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# SEMIGROUP GENERATION AND "HIDDEN" TRACE REGULARITY OF A DYNAMIC PLATE WITH NON-MONOTONE BOUNDARY FEEDBACKS

**IRENA LASIECKA** * and **DANIEL TOUNDYKOV** †

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**Abstract**

The paper establishes well-posedness and semigroup generation for a linear dynamic plate equation with non-monotone boundary conditions. The lack of dissipation prevents applicability of the classical semigroup theory, approximation techniques, or energy methods. Investigation of such systems was originally motivated by applications, but due to the challenging nature of the problem had been essentially limited to 1-dimensional models. A more recent result [BeLa], though still dealing with a (1D) Euler-Bernoulli beam, showed how the wellposedness in absence of dissipativity can be approached using tools of microlocal analysis, potentially applicable in higher dimensions. This paper extends the later work to a two dimensional system. The main difficulties in the 2D setting arise from a substantially increased complexity of boundary operators, and the failure of the uniform Lopatinskii condition, which ultimately necessitates additional control on tangential components of the boundary traces. The latter issue is handled by introducing a suitably constructed boundary feedback which acts as the additional moment present on the boundary of the two-dimensional domain.

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*Irena Lasiecka, Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA. E-mail address: il2v@virginia.edu. The research of Irena Lasiecka was partially supported by the National Science Foundation under Grant DMS-0606682.

†Daniel Toundykov, Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA. E-mail address: dtoundykov@math.unl.edu. The research of Daniel Toundykov was partially supported by the National Science Foundation under Grant DMS-0908270.

# 1 Introduction

## 1.1 The model

Let $w = w(t;x,y)$ be the relative vertical displacement at point $(x,y)$, time $t \geq 0$, of a thin plate, whose mid-surface occupies a smooth domain $\Omega \subset \mathbb{R}^2$. The following equation is a version of the Kirchhoff plate model, sometimes referred to as "Euler-Bernoulli plate":

$$(\boxdot w :=) \quad w_{tt} + \Delta^2 w = f \quad \text{in} \quad Q_T := ]0,T[\times\Omega, \tag{1}$$

The boundary $\Gamma := \partial\Omega$ consists of two segments $\Gamma := \Gamma_0 \cup \Gamma_1$ that are disjoint $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ with the edge $\Gamma_0$ being clamped. For instance, consider a plate with a clamped outer edge $\Gamma_0$, and a thin slit (with boundary $\Gamma_1$) being cut in the middle.

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Sigma_T^{(0)} := ]0,T[\times\Gamma_0. \tag{2}$$

Introduce some boundary operators on $\Gamma_1$:

$$\mathcal{B}_1 \ := \ \Delta + (1-\mu)B_1 \tag{3}$$

$$\mathcal{B}_2 \ := \ \frac{\partial}{\partial \nu}\Delta + (1-\mu)B_2, \tag{4}$$

where $0 < \mu < 1/2$ is Poisson's ratio. The operator $\mathcal{B}_1$ represents the bending moment about the tangent vector (the moment is zero the case of a free, or a simply supported boundary), and $\mathcal{B}_2$ models shear forces [Lag, Chapter 2] Here the modulus of flexural rigidity was re-normalized to 1. The boundary operators $B_i$, are given by

$$B_1 w \ := \ 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx} \tag{5}$$

$$B_2 w \ := \ \frac{\partial}{\partial \tau}\left( (\nu_1^2 - \nu_2^2)\, w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx}) \right). \tag{6}$$

Here the vector fields $\nu = \nu(x)$ and $\tau = \tau(x)$ are smooth, and denote respectively the outward normal field and a tangential frame on the boundary $\Gamma$. For more information on these boundary operators and their properties see [LaTrV1, pp. 296 – 310]

Operators (3), (4), correspond to the *free boundary conditions* which naturally arise in plate theory and, as such, have attracted substantial research efforts [LagLio, Lag, LagLeu, LLS, Las1, LaTrV1, Av]. These boundary conditions share some of the properties of Neumann boundary dynamics for the wave equation, in particular, the uniform Lopatinskii (Kreiss-Sakamoto) condition is not satisfied by such a system, causing the loss of regularity in the boundary traces [Sy, Sa, Tat2] when the dimension of the domain exceeds one. On the other hand, free boundary conditions have a lot of applications in mechanics, particularly in the context of controllability and stability. It is well-known that the absorbing boundary feedback affecting moments, torques and shears:

$$\mathcal{B}_1 w = -k_1 \frac{\partial w}{\partial \nu}_t, \qquad \mathcal{B}_2 w = k_2 w_t, \qquad k_i \geq 0, \quad k_1 + k_2 > 0, \tag{7}$$

causes the overall energy of the plate

$$E(t) \sim \|w_t(t)\|_{L_2}^2 + \|w(t)\|_{H^2}^2$$

to decay to zero at an exponential rate, e.g. [Lag], [LaTrV1, Sec. 3.14].

In fact, due to the dissipativity of conditions (7), the generation of a contraction semigroup corresponding to (1) is straightforward, and the standard energy identity confirms that the energy of solutions is non-increasing. It is more challenging to prove that the energy decays to zero at the uniform rate, but that has been accomplished using multiplier theory in the context of control and inverse problems (for instance see the aforementioned references, as well as an extensive overview of known results in [Las2]).

On the other hand, physical considerations in the context of problems in robotics [LuGu] dictate a different boundary feedback mechanism which in practice appears more suitable for stabilization. The latter corresponds to the case when $\Gamma_1$ is subject to *non-monotone* boundary conditions: the bending moment feedback on $\Sigma_T^{(1)} := ]0,T[\times\Gamma_1$, or its equivalent (dual) version - shear feedback (marked with an asterisk)

$$\mathcal{B}_1 w = -kw_t \qquad (8) \qquad\qquad \mathcal{B}_1 w = 0 \qquad (8^*)$$

$$\mathcal{B}_2 w = 0 \qquad (9) \qquad\qquad \mathcal{B}_2 w = k\frac{\partial w_t}{\partial \nu} \qquad (9^*)$$

for a constant $k > 0$. Elementary calculations, presented in Section 2.2 below, reveal that these boundary conditions destroy the natural symmetry in the problem and its monotonicity. Since the boundary traces involved are of a higher order and cannot be controlled by the energy, a fundamental question of well-posedness (generation of a semigroup) becomes problematic undermining the classical and well-established role of differential calculus and energy methods.

It was only recently that the generation of a semigroup was shown in a *one-dimensional case*: a spectral argument based on Riesz basis generation was applied in [GWY] showing that the model (1), (2), with $(10, 11)$ or $(10^*, 11^*)$ in 1-D leads to a strongly continuous semigroup of contractions.

Treatment of higher-dimensional models, however, necessitates a more thorough approach and a better understanding of trace regularity of solutions, thus leading to developments in microlocal analysis and propagation of singularities theory for hyperbolic equations [BLR, Tay1, Tat1]. With the aid of microanalytic framework and pseudo-differential operators in [BeLa] it was shown that the same (Euler-Bernoulli beam) model not only generates a strongly continuous semigroup, but the semigroup obtained is of *Gevrey's class* which indicates existence of a smoothing effect produced and propagated by the non-monotone boundary feedback. In addition, the pseudo-differential/microlocal methods of [BeLa] reveal, rather unexpectedly, "hidden" regularity of the boundary traces of solutions.

