Communications in **M**athematical **A**nalysis
Special Volume in Honor of Prof. Peter D. Lax
Volume 8, Number 2, pp. 5–15 (2010)
ISSN 1938-9787
www.commun-math-anal.org

SOME COMMENTS ON ALMOST PERIODICITY AND RELATED TOPICS

C. CORDUNEANU*
The University of Texas
Arlington, TX
The Romanian Academy, Bucharest

(Communicated by Toka Diagana)

1 Introduction

The concept of almost periodicity has acquired widespread diffusion in contemporary research, the interest for it reaching more and more fields of investigation, in both pure and applied mathematics.

This paper contains a few comments and results, related mainly to the classical aspects of the theory, but also with incursions in the recent applications and newly generated concepts, such as spaces of almost periodic functions or related concepts like pseudo-almost periodic functions.

A growing number of authors have brought remarkable contributions related to almost periodicity and its applications, and we indicate here a few classical references, other pertaining particularly to authors who relatively recently have published books, survey papers or extended articles treating various aspects concerning the almost periodicity and its related fields.

For general references, see the books/monographs by H. Bohr [4], A.S. Besicovitch [3], J. Favard [13], B.M. Levitan and V.V. Zhikov [17], L. Amerio and G. Prouse [1], C. Corduneanu [5], [11], A.M. Fink [14], S. Zaidman [19], Ch. Zhang [20], and the vast literature therein. Survey papers are numerous, and we send the reader to a recent one by A. Andres et al. [2].

The applications of almost periodicity are numerous, usually scattered in various books and monographs, not to mention those in journals (mathematical, science, engineering). We shall quote here the books by M.A. Krasnoselskii et al. [16], by Y. Hino et al.[15], and C. Corduneanu [11].

It is very significant, for applications to real phenomena, the fact that properties of almost periodic functions are naturally extended to richer classes of functions, such as the

*E-mail address: concord@uta.edu

pseudo almost periodic functions. These functions, introduced and studied by Ch. Zhang [20], and other authors, appear as perturbations of classical almost periodic functions, which enables us to conclude that their behavior is ”mimicking” that of the latter.

2 A theorem “in the first approximation”

Let us begin with a result we have recently established, and show how it can be generalized to other function spaces, whose relationship with the space $AP(R,R^n)$ is rather close.

In our paper [10], the following theorem has been proven, and a few applications have been indicated.

The result concerns functional equations of the form

$$\dot{x}(t) = (Lx)(t) + (fx)(t),\ t \in R, \tag{1}$$

where L is a linear operator on $AP(R,R^n)$, while f is acting on the same space, but is – generally – nonlinear. The equation (1) is regarded as a perturbation of the linear equation

$$\dot{x}(t) = (Lx)(t) + f(t),\ t \in R, \tag{2}$$

with the operator L linear.

Since we have in mind solutions in $AP(R,R^n)$, we do not associate with either equation (1) or (2) an initial condition.

Theorem 1. *Consider equation* (1)*, under the following assumptions:*

1. $L : AP(R,R^n) \to AP(R,R^n)$ *in a linear continuous operator, for fixed* $n \geq 1$.
2. *Equation* (2) *has the property that for any* $f \in AP(R,R^n)$*, there exists a unique solution* x *in* $AP(R,R^n)$.
3. $f : AP(R,R^n) \to AP(R,R^n)$ *is an operator generally nonlinear, satisfying a global Lipschitz condition*
$$|fx - fy|_{AP} \leq L_0 |x-y|_{AP}, \tag{3}$$
on the whole space $AP(R,R^n)$*, with* L_0 *a constant.*

Then, equation (1) *has a unique solution* $x \in AP(R,R^n)$*, provided* L_0 *is small enough.*

The proof has been conducted by the Banach fixed point theorem, without any difficulty after showing the continuity of the map $f \to x$, where x is the unique solution in $AP(R,R^n)$ of equation (2).

The following notations are used above and in the sequel:

- $AP(R,R^n)$ is the space of almost periodic functions in the sense of Bohr.
- $BC(R,R^n)$ will denote the space of all continuous and bounded maps from R into R^n, with the supremum norm:
$$\|x\| = \sup\{|x(t)|;\ t \in R\}, \tag{4}$$
where $|\cdot|$ stands for the Euclidean norm in R^n.

