterizations of inner product spaces have been introduced or developed by numerous mathematicians. Among many significant characterizations for a normed space $V, \|.\|)$ to be inner product we mentioned the following items for instance, see [1, 2, 3, 4, 8, 10, 11, 13, 17, 19, 23] and references therein for more information.

(i) For all $x,y \in V$, $\|x+y\|^2 + \|x-y\|^2 \sim 2(\|x\|^2 + \|y\|^2)$, where $\sim$ is (consistently) one of the relations $\leq, =$ or $\geq$; [22].

(ii) Each Diminnie orthogonally additive functional is additive; [21].

(iii) $x,y \in V, \|x\| = \|y\| = 1$ and $x \perp y$ imply $\|\lambda x + y\| = \|x - \lambda y\|$; [24].

(iv) For fixed $n \in \mathbb{N}$, $n \geq 2$,

$$\sum_{i=1}^{n} \left\| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2$$

for all $x_1, \cdots, x_n \in V$; [15, 20].

(v) For x,y in V and α, β in $\mathbb{R}$ different from 1 (α,β)-orthogonality is either homogeneous or both right and left additive, where , x is said to be (α,β)-orthogonal to y if $\|x-y\|^2 + \|\alpha x - \beta y\|^2 = \|x - \beta y\|^2 + \|y - \alpha x\|^2$; [5].

(vi) For each $x,y \in V$ with $\|x\| = \|y\| = 1$,

$$\inf\{\|tx + (1-t)y\| : t \in [0,1]\} = 2^{-1/2} \Rightarrow x \perp y,$$

where $x \perp y$ means that x is Birkhoff–orthogonal to y, i.e. $\|x\| \leq \|x + \lambda y\|$, $\lambda \in \mathbb{R}$; [7].

In this paper we present a new criterion on characterization of inner product spaces concerning the Euler–Lagrange type identity (see [14])

$$\|r_2 x_1 - r_1 x_2\|^2 + \|r_1 x_1 + r_2 x_2\|^2 = (r_1^2 + r_2^2)(\|x_1\|^2 + \|x_2\|^2).$$

Our result extends that of J.M. Rassias [18].

2 Main Results

We now state our main result.

Theorem 2.1. *Let* $(\mathscr{X}, \|\cdot\|)$ *be a real normed space,* n *be a positive real number and* $r = (r_1, r_2)$ *be a pair of nonnegative real numbers. If*

$$R_{r,n} = \|r_2 x_1 - r_1 x_2\|^n + \|r_1 x_1 + r_2 x_2\|^n,$$

$A_{r,n} = (r_1\|x_1\| + r_2\|x_2\|)^n + \max\{(r_2\|x_1\| - r_1\|x_2\|)^n, (r_1\|x_1\| - r_2\|x_2\|)^n\}$,
and
$B_{r,n} = (r_1\|x_1\| + r_2\|x_2\|)^n + \min\{(r_2\|x_1\| - r_1\|x_2\|)^n, (r_1\|x_1\| - r_2\|x_2\|)^n\}$.
Then a necessary and sufficient condition for that the norm $\|\cdot\|$ *over* $\mathscr{X}$ *is induced by an inner product is that*

(I) $R_{r,n} \le A_{r,n}$ for $n \ge 2$
and
(II) $R_{r,n} \ge B_{r,n}$ for $0 < n \le 2$
for any $x_1, x_2 \in \mathscr{X}$.

Proof. The case $r_1 = r_2$ is known; cf. [18], so let us assume that $r_1 \neq r_2$.
Necessity.
Assume that the norm $\|\cdot\|$ on $\mathscr{X}$ is induced by an inner product $\langle\cdot,\cdot\rangle$. Hence $\|x\|^2 = \langle x, x\rangle$ $(x \in \mathscr{X})$. We have

$$\begin{aligned} R_{r,n} &= \|r_2x_1 - r_1x_2\|^n + \|r_1x_1 + r_2x_2\|^n \\ &= (\|r_2x_1 - r_1x_2\|^2)^{\frac{n}{2}} + (\|r_1x_1 + r_2x_2\|^2)^{\frac{n}{2}} \\ &= (a_1 - b\cos p)^{n/2} + (a_2 + b\cos p)^{n/2} \\ &= R_{r,n}(p)\,, \end{aligned}$$

where $a_1 := r_2^2\|x_1\|^2 + r_1^2\|x_2\|^2$, $a_2 := r_1^2\|x_1\|^2 + r_2^2\|x_2\|^2$, $b := 2r_1r_2\|x_1\|\,\|x_2\|$ and p is defined in such a way that $\langle x_1, x_2\rangle = \|x_1\|\,\|x_2\|\cos p$. Note that $\|x_1\| \le \|x_2\|$ if and only if $a_1 \le a_2$. By the first differentiation we find

$$\begin{aligned} R'_{r,n}(p) = \frac{n}{2}&[(a_1 - b\cos p)^{\frac{n}{2}-1} \\ &-(a_2 + b\cos p)^{\frac{n}{2}-1}]b\sin p\,. \end{aligned}$$

Therefore the critical values of $R_{r,n}$, being the roots of $R'_{r,n}(p) = 0$, are $p = k\pi$ $(k = 0, \pm1, \pm2, \cdots)$. By the second differentiation we get

$$R''_{r,n}(p) = \tfrac{n}{2}[(a_1 - b\cos p)^{\frac{n}{2}-1} - (a_2 + b\cos p)^{\frac{n}{2}-1}]b\cos p + \tfrac{n(n-2)}{4}[(a_1 - b\cos p)^{\frac{n}{2}-2} + (a_2 + b\cos p)^{\frac{n}{2}-2}]b^2\sin^2 p\,.$$