The goal of this paper is to address the same question for a *two-dimensional dynamic plate*. Clearly, the Riesz basis argument developed in [GWY] is no longer applicable. Pseudodifferential operator approach employed in [BeLa] has chances of surviving 2D analysis, provided the main obstacles: calculations in higher dimensions, and the failure of the Lopatinskii condition, are properly addressed. In fact, we show how incorporation of a suitable feedback boundary control via shears affects the dynamics and leads to a well-posed semigroup solution. The obtained solution displays a regularizing effect, often dubbed "hidden regularity", [LLT] on the boundary manifesting itself by the unexpected gain of 1 (time-space) tangential derivative on $\Gamma$.

To wit, we will be studying plate equation with the following feedback operator denoted by $\mathcal{C}$ added to the shear forces:

$$\mathcal{B}_1 w = -kw_t \quad (10) \qquad \mathcal{B}_1 w = 0 \quad (10^*)$$

$$\mathcal{B}_2 w = \mathcal{C} w \quad (11) \qquad \mathcal{B}_2 w = \mathcal{C} w + k\frac{\partial w_t}{\partial \nu} \quad (11^*)$$

where

$$\mathcal{C} = \mathcal{C}_\delta := c_{\mu\delta}\left(\frac{\partial^2}{\partial \tau^2}\right)\frac{\partial}{\partial \nu}, \qquad c_{\mu\delta} := 1-\mu+\delta \tag{12}$$

and $\delta > 0$ is sufficiently small.

**Remark 1.1.** *It should be noted that while* (8), (9) *are dual (adjoint) boundary conditions to* $(8^*)$, $(9^*)$, *that is not the case for* (10), (11) *and* $(10^*)$, $(11^*)$ *when* $\mathcal{C} \neq 0$. *However, it will be shown that solvability of the problem follows from use of the either boundary condition. For this reason the two problems are listed together.*

**Remark 1.2** (Value of $\delta$)**.** *The choice of the "scaling" parameter $\delta$ is sensitive and the precise value depends on the geometry of $\Omega$. Since the feedback operator added to the model is a higher-order differential operator (way outside the perturbation theory for semigroups), it is not surprising that the value of the scaling parameter must be appropriately calibrated. For a half-space version of the problem (flat boundaries) it suffices to choose $\delta \neq \frac{2(1-\mu)}{1+\mu}$; in general, to account for changes in curvature of $\partial\Omega$, $\delta$ should be either suitably small or very large, thus staying away from a certain "singular" region. From optimality considerations it is better to minimize the impact of the correcting control $\mathcal{C}$, hence the result is stated with $\delta$ being sufficiently small. For more details see the proof of Theorem 7.1 in the Appendix.*

## 1.2 Previous work

The study of dynamic beam and plate equations with various boundary conditions and the associated feedbacks has gained great attention over the recent decades. Questions of generation of semigroups, related stability and control were thoroughly considered with many results available in [Lag, Ko, LagLio, LLS, Las2] as well as references therein.

Boundary (velocity) feedbacks applied at the end of a beam or the edge of a plate are typical mechanisms enabling control or stabilization of elastic structures. An almost universal feature of feedbacks considered in the prior literature is monotonicity (dissipativity) - a fundamental property that permits applicability of the linear and nonlinear semigroup theories. In the absence of monotone dissipative behavior the problem is poorly structured since the presence of boundary velocity traces leads to perturbations that are not even relatively bounded; proving generation of a well-posed dynamics becomes problematic.

The main feature of this paper is that all of the boundary conditions listed above: (8) - (9) (or their dual counterparts $(8^*)$, $(9^*)$), just like (10), (11) (and the respective dual version $(10^*)$, $(11^*)$) lead to principal non-monotonicity in the underlying evolution generator, preventing one from employing standard semigroup methods to study wellposedness of this system, or even formally representing it as a perturbation of a dissipative equation. In Section 2 it will be shown that essentially there is no apparent candidate Lyapunov function, neither based on the finite energy, nor on any perturbation of it. As a result, it becomes difficult to merely determine whether the energy associated with the system remains finite, even over short time-intervals, not to mention prove any kind of global behavior.

As was already stated, the above boundary conditions destroy the natural symmetry in the problem, however, introduction of such feedbacks stems from practical considerations: e.g. [LuGu] and references therein, which proposed to use the 1D version of ($8^*$) and ($9^*$), for shear force stabilization of flexible robot arms with rotational joints. The issue of semigroup generation in the 1D case gathered attention of several authors [GWY, Sh] where it was studied by means of Riesz basis techniques. In [GWY] the authors showed that despite the lack of dissipation, the corresponding Euler-Bernoulli beam system generates an exponentially stable differentiable semigroup.

However, the Riesz basis approach, besides computational complexity requires one to consider the uniform gap condition (e.g. see [KoLo]), which makes the technique essentially inapplicable to models whose the space-dimension exceeds 1, unless a very special geometry is involved. This fact motivated the work [BeLa] which offered a very different approach and framework to study this type of non-dissipative dynamics by employing tools of microlocal analysis developed in [Tat1]. Some related results on trace and Gevrey's regularity can also be found in [HoLi, Las1]. In [BeLa] the authors ultimately prove that an Euler-Bernoulli beam with the non-monotone shear feedback generates a continuous semigroup which, moreover, possesses Gevrey's regularity along with "hidden" boundary regularity. Geverey's regularity is deduced from abstract results of [ChTr2, ChTr1] supported by spectral-microlocal estimates obtained in [BeLa].

The most important aspect of the analysis in [BeLa] is that it is not intrinsically limited to one space dimension or any special geometric configurations (rectangles etc.). In fact, the present paper takes the approach further by generalizing the model from beams to thin plates defined on arbitrary smooth domains in $\mathbb{R}^2$.

The treatment of a two dimensional framework, however, does not readily follow from previous work and poses new challenges related exclusively to the "extra" tangential direction (the reason for the failure of the Lopatinskii condition). One must now deal with:

- increased complexity in the structure of the boundary operators and their associated symbols,
- instability due to the lack of control on the tangential components of the traces.

This work overcomes these obstacles showing that the system (1) with clamped conditions (2) on $\Gamma_0$ and subject to non-dissipative feedback on $\Gamma_1$, generates a $C_0$ semigroup provided the non-dissipative boundary conditions are controlled by a feedback operator affecting shears or moments $\mathcal{C}w$, as in (10), (11), or ($10^*$), ($11^*$) (see also the dual version (21),(22) or ($21^*$), ($22^*$)).

Let us briefly preview the strategy for the argument. By using a microlocal decomposition of the principal symbol governing the dynamics, the generator is split into a principal part that has smoothing effect, and lower order terms that do not affect wellposedness [Tay2, Tat1]. More specifically, the fourth order (anisotropic) dynamic operator is decomposed algebraically into four branches according to the roots of the fourth order characteristic polynomial. As shown in [Tat1] the two real roots correspond to "conservative" dynamics while complex roots produce forward and backward diffusion. It is the forward diffusion that provides regularizing effect producing lower order terms in the estimates (see Sect. 4.2.3 and 4.4 below, in particular Lemma 4.2). Then the boundary operators are rewritten according to this splitting of the principal symbol - see Section 4.3 including Lemma 4.1. The decomposition is critical and employs the technique introduced in the seminal work [Tat1] where the author shows that the Cauchy data imposed on three boundary conditions (rather than four) in the plate equation allow to reconstruct plate dynamics modulo a "smooth" part. The decomposition is microlocal only, which explains the difficulties faced when dealing with the differential version of the problem where energy methods seem to be of little avail. Albeit technical, the microlocal decomposition permits to derive energy inequality which provides quantitative information on the "hidden" boundary regularity of the traces. At the end of the day, we obtain a well-posed semigroup solution, defined for all times, which, however, is not of a dissipative type (unlike classical problems with absorbing feedback boundary conditions [Lag, Las2, Ko] and references therein).