- $BUC(R,R^n)$ will designate the subspace of $BC(R,R^n)$, consisting of those elements which are uniformly continuous on R; and the norm (4).
- $E(R,R^n)$ will stand for a subspace (hence, closed) of $BC(R,R^n)$, such as $AP(R,R^n)$, $PAP(R,R^n)$ or $BUC(R,R^n)$.
- $PAP(R,R^n)$ will be the space of pseudo almost periodic functions.

The following generalization of Theorem 1 above, can be proven in the same manner as in our paper [10].

Theorem 2. *Consider equation* (1)*, under the following assumptions:*

1. $L : E(R,R^n) \to E(R,R^n)$ *is a linear continuous operator, where* E *is a subspace of* $BC(R,R^n)$.
2. *Equation* (2) *has a unique solution* $x \in E(R,R^n)$ *for each* $f \in E(R,R^n)$.
3. *The map* $f : E(R,R^n) \to E(R,R^n)$ *is satisfying the global* (*i.e., on* E) *Lipschitz condition*
$$\|fx - fy\| \le L_0\|x - y\|, \tag{5}$$
with L_0 *sufficiently small.*

Then equation (1) *has a unique solution* x *in* $E(R,R^n)$.

Proof. It is completely similar to the proof of Theorem 2 in [10], and we shall present it concisely. □

First, let us denote by $E^{(1)}(R,R^n)$ the space (Banach) of those $x \in E(R,R^n)$, such that $\dot{x} \in E(R,R^n)$, the norm being defined by

$$\|x\|_1 = \|x\| + \|\dot{x}\|, \tag{6}$$

as it is usually considered (with $\|\cdot\|$ defined by (4)). Then we consider the operator T, given by

$$(Tx)(t) = \dot{x}(t) - (Lx)(t) \tag{7}$$

from $E^{(1)}$ into E, and notice that the inverse operator $T^{-1} : E \to E^{(1)}$, $x = T^{-1}f$, is well defined on E if we agree that x and f are related by equation (2), taking into account the uniqueness assumption.

From (7) one derives

$$\|Tx\| \le \|\dot{x}\| + \|Lx\|,\ x \in E^{91)}, \tag{8}$$

which combined with

$$\|Lx\| \le K\|x\|,\ x \in E, \tag{9}$$

for some $K > 0$ (the continuity of L!), leads to

$$\|Tx\| \le K_1(\|\dot{x}\| + \|x\|), \tag{10}$$

where $K_1 = \max\{1, K\}$. One can rewrite (10) as

$$\|Tx\| \le K_1 \|x\|_1,\ x \in E^{(1)}, \tag{11}$$

from which we obtain the continuity of $T : E^{(1)} \to E$.

Hence, T is a continuous operator from $E^{(1)}$ to E (because (2) is soluble for each $f \in E$), and it is one to one.

According to a well known theorem of Banach, the operator T^{-1} is also continuous. Therefore, from $x = T^{-1}f$, one obtains $\|x\|_1 \le M\|f\|$, $f \in E$, for some $M > 0$, which leads on behalf of (6) to

$$\|x\| \le M\|f\|,\ f \in E. \tag{12}$$

This is what we need to apply the Banach contraction principle, to derive existence and uniqueness for the nonlinear equation (1).

Indeed, one can define on $E(R,R^n)$ the operator U, by letting $x = Uy$ iff

$$\dot{x}(t) - (Lx)(t) = (fy)(t),\ t \in R, \tag{13}$$

with $x = x(t)$ the unique solution in E to equation (13). With x_1 and x_2 given by $x_1 = Uy_1$ and $x_2 = Uy_2$, one an write

$$(x_1 - x_2)^{\cdot} - L(x_1 - x_2) = fy_1 - fy_2, \tag{14}$$

which combined with (12) provides

$$\|Uy_1 - Uy_2\| \le M\|fy_1 - fy_2\|,\ y_1, y_2 \in E_1,$$

and relying on (5)

$$\|Uy_1 - Uy_2\| \le ML_0\|y_1 - y_2\|, \tag{15}$$

from which we find that U is a contraction on E, as soon as

$$L_0 < M^{-1}. \tag{16}$$

Theorem 2 is thereby proven, the inequality (16) showing the smallness of L_0.

Remark 3. The result established in [10], stated in Theorem 1 above, corresponds to the choice $E = AP = AP(R,R^n)$, i.e., to the case when E is the space of Bohr almost periodic functions (also known as uniformly almost periodic).

Remark 4. The case $E = PAP(R,R^n)$ leads to pseudo almost periodic functions, whose theory can be found in C. Zhang [20] as well as in [12] by Toka Diagana, in the Banach space framework.