If $p = 2k\pi$, then

$$R''_{r,n}(2k\pi) = \frac{n}{2}\left[(a_1 - b)^{\frac{n}{2}-1} - (a_2 + b)^{\frac{n}{2}-1}\right]b$$

$$= \begin{cases} < 0 & a_1 \ge a_2,\ n > 2,\ b > \frac{a_1 - a_2}{2} \\ < 0 & a_1 \ge a_2,\ 0 < n < 2,\ 0 < b < \frac{a_1 - a_2}{2} \\ < 0 & a_1 \le a_2,\ n > 2,\ 0 < b \\ > 0 & a_1 \ge a_2,\ n > 2,\ 0 < b < \frac{a_1 - a_2}{2} \\ > 0 & a_1 \ge a_2,\ 0 < n < 2,\ b > \frac{a_1 - a_2}{2} \\ > 0 & a_1 \le a_2,\ 0 < n < 2,\ 0 < b \end{cases}$$

If $p = (2k+1)\pi$, then

$$R''_{r,n}((2k+1)\pi) = \frac{n}{2}\left[(a_2 - b)^{\frac{n}{2}-1} - (a_1 + b)^{\frac{n}{2}-1}\right]b$$

$$= \begin{cases} < 0 & a_1 \le a_2,\ n > 2,\ b > \frac{a_2 - a_1}{2} \\ < 0 & a_1 \le a_2,\ 0 < n < 2,\ 0 < b < \frac{a_2 - a_1}{2} \\ < 0 & a_1 \ge a_2,\ n > 2,\ 0 < b \\ > 0 & a_1 \le a_2,\ 0 < n < 2,\ b > \frac{a_2 - a_1}{2} \\ > 0 & a_1 \le a_2,\ n > 2,\ 0 < b < \frac{a_2 - a_1}{2} \\ < 0 & a_1 \ge a_2,\ 0 < n < 2,\ 0 < b \end{cases}$$

For $n > 2$, by utilizing the second differentiation test, we infer that

$$\begin{aligned} & A_{r,n}(2k\pi) \\ = \; & (r_1\|x_1\| + r_2\|x_2\|)^n + \\ & \max\{(r_2\|x_1\| - r_1\|x_2\|)^n, (r_1\|x_1\| - r_2\|x_2\|)^n\} \\ = \; & (a_2+b)^{\frac{n}{2}} + \max\{\left\{(a_1-b)^{\frac{n}{2}}, (a_2-b)^{\frac{n}{2}}\right\} \\ = \; & \max\{R_{r,n}(2k\pi), R_{r,n}((2k+1)\pi)\} \\ = \; & \max R_{r,n}(p) \end{aligned}$$

which yields (I). For $0 < n < 2$, by applying the second differentiation test, we deduce that

$$\begin{aligned} & B_{r,n}(2k\pi) \\ = \; & (r_1\|x_1\| + r_2\|x_2\|)^n + \\ & \min\{(r_2\|x_1\| - r_1\|x_2\|)^n, (r_1\|x_1\| - r_2\|x_2\|)^n\} \\ = \; & (a_2+b)^{\frac{n}{2}} + \min\{\left\{(a_1-b)^{\frac{n}{2}}, (a_2-b)^{\frac{n}{2}}\right\} \\ = \; & \min\{R_{r,n}(2k\pi), R_{r,n}((2k+1)\pi)\} \\ = \; & \min R_{r,n}(p) \end{aligned}$$

which yields (II).

Sufficiency.

Assume that condition (I) to be held. The continuity of the function $n \mapsto \|\cdot\|^n$ implies that

$$R_{r,2} \leq A_{r,2} = 2(r_1^2 + 2r_2^2)$$

for $\|x_1\| = \|x_2\| = 1$. From the pertinent sufficient condition of M.M. Day, it can be proved the following criterion:

"The necessary and sufficient condition for a norm defined over a vector space $\mathscr{X}$ to spring from an inner product is that $R_{r,2} \leq 2(r_1^2 + 2r_2^2)$ where r_1, r_2 are positive numbers and $\|x_1\| = \|x_2\| = 1$". Due to the fact that this condition holds, we conclude that the norm $\|\cdot\|$ on $\mathscr{X}$ can be deduced from an inner product. Similarly, if condition (II) holds, then we get

$$R_{r,2} \geq A_{r,2} = 2(r_1^2 + 2r_2^2)$$

for $\|x_1\| = \|x_2\| = 1$. Applying the same statement as the above criterion except that $R_{r,2} \geq 2(r_1^2 + 2r_2^2)$, we conclude that the norm $\|\cdot\|$ can be deduced from an inner product. $\square$

Corollary 2.2. *A normed space $(\mathscr{X}, \|\cdot\|)$ is an inner product space if and only if*

$$\|r_2x_1 - r_1x_2\|^2 + \|r_1x_1 + r_2x_2\|^2 = (r_1^2 + r_2^2)(\|x_1\|^2 + \|x_2\|^2)$$

for any non-negative real numbers r_1, r_2 and any $x_1, x_2 \in \mathscr{X}$.

We can have an operator version of Corollary above. In fact a straightforward computation shows that

Corollary 2.3. *Let T_1, T_2 be bounded linear operators acting on a Hilbert space and r_1, r_2 be real numbers. Then*

$$|r_2T_1 - r_1T_2|^2 + |r_1T_1 + r_2T_2|^2 = (r_1^2 + r_2^2)(|T_1|^2 + |T_2|^2),$$

where $|T| = (T^*T)^{1/2}$ denotes the absolute value of T.

Acknowledgments

The first author was supported by a grant from Ferdowsi University of Mashhad; (NO. MP88061MOS).

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