Technical complications in this program stem from complexity of boundary symbols representing free boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ in two dimensions [Lag] whose algebraic structure is rather complicated (see [Las1] and Sect. 4.3 below). Once the procedure is carried out, energy methods evince (at the microlocal level) a "well-behaved" monotone part corresponding to the principal term in the decomposition, polluted, however, by unstructured terms. The subsequent challenge is to show that these latter quantities can ultimately be treated as being of a lower order with respect to the topology of the principal dynamics.

**Remark 1.3.** *It is strongly hoped that the methodology developed in this paper may have further bearing and be applicable to other problems of dynamic elasticity, such as shells or linked structures with non-monotone terms of higher energy.*

## 1.3 Notation

For convenience define the following space-time regions

$$Q_{t,T} := ]t,T[\times\Omega, \qquad \Sigma^{(i)}_{t,T} := ]t,T[\times\Gamma_i$$

and

$$Q := Q_{-\infty,\infty}, \qquad \Sigma^{(i)} := \Sigma^{(i)}_{-\infty,\infty}.$$

When considering a localized version of the problem we will be applying coordinate changes that transform a neighborhood of the boundary $\Gamma_i$ into an open subset of a half-space. Then employ the following notation:

$$\mathbf{\Omega} := \mathbb{R}_y \times \mathbb{R}^+_x, \qquad \mathbf{\Gamma}_i := \mathbb{R}_y, \qquad \mathbf{Q} := \mathbb{R}_t \times \mathbf{\Omega} \qquad \mathbf{\Sigma} := \mathbb{R}_t \times \mathbb{R}_y.$$

Here the half-axis $\mathbb{R}^+_x$ corresponds to the normal direction (unit normal $\nu \equiv (-1,0)$). It will be clear from the context which $\Gamma_i$ is being transformed, but most of the discussion will focus on $\Gamma_1$ where the feedback is applied.

Notation $\|\cdot\|_{r,X}$ or $|\cdot|_{r,X}$, depending on whether $X$ has respectively an interior (in $\mathbb{R}^n$) or a boundary (in $\mathbb{R}^{n-1}$) domain as one of its factor spaces, will denote the norm in the Sobolev space $H^r(X)$. The form $(\cdot,\cdot)$ will denote the inner product on $L^2$ functions defined in the interior, whereas $\langle,\rangle$ will be the corresponding bilinear form for boundary-based spaces. Unless stated otherwise the boundary $\Gamma_1$ is understood.

### 1.4 Function spaces

Throughout the paper we will make use of the special classes of anisotropic Sobolev spaces. Following [Ho, Vol. III] define the $H^{r,s}(Q)$ space which, roughly speaking, consists of functions with $r$ derivatives in the normal direction to the boundary with values in $H^s(\Sigma)$. More precisely, use a smooth cutoff to localize a given function to a neighborhood of the boundary $\Sigma$. Use a partition of unity and introduce local coordinates that carry a neighborhood of a point $(t,x_0)\in\Sigma$ to a subset of $\mathbf{Q}=\mathbb{R}^+_x\times\mathbf{\Sigma}$ where $x_0$ is identified with the origin, and $\mathbb{R}^+_x$ with the (inward) normal space over $x_0$. Then let

$$H^{m,r}(\mathbf{Q}) := \bigcap_{i=0}^{m} H^{m-i}\left(\mathbb{R}^+;H^{r+i}(\mathbf{\Sigma})\right), \qquad m\in\mathbb{N}\cup\{0\}, \quad r\in\mathbb{R},$$

or, equivalently,

$$\|v\|^2_{H^{m,r}(\mathbf{Q})} = \sum_{j=0}^{m} \|v\|^2_{H^j(\mathbb{R}^+;H^{r+m-j}(\mathbf{\Sigma}))}.$$

Of course, the norm is not unique and depends on the choice of local coordinates, and does not preserve information on what happens in the interior of the domain away from the chosen boundary $\Gamma_1$-collar.

In addition, we will refer to anisotropic spaces [Is, Tat1] denoted by $H^r_a(\Sigma)$ which for $r\geq 0$ are equivalent to

$$H^r_a(\Sigma)\cong L^2(0,T;H^r(\Gamma))\cap H^{r/2}(0,T;L^2(\Gamma)) \tag{13}$$

and for $r<0$ are defined by duality with respect to the $L^2(\Sigma)$ inner product. By $H^{m,s}_a$ we will henceforth indicate the space $H^{m,r}$ where the normal derivatives take their values in $H^r_a(\mathbf{\Sigma})$.

## 2 Boundary conditions

To illustrate the effect of the boundary conditions (10, 11) or (10*, 11*) let us formally construct and examine the energy identity for the corresponding system. We will often invoke the following identity (see [LaTrV1, p. 310, Proposition C.12])

$$(\Delta^2 u,v) = a(u,v) + \langle \mathcal{B}_2 u, v\rangle - \left\langle \mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right\rangle, \tag{14}$$

where the bilinear form $a$ is given by

$$a(w,v) = \int_\Omega \left[ w_{xx}v_{xx} + w_{yy}v_{yy} + 2(1-\mu)w_{xy}v_{xy} + \mu\Big(w_{xx}v_{yy}+w_{yy}v_{xx}\Big)\right].$$

Note that $a(w,w)$ yields a semi-norm $\|\Delta w\|_{0,\Omega}$ on $H^2(\Omega)$, which for the $H^2$ functions clamped on the boundary as in (2) is equivalent to $\|w\|_{2,\Omega}$.

Now formally take the $L^2(\Omega)$ inner product of (1) with $w_t$:

$$\tfrac{1}{2}\partial_t\left(\|w_t\|^2_{0,\Omega}+a(w,w)\right)+\langle \mathcal{B}_2 w, w_t\rangle-\left\langle \mathcal{B}_1 w, \frac{\partial w_t}{\partial \nu}\right\rangle=(f,w_t). \tag{15}$$

Define the quadratic energy as

$$E_w(t):=\tfrac{1}{2}\,a(w(t),w(t))+\tfrac{1}{2}\,\|w_t(t)\|^2_{0,\Omega}.$$

Integrate (15) from $t=0$ to $t=T$ and apply the boundary conditions (10)-(11) (or their dual (10$^*$)-(11$^*$)) to obtain the energy identity

$$E_w(T)+\int_0^T\langle \mathcal{C}w, w_t\rangle+k\int_0^T\left\langle w_t,\frac{\partial w_t}{\partial\nu}\right\rangle=E_w(0)+\int_0^T(f,w_t) \tag{16}$$

## 2.1 Monotone feedback

To illustrate the implications of (16) recall the standard *monotone feedback* for the above model is:

$$\mathcal{B}_1 w=-k\frac{\partial w_t}{\partial\nu},\qquad\qquad \mathcal{B}_2 w=0. \tag{17}$$

Use the latter boundary condition, and integrate (15) over $t\in[0,T]$

$$E_w(T)+k\int_0^T\left|\frac{\partial w_t}{\partial\nu}\right|^2_{0,\Gamma}=E_w(0)+\int_0^T(f,w_t).$$

Thus, when $f=0$, the energy is non-increasing and the rate of dissipation is proportional to $(\partial_\nu w_t)^2$. The adjoint version of this *monotone* feedback is

$$\mathcal{B}_1 w=0,\qquad\qquad \mathcal{B}_2 w=kw_t, \tag{18}$$

in which case one likewise gets (when $f=0$) non-increasing quadratic energy

$$E_w(T)+k\int_0^T|w_t|^2_{0,\Gamma}=E_w(0).$$

## 2.2 Non-dissipative feedback

In contrast, conditions (8)-(9), lead to the energy identity

$$E_w(T)+k\int_0^T\left\langle w_t,\frac{\partial w_t}{\partial\nu}\right\rangle=E_w(0)$$

The “high-order” trace $\langle w_t,\frac{\partial w_t}{\partial\nu}\rangle$ is of an undetermined sign and does not provide any a priori bounds on the energy functional. Similarly from (16) it is not apparent whether the energy, if well-posed, is bounded, due to the absence of monotonicity in the trace dynamics.