3 Discussion of the second hypothesis in Theorem 2

The most important assumption in Theorem 2 is the second one, stating the existence of a unique solution to equation (2), for each $f \in E$. While it appears to be a rather difficult problem in the general case (i.e., when the operator L is arbitrary), we must stress the fact that there are several useful results available when E is a particular space, or L has a special form.

A first example, also historically, is when $(Lx)(t) = Ax(t)$, with A a constant matrix of type $n \times n$, while $E = AP(R, R^n)$. As a corollary of the Bohr–Neugebauer Theorem (see Corduneanu [5], for instance), in the case $\det(i\lambda I - A) \neq 0$ for $\lambda \in R$, the equation/system $\dot{x}(t) = Ax(t) + f(t)$ has a unique solution $x \in AP(R, R^n)$.

A second example, that has been treated by J.L. Massera, regards the linear systems for which $(Lx)(t) = A(t)x(t)$, with $A(t)$ an upper diagonal matrix with almost periodic entries (i.e., in $AP(R, R)$). If $M\{a_{ii}(t)\} \neq 0$ for $i = 1, 2, ..., n$, then (2) also satisfies the requirements of Theorem 2. The case $n = 1$ has been treated by R.H. Cameron.

The third example is given in Ch. Zhang [20], and is concerned with the case $(Lx)(t) = Ax(t)$, with constant $A = (a_{ij})$, $i, j = 1, 2, ..., n$, while $E = PAP(R, R^n)$. Proceeding by a linear transformation T, which reduces to the case of upper triangular matrix $(T^{-1}AT) = (b_{ij})$, $b_{ij} = 0$ for $i > j$, the discussion is brought to the case $n = 1$, i.e., to deal with a single equation $\dot{y} = \lambda y + f(t)$, with $\mathrm{Re}\,\lambda \neq 0$. Since $f(t) = g(t) + h(t)$, with $f \in AP(R, R)$ and h such that

$$\lim_{t \to \infty} (2t)^{-1} \int_{-t}^{t} |h(s)| ds = 0, \tag{17}$$

it can be seen that hypothesis 2 of Theorem 2 holds, when $E = PAP(R, R^n)$. Actually, as shown in [20], the property remains valid even in case of spaces $AP(R, H)$, with H a Hilbert space, but under stronger assumptions on the linear part.

A fourth case, that needs some consideration, is $A = A(t)$, with almost periodic or pseudo almost periodic entries, and has the upper triangular form. The conditions $M\{a_{ii}(t)\} \neq 0$, $i = 1, 2, ..., n$, which are sufficient for the validity of hypothesis 2 in case of almost periodicity, could be also sufficient in the case of the space $PAP(R, R^n)$. Or, maybe something else should be added in order to obtain the validity of condition 2 for equation (2). This problem is open, and its investigation could lead to another important case of validity for condition 2 in Theorem 2.

Of course, it would be interesting to extend the investigation to the case of infinite-dimensional spaces. See J.L. Massera and J.J. Schäffer [18] for suggestions and background.

A fifth situation is treated in our paper [6], when the space under consideration is the space $BC(R, R^n)$. After reducing the matrix of the linear system $\dot{x}(t) = A(t)x(t) + f(t)$, one obtains necessary and sufficient conditions for existence of a solution in $BC(R, R^n)$, for each $f \in BC(R, R)$. The conditions involved are extension to the real axis R, of some classical results of O. Perron for the semi-axis R_+.

4 The case of functional equations

In this section we shall investigate equations of the form

$$x(t) = (Lx)(t) + (fx)(t),\ t \in R, \tag{18}$$

with the same meaning for L and f as in the preceding sections, related to the linear associate

$$x(t) = (Lx)(t) + f(t),\ t \in R, \tag{19}$$

looking for existence of solutions and the connection between (18) and (19).

We have investigated the problem in [7], but we have limited our considerations to the case when the interval of definition is finite. This time we shall try to obtain some results when the solutions are defined on the whole real axis R, as it is the case with almost periodic functions or pseudo almost periodic functions.

If one considers the existence of solutions to equation (19), say in the case $L : AP(R,R^n) \to AP(R,R^n)$, we do not have a general result, i.e., valid for any continuous operator. Excepting, perhaps, the obvious statement that the operator $I-L$ is invertible on $AP(R,R^n)$. Moreover, the inverse should be continuous in order to further the investigation to the nonlinear case of equation (18). Certainly, this property for (19) holds in the case the norm of L is less than 1, which guarantees the existence and continuity of $(I-L)^{-1}$.