However, since the problem is linear, one approach to prove generation of a semigroup associated with the system (1), (2), with $(10, 11)$ or $(10^*, 11^*)$ would be to establish an a priori bound on $E_w(t)$, and then employ an approximation argument (e.g. [Mi, pp. 480 – 481]). Thus the main challenge is precisely to acquire an energy-related bound on the higher-order "non-monotone" trace product $\left\langle \frac{\partial w_t}{\partial \nu}, w_t \right\rangle$, which arises in the identity (16). This program will be accomplished by decomposing (microlocally) the normal derivative into two parts, where one part has a positive symbol and the associated smoothing effect, while the other is of the lower order, hence can be absorbed by the energy.

# 3 Main Results

For convenience let $\mathcal{A}$ denote the corresponding bi-Laplacian operator

$$\mathcal{A}h = \Delta^2 h$$

$$\mathscr{D}(\mathcal{A}) = \left\{ h \in H^4(\Omega) \quad : \quad \left[ h = \frac{\partial h}{\partial \nu} \right]_{\Gamma_0} \equiv 0; \quad \left[ \mathcal{B}_1 h = \mathcal{B}_2 h \right]_{\Gamma_1} \equiv 0 \right\}.$$

**Theorem 3.1.** *The system* (1), (2), *with* $(10, 11)$ *or* $(10^*, 11^*)$ *generates a strongly continuous semigroup* $t \mapsto S(t)$ *on the state space* $\mathscr{H} = \mathscr{D}(\mathcal{A}^{1/2}) \times L^2(\Omega)$. *Furthermore, if* $\{w, w_t\}$ *is any semigroup solution and* $T > 0$, *then there is a constant* $C_T = C(E_w(0), T) > 0$ *so that the following trace estimate holds:*

$$|w_t|^2_{\frac{1}{2},a,\Sigma_T^{(1)}} + \left| \frac{\partial w_t}{\partial \nu} \right|^2_{-\frac{1}{2},a,\Sigma_T^{(1)}} + |w|^2_{\frac{5}{2},a,\Sigma_T^{(1)}} + \left| \frac{\partial w}{\partial \nu} \right|^2_{\frac{3}{2},a,\Sigma_T^{(1)}} \leq C_T \left( E_w(0) + \|f\|^2_{0,Q_T} \right)$$

The trace estimate in the above Theorem 3.1 demonstrates the smoothing effect of the semigroup on the boundary. According to the trace theory $H^2(\Omega)$ regularity of solutions merely implies $H^{1/2}(\Gamma)$ regularity of the normal derivative - one derivative less than the $H^{3/2}(\Gamma)$ regularity predicted by the Theorem. The same numerology applies to other traces, including anisotropic regularity of time derivatives (rescaling gives 2 space derivatives for one time derivative as defined in (13)).

As subsequent sections will show, this "hidden" boundary regularity stems from a monotone "sub-component" (on the microlocal scale) of the principal symbol and a parabolic smoothing effect provided by one of the symbol factors. In fact, it was first observed in [Tay1, Ch. V, Ch. IX ] that near the boundary a second-order operator can be decoupled into branches corresponding to backward and forward diffusion equations along the normal direction. The smoothing of the traces results from parabolic diffusion towards the boundary. By solving the Cauchy problem for each branch it was shown in [Tat1] that only three boundary conditions determine the smoothness of the dynamics. The fourth boundary condition has no effect on the smoothness. This discovery was used to show that plate equations can be controlled or stabilized by using a feedback operator acting on one boundary condition only (thus giving an over-determination of three traces - rather than the customary four). By decomposing boundary conditions according to the four branches of the principal symbol, it will be shown that the three pieces of information encoded in symbols of $\mathcal{B}_1 w$, $\mathcal{B}_2 w - \mathcal{C} w$, and $\partial_\nu w_t$ suffice to claim the needed regularity.

**Remark 3.1.** *When $\mathcal{C} = 0$ the result of Theorem 3.1 recovers the one-dimensional result obtained in [BeLa].*

**Remark 3.2** (Global stability)**.** *As was remarked in the introduction, classical absorbing boundary conditions (17 or 18) lead to an exponential decay of the semigroup. An interesting question to raise is whether some sort of decay also holds in the present case. In the absence of global stability (as $t \to \infty$); it is difficult to make any conjecture. The method of proof of existence of solutions does not control the size of lower order terms. On the other hand, stability of the model in one-dimensional case is known [GWY] and obtained by spectral analysis that depends on the Riesz basis generation. Thus, an open question whether long-time behavior of solutions is bounded or decaying seems a legitimate and interesting open problem.*

It appears plausible that the method presented in this paper may be applied to a variety of non-monotone problems where natural symmetry of boundary conditions goes away, yet intrinsic properties of the dynamics are still encoded in the formulation.

# 4 PROOFS I: Microlocal analysis

## 4.1 Outline

It was observed in [Tay1, Tat1] that interior hyperbolic dynamics provides parabolic-like smoothing propagating from the interior to the boundary. As a consequence, for the plate problem one of the four boundary traces (roughly speaking, three space derivatives, and a trace of the velocity) can be controlled by the other three modulo some lower order terms, and a certain microlocal perturbation.

Following [BeLa] we consider the higher-order non-dissipative trace

$$\left(\frac{\partial}{\partial \nu}\frac{\partial}{\partial_t}\right) w$$

and, on a microlocal level, algebraically express it as a perturbation of the "natural" monotone feedback $w_t$. However, due to the fact that the problem is two-dimensional, the coefficient of the monotone part will be degenerate (precisely because of the presence of the tangential component) and the purpose of the correcting feedback $\mathcal{C}w$ will be to handle that issue.

At the end of the day (10, 11) or (10*, 11*) will be represented as a tractable perturbation of monotone boundary conditions. As an additional technicality, to justify the microlocal approach, in particular the Fourier(-Laplace) transforms in the tangential and time-variables the analysis will be performed on the backward adjoint problem and then extended to the original one by duality.

## 4.2 Preliminary steps

### 4.2.1 Backward adjoint problem

Consider the (backward) adjoint system of (1), (2), with $(10, 11)$ or $(10^*, 11^*)$ with a right-hand side that will be specified later:

$$\boxdot z = \mathrm{rhs}(z) \qquad \text{in} \qquad ]-\infty, T[\times\Omega \tag{19}$$

$$z(T) = z_t(T) = 0 \tag{20}$$

The dynamics is clamped on $\Gamma_0$, whereas on $\Gamma_1$ we have:

$$\begin{aligned} \mathcal{B}_1 z &= \mathcal{C}^* z + h(t) \qquad (21) \\ \mathcal{B}_2 z &= -k\frac{\partial z_t}{\partial \nu} + g(t) \qquad (22) \end{aligned}$$

$$\mathcal{B}_1 z = \mathcal{C}^* z + k z_t + h(t) \tag{21*}$$

$$\mathcal{B}_2 z = g(t) \tag{22*}$$

We will show how to come up with the adjoint auxiliary feedback $\mathcal{C}^*$, but running ahead one can think of it as

$$\mathcal{C}^* = -c_{\mu\delta}\frac{\partial^2}{\partial \tau^2}$$

which by duality will then yield the announced earlier definition of $\mathcal{C}$ in (12). Extend $z$ by zero for $t > T$, so now we can work on space-time cylinders $Q$ and $\Sigma_i$ (with the time domain now being all of $\mathbb{R}_t$).

**Remark 4.1.** *We note that the boundary conditions in (21,22) and (21*,22*) are not formally dual problems, unless $\mathcal{C} = 0$.*

### 4.2.2 Localization near the boundary

Let $x_0 \in \Gamma_1$, and $U$ be some open neighborhood of $x_0$. Introduce a smooth cutoff $\phi$ compactly supported in $U$ and a local change of coordinates $\psi : U \cap \Omega \to \mathbb{R}^n_+$. Set

$$\tilde{z} = (\phi z) \circ \psi^{-1}$$

Let $P$ and $F$ denote respectively the operators $\boxdot = (\partial_{tt} + \Delta^2)$ and $\mathrm{rhs}(\cdot)$ expressed in the local coordinates. With a slight abuse of notation we shall omit "$\sim$" over the boundary data $h$ and $g$ assuming they have been modified accordingly whenever local coordinates are involved. Using a partition of unity to generate such cutoffs, the system (19) is locally equivalent (see [Ho, Vol I., Sect. 6.4]) to:

$$P(x,y,D_x,D_y,D_t)\tilde{z} = F(x,y,D_x,D_y)\tilde{z} + [\![P - F,\, \phi \circ \psi^{-1}]\!]\left(z\big|_{U\cap\Omega} \circ \psi^{-1}\right)$$

The adjoint system (19) is locally equivalent to the half-space problem:

$$P\tilde{z} = F\tilde{z} + [\![P - F, \phi]\!] z \tag{23}$$

defined on $\boldsymbol{\Omega} := \mathbb{R}_y \times \mathbb{R}^+_x$ with $\mathbb{R}_y$ corresponding to the tangential component (the image of a neighborhood of $x_0$ in $\Gamma_1$), and the half-axis $\mathbb{R}^+_x := ]0, \infty[$ collinear to the normal direction, with the unit normal vector being $(-1, 0)$.

### 4.2.3 Principal symbol of $P$

Formally (assuming a sufficiently rapid decay in the time variable) apply the Fourier transforms in tangential and time directions: $\partial_y \to i\eta$, $\partial_t \to is$. Symbolically substitute $-i\xi$ for the normal derivative (i.e. $\partial_x \to i\xi$). Differential operator $P(x,y,D_x,D_y,D_t)$ in (23) has principal symbol of the form

$$p(x,y,\xi,\eta,s) = -a(x,y)s^2 + (\xi^2 + r(x,y,\eta))^2 \tag{24}$$

where $a(x,y) \geq a_0 > 0$ in $\overline{\mathbf{\Omega}}$, and the positive real symbol $r(x,y,\eta)$ corresponds to a strongly-elliptic 2nd-order tangential operator. Henceforth, without loss of generality we assume that our symbols vanish in some neighborhood of the zero-section of the cotangent bundle, thus avoiding singularities, if any, contained within a bounded set. The latter assertion can be formalized by introducing suitable cutoffs near the origin in the frequency variables.

Following [Tat1] and [BeLa] we factor

$$p = p^- p^+ \tag{25}$$

$$p^-(x,y,\xi,\eta,s) = \xi + i\sqrt{r(x,y,\eta) + \sqrt{a(x,y)}\,|s|}$$

$$p^+(x,y,\xi,\eta,s) = \left(\xi - i\sqrt{r(x,y,\eta) + \sqrt{a(x,y)}|s|}\right)\left(\xi^2 + r(x,y,\eta) - \sqrt{a(x,y)}|s|\right)$$

The factor $p^-$, corresponding to the root with negative imaginary part, exhibits parabolic-like behavior form the interior towards the boundary and provides additional smoothing on the traces of $p^+$.

## 4.3 Algebraic decomposition of the trace $\partial_\nu z_t$

**Lemma 4.1.** *If $z$ is a finite energy solution to* (19)*, with* $(21,22)$ *or* $(21^*,22^*)$*, integrable in time variable* $z \in L^2(\mathbb{R}_t;Q)$*,* $z\big|_{\Gamma_1} \in L^2(\mathbb{R}_t;\Sigma^{(1)})$*, then*

$$\frac{\partial z_t}{\partial \nu} = M_1 z_t + A_1 h + A_0 P^+ z + iA_0 g + l.o.t.(z) \tag{26}$$

*where*

$$M_1 \in OPS^1_a(\Sigma^{(1)}), \qquad A_1 \in OPS^1_a(\Sigma^{(1)}), \qquad A_0 \in OPS^0_a(\Sigma^{(1)})$$

*and* $l.o.t.(z)$ *stands for the lower order terms: those which can be estimated by norms of the solution in spaces below the finite energy level*

$$l.o.t.(z) \leq C\int_0^T \left(\|w(t)\|^2_{2-\varepsilon,\Omega} + \|w_t(t)\|^2_{-\frac{1}{2}+\varepsilon,\Omega}\right) dt$$

*for some* $0 < \varepsilon < 1/2$.

*Furthermore, if* $\mathcal{C}^* = -c_{\mu\delta}\frac{\partial^2}{\partial \tau^2}$*, then the operator* $M_1$ *is positive and* strongly elliptic *(of anisotropic order* 1*).*

*Proof.* **Step1. Principal symbols for the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$.** We shall use the following representation for $B_1$ and $B_2$ ([LaTrV1, pp. 296, 307])

$$B_1 = -\left(\frac{\partial}{\partial\tau}\right)^2 - \kappa(x)\frac{\partial}{\partial\nu}, \qquad B_2 = \frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\nu} - \kappa(x)\frac{\partial}{\partial\tau}\right)$$

where $\kappa(x)$ stands for the mean curvature $\operatorname{div}(\nu(x))$. The respective *principal* symbols are:

$$b_1 = r, \qquad b_2 = i\xi r$$

Hence the principal symbols of the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ have the form:

$$\beta_1(x,y,\xi,\eta) = \quad -\xi^2 - r + (1-\mu)r \quad = -\xi^2 - \mu r \tag{27}$$

$$\beta_2(x,y,\xi,\eta) = \quad -i\xi\big(-\xi^2 - r\big) + (1-\mu)i\xi r \quad = i\xi^3 + (2-\mu)i\xi r \tag{28}$$

**Step 2. Microlocal decomposition of the higher-order trace.**
We factored the principal symbol of $P$ in (23) as $p = p^- p^+$ with

$$p^+ = \left(\xi - i\sqrt{r + \sqrt{a}|s|}\right)\left(\xi^2 + r - \sqrt{a}|s|\right).$$

Now decompose $s\xi$ (formally of anisotropic order 3) in terms of the other traces, in particular the dissipative feedback $(is) \sim \partial_t$. Solving for the coefficients find

$$s\xi = a_0 p^+ + a_1\beta_1 + ia_0\beta_2 + a_3 \tag{29}$$

$$\begin{aligned}
a_0 &= \frac{-s}{(1-\mu)r + \sqrt{a}|s|} && \in S_a^0(T^*\Sigma^{(1)}) \\
a_1 &= -ia_0\sqrt{r + \sqrt{a}|s|} && \in S_a^1 \\
a_3 &= a_1\big(\sqrt{a}|s| - (1-\mu)r\big) && \in S_a^3
\end{aligned}$$

The boundary conditions (21), (22) or their dual versions (21*), (22*) on the principal symbol level translate into:

$$\beta_1 = \{\hat{h}\} + \theta_1 \tag{30}$$

$$\beta_2 = -ks\xi + \{\hat{g}\} + \theta_2 \tag{31}$$

$$\beta_1 = iks + \{\hat{h}\} + \theta_1 \tag{30*}$$

$$\beta_2 = \{\hat{g}\} + \theta_2 \tag{31*}$$

where $\theta_1$ is the symbol corresponding to $\mathcal{C}^*$. The extra variable $\theta_2$ has been introduced to investigate what happens when the auxiliary feedback is applied at the other boundary condition. To simplify the subsequent calculations the reader may simply ignore $\theta_2$ and its coefficients.