We will discuss in the sequel cases when the existence to (19) is assumed for any $f \in AP$. Or, if we want a result similar to the one in Theorem 2, for each $f \in E$, with $E(R,R^n)$ a subspace of $BC(R,R^n)$.

Consequently, we shall use again a hypothesis similar to condition 2 in Theorem 2, related this time to equation (19).

The following result can be proven using the Banach contraction principle.

Theorem 5. *Consider equation* (18)*, under the following hypotheses:*

1. *The operator $L : E(R,R^n) \to E(R,R^n)$ is a linear continuous operator, with E a subspace of $BC(R,R^n)$.*
2. *Equation* (19) *is uniquely soluble in E, for each $f \in E(R,R^n)$.*
3. *The same as in Theorem 2.*

Proof. It can be conducted exactly as in the case of Theorem 2. We must only prove the continuity of the operator $(I-L)^{-1}$ on E. This is again the immediate consequence of the Banach theorem on the continuity of the inverse operator (when it exists). We leave details to the reader. □

Remark 6. As noticed above, the condition 2 holds true when the operator L is such that $\|L\| < 1$ (on the space E, which is a subspace of $BC(R,R^n)$). Of course, it is sufficient to assume that $\|L\| < 1$, the norm being that corresponding to L on the Banach space E (i.e., a closed subspace of $BC(R,R^n)$, with the same norm as in the latter space).

The above restriction on L represents a rare occurrence in applications, and this feature makes desirable to describe other cases when the continuity of the inverse is assured.

Remark 7. The following result is part of Theorem 14.1 in our book [7].

Consider the convolution equation

$$x(t) = \int_R k(t-s)x(s)ds + f(t),\ t \in R, \tag{20}$$

under the following assumptions:

1. $k \in L^1(R, \mathcal{C})$, with $\mathcal{C}$ the complex field.
2. The Fourier transform of k

$$\widehat{k}(s) = \int_R k(t)e^{its}dt,\ s \in R, \tag{21}$$

satisfies the condition

$$\widehat{k}(s) \neq 1,\ s \in R. \tag{22}$$

Then, there exists a unique solution to equation (20), for each $f \in E$, with E standing for any of the spaces $BC(R,\mathcal{C})$, $AP(R,\mathcal{C})$, $PAP(R,\mathcal{C})$ or $A_\omega(R,\mathcal{C})$. By $A_\omega(R,\mathcal{C})$ one denotes the space of continuous periodic functions, with period ω. Moreover, the solution of (20) belongs to the same space as f.

Actually, for space E one could choose other function spaces, which are not subspaces of $BC(R,\mathcal{C})$, but are invariant with respect to the integral operator

$$x(t) \to \int_R k(t-s)x(s)ds.$$

In particular, such spaces are $M(R,\mathcal{C})$, $S(R,\mathcal{C})$ and $P_\omega(R,\mathcal{C})$. See [8] for the meaning of these notations.

Of course, the considerations above, in Remark 7, can be extended to the case of n-dimensional vector functions.

The discussion above offers a nontrivial example when assumption 2 of Theorem 5 is verified. Let us notice that the solution can be expressed by means of the resolvent kernel.

Remark 8. Another case we want to discuss is regarding the functional equation

$$x(t) = px(\lambda t) + f(t),\ t \in R, \tag{23}$$

where $p \in \mathcal{C}$, $\lambda \in R$, $|p| \neq 1$, $\lambda \neq 0$, and the underlying space is the space $BC(R,\mathcal{C})$. We shall be interested only in the case $E = AP(R,\mathcal{C})$, but other choices are possible.

This example is due to Mr. Peter Jossen, who was one of the participants in a short course on almost periodic functions I have thought in September 2006 at the Central European University in Budapest.

The existence and uniqueness of the solution $x(t) \in AP(R,\mathcal{C})$, for each $f(t) \in AP(R,\mathcal{C})$, is elementary application of the contraction mapping principle when $|p| < 1$.

In the case $|p| > 1$, one can reduce the discussion to the preceding case by writing (23) in the equivalent form

$$x(\lambda t) = p^{-1}x(t) - p^{-1}f(t),$$

and changing the variable t in $\tau = \lambda t$:

$$x(\tau) = p^{-1}x(\lambda^{-1}\tau) - p^{-1}f(\lambda^{-1}\tau). \tag{24}$$

Equation (24) is of the form (23), and $|p^{-1}| < 1$. Since $f(\lambda^{-1}\tau)$ is almost periodic whenever $f(t)$ is, the discussion is complete.