Substituting the boundary conditions into (29) respectively get:

$$s\xi = a_0 p^+ + a_1\left(\{\hat{h}\} + \theta_1\right) + ia_0\left(-ks\xi + \{\hat{g}\} + \theta_2\right) + a_3 \tag{32}$$

$$s\xi = a_0 p^+ + a_1 iks + a_1\left(\{\hat{h}\} + \theta_1\right) + ia_0\left(\{\hat{g}\} + \theta_2\right) + a_3 \tag{32*}$$

**Step 3. Express $s\xi$ as a perturbation of the "dissipative" symbol $is$.**
The goal now is to rewrite $s\xi$ as a multiple of $(is)$ plus terms dependent only on the boundary data $\{\hat{h}\}, \{\hat{g}\}$ and $p^+$. From (32) and (32*) respectively obtain

$$\begin{aligned}
s\xi =& \frac{1}{(1+ia_0k)}\left((a_3+a_1\theta_1+ia_0\theta_2)+a_1\{\hat{h}\}+a_0p^+ + ia_0\{\hat{g}\}\right)\\
=& \frac{-i(a_3+a_1\theta_1+ia_0\theta_2)}{(1+ia_0k)s}(is)+\frac{1}{(1+ia_0k)}\left(a_1\{\hat{h}\}+a_0p^+ + ia_0\{\hat{g}\}\right)\\
=& \frac{-i\big((1-\mu)r+\sqrt{a}|s|\big)(a_3+a_1\theta_1+ia_0\theta_2)}{(1-\mu)rs+(\sqrt{a}|s|-iks)s}(is)\\
& \qquad + \frac{1}{(1+ia_0k)}\left(a_1\{\hat{h}\}+a_0p^+ + ia_0\{\hat{g}\}\right)\\
=& \boxed{\frac{-\sqrt{r+\sqrt{a}|s|}}{(1-\mu)r+\sqrt{a}|s|-iks}\left((1-\mu)r-\sqrt{a}|s|-\theta_1+\frac{\theta_2}{\sqrt{r+\sqrt{a}|s|}}\right)}(is)\\
& \qquad + \frac{1}{(1+ia_0k)}\left(a_0p^+ + a_1\{\hat{h}\}+ia_0\{\hat{g}\}\right)
\end{aligned} \tag{33}$$

$$\begin{aligned}
s\xi =& a_1k(is)+a_1\theta_1+ia_0\theta_2+a_3+a_0p^+ + a_1\{\hat{h}\}+ia_0\{\hat{g}\}\\
=& \left(a_1k-\frac{ia_1\theta_1-a_0\theta_2+ia_3}{s}\right)(is)+a_0p^+ + a_1\{\hat{h}\}+ia_0\{\hat{g}\}\\
=& \boxed{\frac{-\sqrt{r+\sqrt{a}|s|}}{(1-\mu)r+\sqrt{a}|s|}\left((1-\mu)r-\sqrt{a}|s|-iks-\theta_1+\frac{\theta_2}{\sqrt{r+\sqrt{a}|s|}}\right)}(is)\\
& + a_0p^+ + a_1\{\hat{h}\}+ia_0\{\hat{g}\}
\end{aligned} \tag{33*}$$

Note that as long as $\theta_1 \in S^2_a$ and $\theta_2 \in S^3_a$ the (boxed) coefficients of $(is)$ in equations (33), (33*) have regularity $S^1_a$. Since $\partial_\nu\partial_t$ corresponds to the symbol $(\xi s)$, then from either expression conclude

$$\frac{\partial \tilde{z}_t}{\partial\nu} = M_1\tilde{z}_t + A_1h + A_0P^+\tilde{z} + iA_0g + \text{l.o.t.}(\tilde{z})$$

with

$$\text{Symb}(M_1)\in S^1_a(T^*\Sigma^{(1)}), \qquad \text{Symb}(A_1)\in S^1_a, \qquad \text{Symb}(A_0)\in S^0_a$$

and the lower order terms l.o.t.$(z)$ originating from those in $\mathcal{B}_1$, $\mathcal{B}_2$ and from commutators of operators corresponding to $S^m_{1,0}$ symbols.

**Step 4. Strong ellipticity of $M_1$.** Finally we construct the auxiliary feedbacks $\theta_i$ in order to ensure ellipticity of $M_1$. We may assume that $\theta_i$ are real. Then computing $\Re(\text{Symb}(M_1))$, which corresponds to the coefficient of $(is)$ in (33) or (33*), it becomes apparent that in order to have

$$\Re(\text{Symb}(M_1)) \geq C(\sqrt{r+|s|})$$

it suffices to ensure that

$$(1-\mu)r-\sqrt{a}|s|-\theta_1+\frac{\theta_2}{\sqrt{r+\sqrt{a}|s|}} \leq -c\sqrt{r+|s|}$$

at infinity in the $(s,\eta)$-plane. Here it becomes apparent that adding the correcting feedback via $\theta_2$ requires an extra derivative in time and space,

$$\theta_2 \sim \frac{\partial^3}{\partial\tau^3}\partial_t,$$

which is a viable strategy, however a simpler approach would be to choose

$$\theta_2 = 0, \qquad \theta_1 = (1-\mu+\delta)r$$

Thus $\theta_1$ microlocally corresponds to $-c_{\mu\delta}\partial_\tau^2$. Change to the original coordinates and "piece together" the identities from each coordinate patch. Accordingly redefine (preserving the order and ellipticity) the operators $M_1$, $A_0$, $A_1$ to obtain the statement of Lemma 4.1.