The nonlinear associate to (23) will be the equation

$$x(t) = px(\lambda t) + (fx)(t),\ t \in R, \tag{25}$$

where $f : AP(R,\mathcal{C}) \to AP(R,\mathcal{C})$ is a globally Lipschitz operator.

It is obvious that other possible choices for the underlying space, are $BC(R,\mathcal{C})$, $BUC(R,\mathcal{C})$, $PAP(R,\mathcal{C})$. By changing the underlying space, one obtains other interesting cases. For instance, if we substitute to $BC(R,\mathcal{C})$ the space $M(R,\mathcal{C})$ of locally integrable functions bounded in the mean

$$M = \left\{ x : R \to \mathcal{C},\ x \in L_{\text{loc}},\ \sup_{t\in R} \int_t^{t+1} |x(s)|ds < \infty \right\},$$

one can deal with the Stepanov's space of almost periodic functions $S(R,\mathcal{C})$ instead of $AP(R,\mathcal{C})$.

5 The case of neutral functional equations

We shall consider in this section some neutral functional equations of the form

$$(Vx)(t) = (Wx)(t),\ t \in R, \tag{26}$$

where $V,W : AP(R,R^n) \to AP(R,R^n)$ are continuous operators, satisfying some extra conditions which will assure the existence of solutions.

In the Appendix of our book [9], we dealt with equation (26), limiting our considerations to the case of the space $AP(R,R^n)$. Actually, the result established there, concerning the existence of solutions, can be extended to other spaces E, with $E = BC(R,R^n)$, $BUC(R,R^n)$, $PAP(R,R^n)$, or E = any subspace of $BC(R,R^n)$. Of course, instead of equations on a space whose elements are taking values in a finite-dimensional space (R^n or $\mathcal{C}^n$), one may consider the case of infinite-dimensional spaces. See, for instance, Ch. Zhang [20] or T. Diagana [12], for problems in infinite dimension and the use of semigroups theory, with the remark that only time-invariant operators lead to satisfactory treatment.

Proceeding on the same scheme as in [9], one can obtain the following results.

Lemma 9. *Consider the "first kind" functional equation*

$$(Vx)(t) = f(t),\ t \in R, \tag{27}$$

on a space E, from the list above. Let us assume that $V : E \to E$ satisfies the monotonicity condition

$$m\|x-y\|^2 \le \langle Vx - Vy, x-y \rangle,\ m > 0, \tag{28}$$

for any $x,y \in R^n$. Also, let V be globally Lipschitz continuous on E:

$$\|Vx - Vy\| \le M\|x-y\|,\ M > 0. \tag{29}$$

Then, there exists a unique solution $x \in E$ of equation (27)*, for any $f \in E$.*

The proof is detailed in [9]. We send the reader at this reference (in book form).

The neutral equation (26) suggests the following scheme, in view of obtaining existence of its solution: for each $u \in E$, one considers the equation of the form (27), with $f(t) = (Wu)(t)$. As above, E stands for one of the spaces listed in this section. In other words, we deal with the equation

$$(Vx)(t) = (Wu)(t),\ t \in R. \tag{30}$$

According to Lemma 9, there exists a unique $x \in E$, say $x = Tu$, satisfying (30), which implies that $T : E \to E$ is defined on the whole space E.

Since we do not want to use here the global Lipschitz condition for W, and intend to use Schauder's fixed point theorem on E, we will make the following assumption on W:

(a) $W : E \to E$ is a compact operator, i.e., it is continuous and takes bounded sets into compact sets of E.

We notice that (a) is a different assumption than Lipschitz continuity (neither one implying the other). On the other hand, the compactness requirement may be difficult to be established in any space E, or impracticable, because of its complexity. Nevertheless, we know this condition in the case of $AP(R,R^n)$.

Another condition of growth is necessary in view of the application of Schauder's theorem:

(b) If $w(r) = \sup\{\|Wx\|;\ x \in E,\ \|x\| \le r\}$, then

$$\limsup_{r\to\infty} \frac{w(r)}{r} = \lambda, \tag{31}$$

with $\lambda > 0$ sufficiently small (to be precised).