**Remark 4.2.** *It may be possible to extend the argument to the case $\delta = 0$ in (12). Then $M_1$ is not, strictly speaking, strongly elliptic since its symbol degenerates as $r\to\infty$ provided $|s|$ remains bounded; however, in this situation we enter the "elliptic" sector of the phase space where the tangential symbol $\eta^2$ dominates the time $|s|$. One may be able to exploit the elliptic regularity in order to obtain a priori estimates for this scenario.*

□

### 4.4 Parabolic smoothing from the interior

**Lemma 4.2.** *Suppose $z$ is a finite-energy time-integrable solution to* (19)*, with* $(21,22)$ *or* $(21^*,22^*)$*. Decompose $P = P^-P^+$ so that the principal symbol of $P^+$ is the polynomial $p^+$ in the factorization (25). Assuming* $\mathrm{rhs}(\cdot)$ *corresponds to a linear operator in $OPS^m_a(\Sigma^{(1)})$, the following estimate holds:*

$$|A_0P^+z|_{-\frac{1}{2},a,\Sigma^{(1)}} \le C\left(\|z\|_{2,a,Q} + \|\,\mathrm{rhs}(z)\|_{H_a^{0,-1}(Q)} + l.o.t.(z)\right) \tag{34}$$

*Proof.* The result is a special case of [Tat1, Lemma 3.4]. In order to have a self-contained presentation we provide a specialization of that argument to our case (see also [Las1]). Recall that $A_0 \in S^0_a$, so the task is to estimate $P^+z$. As before carry out the analysis on the half-space problem (23)

$$P\tilde{z} = F\tilde{z} + [[P-F,\,\phi]]z =: \mathcal{R}(z)$$

hence

$$P^-P^+\tilde{z} = \mathcal{R}(z) + \mathrm{l.o.t.}(z)$$

For a fixed $y$, set $v(x) = iP^+\tilde{z}(x,y)$ and without loss of generality ignore the lower order terms, arriving at:

$$-iP^-v = \mathcal{R}(z)$$

Since the principal symbol of $-iP^-$ is $-ip^- = -i\xi + \sqrt{r+\sqrt{a}|s|}$, we have a parabolic system in variable $x$ with evolution directed along the outward normal ($\nu = -1$ in these local). Denote $v' = -\partial_x v$

$$v' + \Lambda_1 v = \mathcal{R}(z), \qquad\qquad \Lambda_1 \sim \sqrt{r+\sqrt{a}\,|s|} \in S^1_a \tag{35}$$

For the half-space formulation corresponding to a collar of the boundary we can fix $x$ and take the inner product of (35) with $v(x)$ in $H_a^p(\mathbf{\Sigma}^{(1)})$:

$$
\begin{aligned}
-\frac{d}{dx}|v|^2_{p,a,\mathbf{\Sigma}^{(1)}} + 2\left|\Lambda_1^{1/2}v\right|^2_{p,a,\mathbf{\Sigma}^{(1)}} &= 2\langle \mathcal{R}(z), v\rangle_{p,a,\mathbf{\Sigma}^{(1)}} \\
&\leq \left|\Lambda_1^{1/2}\mathcal{R}(z)\right|^2_{p,a,\mathbf{\Sigma}^{(1)}} + \left|\Lambda_1^{-1/2}v\right|^2_{p,a,\mathbf{\Sigma}^{(1)}}
\end{aligned}
$$

Recall from Section 1.4 that $H_a^{0,p}(\mathbf{Q}) := L^2\left(0,\infty; H_a^p\left(\mathbf{\Sigma}^{(1)}\right)\right)$. Integrate on $x \in (0,\infty)$

$$
|v(0)|^2_{p,a,\mathbf{\Sigma}^{(1)}} + 2\left\|\Lambda_1^{1/2}v\right\|^2_{H_a^{0,p}} \leq \left\|\Lambda_1^{-1/2}\mathcal{R}(z)\right\|^2_{H_a^{0,p}} + \left\|\Lambda_1^{1/2}v\right\|^2_{H_a^{0,p}}.
$$

From here

$$
|v(0)|^2_{p,a,\mathbf{\Sigma}^{(1)}} \leq \left\|\Lambda_1^{-1/2}\mathcal{R}(z)\right\|^2_{H_a^{0,p}}.
$$

Since $\Lambda_1$ is a first-order (anisotropic) strongly elliptic operator on $\mathbf{\Sigma}^{(1)}$, get

$$
|P^+\tilde{z}|_{p,a,\mathbf{\Sigma}^{(1)}} = |v(0)|^2_{p,a,\mathbf{\Sigma}^{(1)}} \leq \|\mathcal{R}(z)\|^2_{H_a^{0,p-1/2}(\mathbf{Q})}.
$$

Set $p=-1/2$; the principal part is given by the commutator $[\![P,\phi]\!]z$ which we need to estimate in $H_a^{0,-1}(\mathbf{Q})$ (recall that $z$ here is just a shorthand for $z\big|_{U\cap\Omega}\circ\psi^{-1}$ for a local coordinate map $\psi$). Since $\phi$ is time-independent $[\![P,\phi]\!] = [\![P_0,\phi]\!]$ and the principal symbol of $P_0$ is $(\xi^2+r)^2$.

**Remark 4.3.** Formally $P_0$ *is of (anisotropic) order* 4*, hence the commutator could be thought of as being in* $OPS_a^3$ *and the desired norm would then coincide with the space* $L^2(\mathbb{R}_x^+;S_a^2)$ *which corresponds to the* $L^2(\mathbb{R}_x^+)$ *bound on one time and two space derivatives, as desired.*

To make the argument rigorous, however, directly estimate the normal derivatives in the principal symbol of the commutator (up to the order 3, since $\phi$ is smooth):

$$
\|\,[\![P_0,\phi]\!]z\,\|^2_{H_a^{0,-1}(\mathbf{\Sigma}^{(1)})} \leq C\sum_{i=0}^{3}|\tilde{z}|^2_{H^i\left(\mathbb{R}_x^+;H_a^{-1}(\mathbf{\Sigma}^{(1)})\right)} \leq C|\tilde{z}|^2_{H_a^{3,-1}(\mathbf{Q})}.
$$

The latter norm corresponds to the space $H^{(3,p=-1)}$ of [Tat1, (1.11)]; now we can use [Tat1, Appendix B, and Proposition B.9a] since $z$ satisfies the problem $Pz=\text{rhs}$, to infer that

$$
\|\,[\![P_0,\phi]\!]z\,\|^2_{H_a^{(3,-1)}(\mathbf{Q})} \leq C\|z\|^2_{H_a^2(\mathbf{Q})} + \|\,\text{rhs}(z)\|_{H_a^{0,-1}(\mathbf{Q})}.
$$

Now pass to the original coordinates to obtain the statement of the lemma. Note that the coordinate patches away from the boundary will create additional lower order terms. $\square$

# 5 PROOFS II: Estimates for the backward adjoint problem

**Lemma 5.1.** *Assume $z$ is a finite-energy solution to the adjoint problem* (19)*, with* $(21,22)$ *or* $(21^*,22^*)$ *with* $\mathrm{rhs}(z)\equiv 0$*. There exists a constant $C_{t,T}>0$ such that*

$$|z_t|_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}}+|z|_{\frac{5}{2},a,\Sigma^{(1)}_{t,T}}+\left|\frac{\partial z_t}{\partial\nu}\right|_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}}+\left|\frac{\partial z}{\partial\nu}\right|_{\frac{3}{2},a,\Sigma^{(1)}_{t,T}}\leq C_{t,T}\,\mathbf{bt}_{t,T} \tag{36}$$

$$|\mathcal{B}_1 z|_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}}+|\mathcal{B}_2 z|_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}}\leq C_{t,T}\,\mathbf{bt}_{t,T} \tag{37}$$

*and*

$$E_z(t)\leq C_{t,T}\,\mathbf{bt}_{t,T} \tag{38}$$

*where*

$$\mathbf{bt}_{t,T}:=|h|_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}}+|g|_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}}$$

The rest of this section will be devoted to the proof of Lemma 5.1. Overall one can subdivide the argument into two major parts: (i) derive the fundamental identity which sets up the foundation the estimates and identify the "critical" terms present therein which must be assessed in order to verify the statement of the Lemma; and part (ii): the estimates themselves, taking advantage of Lemmas 4.1 and 4.2.