Finally, one more condition has to be imposed in order to carry out the proof of existence. Namely,

(c) V is a continuous linear operator.

On behalf of (c), condition (29) is automatically satisfied, while condition (28) implies $\|Vx\| \ge m\|x\|$, $x \in E$. This is because (28) leads to $m\|x-y\|^2 \le \|Vx - Vy\|\,\|x-y\|$, for any $x, y \in E$. Hence, V^{-1} exists and is continuous on $E : \|V^{-1}x\| \le m^{-1}\|x\|$.

Since equation (30) can be rewritten as

$$x = Tx, \tag{32}$$

with $T = V^{-1}W$, or $Tx = V^{-1}(Wx),\ x \in E$, there results that T is compact. Indeed, T is continuous since both V^{-1} and W are continuous (on E), and their product is compact because one factor is continuous and another is compact. We can proceed to applying the Schauder theorem. All that remains to prove is that a ball $\Sigma_r = \{x;\ x \in E,\ \|x\| \le r,\ r > 0\}$ is taken into itself by T:

$$T\Sigma_r \subset \Sigma_r, \tag{33}$$

for some $r > 0$. And we see that this is possible when $\lambda < 1$.

We shall now summarize the discussion about equation (26), carried above, in the following result.

Theorem 10. *Consider the neutral equation* (26)*, under assumptions* (a), (b) *and* (c)*. Then, condition* (31)*, with* $\lambda < 1$*, assures for some sufficiently large r,* $a(r)/r < 1$*, and* $T\Sigma_r \subset \Sigma_r$*. This inclusion means, together with the conditions assumed, the existence of a fixed point to the operator T, i.e., an* $x(t) \in \Sigma_r$*, such that* (26) *is satisfied.*

Remark 11. The monotonicity of the operator V is used to prove the existence of the inverse operator V^{-1} under Lipschitz continuity without the linearity assumption. Therefore, for $(Vx)(t) = f(t)$ under monotonicity of V, one assures the existence of V^{-1}, in general, nonlinear.

References

[1] L. Amerio and G. Prouse, *Almost Periodic Functions and Functional Equations*, Van Nostrand, N.Y., 1971.

[2] J. Andres, A. M. Bersani and R. F. Grande, *Hierarchy of almost periodic functions spaces,* Rend. Mat., Serie VII, **26** (2006), 121-188.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Cambridge Univ. Press, 1932.

[4] H. Bohr, *Almost Periodic Functions*, Chelsea, N.Y., 1951.

[5] C. Corduneanu, *Almost Periodic Functions*, John Wiley & Sons, N.Y., 1968.

[6] C. Corduneanu, *Sur l'existence des solutions bornées des systèmes différentiels non-linéaires*, Annales Polonici Math., V (1958), 103-106.

[7] C. Corduneanu, *Perturbations of linear abstract Volterra equations,* Journal Integral Equations and Appl., **2** (1990), 393-401.

[8] C. Corduneanu, *Integral Equations and Stability of Feedback Systems,* Academic Press, N.Y., 1973.

[9] C. Corduneanu, *Functional Equations with Causal Operators.* Taylor & Francis, London, 2002.

[10] C. Corduneanu, *Almost periodic solutions for a class of functional diffrential equations*, Functional Differential Equations, **14** (2007), 223-229.

[11] C. Corduneanu, *Almost Periodic Oscillations and Waves,* Springer, 2009.

[12] T. Diagana, *Pseudo-Almost Periodic Functions in Banach Spaces,* Nova Sci. Publishers, 2007.

[13] J. Favard, *Leçons sur les fonctions presque-périodiques,* Gauthier–Villars, 1933.

[14] A. M. Fink, *Almost Periodic Differential Equations,* Lecture Notes in Mathematics, vol. **377**, Springer–Verlag, 1974.

[15] Y. Hino, T. Naito, Nguyen V. Minh and J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces,* Taylor & Francis, London, 2002.

[16] M. A. Krasnoselskii, V. S. Burd and Yu. S. Kolesov, *Nonlinear Almost Periodic Oscillations.* John Wiley & Sons, 1973.

[17] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations.* Cambridge Univ. Press, 1982.

[18] J. L. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces.* Academic Press, N.Y., 1966.

[19] S. Zaidman, *Almost Periodic Functions in Abstract Spaces,* Pitman, London, 1985.

[20] Ch. Zhang, *Almost Periodic Type Functions and Ergodicity,* Science Press, Kluwer, Academic Publications, 2003.