## 5.1 Proving Lemma 5.1 - Part I: Fundamental inequalities

### 5.1.1 Step 1: Introducing decay in time and the frequency cutoff

To take advantage of Lemmas 4.1 and 4.2 we must ensure $L^2$ integrability of the solution in time. In addition, the phase space will be (microlocally) partitioned into "elliptic" and "hyperbolic" sectors and the final estimates will be pieced together from the information obtained in each subregion. For these purposes introduce respectively an exponential weight $e^{\gamma t}$ with a large parameter $\gamma>0$ (for the backward-in-time problem), and let $\boldsymbol{\chi}$ denote a zero-order operator, which (up to a change of coordinates) resides in the class $OPS^0_a(T^*\boldsymbol{\Sigma})$, i.e. $\boldsymbol{\chi}$ could be thought of a tangential near the boundary with arbitrary smooth extension to the interior. Introduce

$$\bar{z}:=e^{\gamma t}z,\qquad v:=\boldsymbol{\chi}z$$

and combining the two

$$\bar{v}:=e^{\gamma t}\boldsymbol{\chi}z$$

Similarly let

$$\bar{h}=e^{\gamma t}h,\qquad \bar{g}=e^{\gamma t}g$$

This proof will focus on the boundary conditions (21)-(22) since the analysis for (21$^*$)-(22$^*$) is analogous. Apply $e^{\gamma t}\boldsymbol{\chi}$ to the adjoint system (19), with $(21,22)$ or $(21^*,22^*)$ with $\mathrm{rhs}(z)\equiv 0$:

$$\begin{aligned}
\boxdot\bar{v}+[[e^{\gamma t}\boldsymbol{\chi},\,\boxdot]]z &= 0\\
\mathcal{B}_1\bar{v}+[[\boldsymbol{\chi},\,\mathcal{B}_1]]\bar{z} &= \mathcal{C}^*\bar{v}+[[\boldsymbol{\chi},\,\mathcal{C}^*]]\bar{z}+\boldsymbol{\chi}\bar{h}\\
\mathcal{B}_2\bar{v}+[[\boldsymbol{\chi},\,\mathcal{B}_2]]\bar{z} &= -k\left(\partial_\nu\bar{v}_t+[[e^{\gamma t}\boldsymbol{\chi},\,\partial_t\partial_\nu]]z\right)+\boldsymbol{\chi}\bar{g}
\end{aligned}$$

Rewrite the commutator terms:

$$[\![e^{\gamma t}\boldsymbol{\chi},\boxdot]\!]z = [\![e^{\gamma t},\partial_{tt}]\!]v - [\![\boldsymbol{\chi},\Delta^2]\!]\bar{z} = -2\gamma\bar{v}_t + \gamma^2\bar{v} - [\![\boldsymbol{\chi},\Delta^2]\!]\bar{z}$$

and

$$\begin{aligned}[\![e^{\gamma t}\boldsymbol{\chi},\partial_t\partial_\nu]\!]z =& e^{\gamma t}\boldsymbol{\chi}\partial_\nu z_t - \partial_\nu\bar{v}_t = e^{\gamma t}\left(\partial_\nu v_t + [\![\boldsymbol{\chi},\partial_\nu]\!]z_t\right) - \partial_\nu\bar{v}_t = \\ =& \partial_\nu(e^{\gamma t}v_t) + [\![\boldsymbol{\chi},\partial_\nu]\!](e^{\gamma t}z_t) - \partial_\nu\bar{v}_t \\ =& -\gamma\partial_\nu\bar{v} + [\![\boldsymbol{\chi},\partial_\nu]\!](\bar{z}_t - \gamma\bar{z}) = -\gamma\partial_\nu\bar{v},\end{aligned}$$

where in the last step we used the fact that on the boundary $\boldsymbol{\chi}$ acts as a tangential operator (by construction). Thus, $\bar{v}$ satisfies the following system

$$\begin{aligned}\boxdot\bar{v} - 2\gamma\bar{v}_t + \gamma^2\bar{v} &= \mathbf{K}_3\bar{z} && (39)\\ \mathcal{B}_1\bar{v} &= \mathcal{C}^*\bar{v} + \boldsymbol{\chi}\bar{h} + \mathbf{K}_1\bar{z} && (40)\\ \mathcal{B}_2\bar{v} &= (k\gamma)(\partial_\nu\bar{v}) - k(\partial_\nu\bar{v}_t) + \boldsymbol{\chi}\bar{g} + \mathbf{K}_2\bar{z} && (41)\\ \bar{v}(T) = \bar{v}_t(T) &= 0. && (42)\end{aligned}$$

Operators $\mathbf{K}_i$ denote the commutators:

$$\begin{aligned}\mathbf{K}_1 &= [\![\boldsymbol{\chi},\mathcal{C}^* - \mathcal{B}_1]\!] && (43)\\ \mathbf{K}_2 &= [\![\mathcal{B}_2,\boldsymbol{\chi}]\!]. && (44)\end{aligned}$$

As for $\mathbf{K}_3$, recall that $\boldsymbol{\chi}$ is tangential operator (and locally smoothing, e.g. vanishing in the interior) consequently $[\![\chi,\xi]\!] = 0$, so $\mathbf{K}_3$ is a third order operator in tangential and time variables only

$$\mathbf{K}_3 = [\![\boldsymbol{\chi},\Delta^2]\!] \in_{\text{loc}} OPS^3_a(\boldsymbol{\Sigma}) \tag{45}$$

From the principal symbols of $\mathcal{B}_1$, $\mathcal{B}_2$ ((27), (28)) and $[\![\boldsymbol{\chi},\xi]\!] = 0$ derive

$$\mathbf{K}_1 \in OPS^1_a(T^*\Sigma) \qquad \text{and} \qquad \mathbf{K}_2 = \frac{\partial}{\partial\nu}\circ\left[OPS^1_a(T^*\Sigma)\right]. \tag{46}$$

### 5.1.2 Step 2: The fundamental identity for the $\bar{v}$-system.

At this stage use standard multiplier techniques to relate the interior and boundary dynamics. Multiply (39) - (42) by $\bar{v}_t$ in $L^2(\Omega)$ to procure an energy identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\bar{v}_t\|^2_{0,\Omega} + a(\bar{v},\bar{v}) + \gamma^2\|\bar{v}\|^2_{0,\Omega}\right) - 2\gamma\|\bar{v}_t\|^2_{0,\Omega} + \langle\mathcal{B}_2\bar{v},\bar{v}_t\rangle - \left\langle\mathcal{B}_1\bar{v},\frac{\partial\bar{v}_t}{\partial\nu}\right\rangle =\\ = (\mathbf{K}_3\bar{z},\bar{v}_t)\end{aligned} \tag{47}$$

Recall that by assumption $\bar{v}$ is a semigroup solution (vanishing for $t > T$), so for $\gamma$ sufficiently large $\bar{v}$ tends to 0 as $t \to -\infty$. Integration in time yields

$$\int_{\mathbb{R}_t} 2\gamma\|\bar{v}_t\|^2_{0,\Omega} = \int_{\mathbb{R}_t}\left[\langle\mathcal{B}_2\bar{v},\bar{v}_t\rangle - \left\langle\mathcal{B}_1\bar{v},\frac{\partial\bar{v}_t}{\partial\nu}\right\rangle\right] - \int_{\mathbb{R}_t}(\mathbf{K}_3\bar{z},\bar{v}_t). \tag{48}$$

Similarly, multiplication by $\bar{v}$ produces the "equipartition" equation:

$$\int_{\mathbb{R}_t} \|\bar{v}_t\|_{0,\Omega}^2 = \int_{\mathbb{R}_t} a(\bar{v},\bar{v}) + \gamma^2 \int_{\mathbb{R}_t} \|\bar{v}\|_{0,\Omega}^2 + \int_{\mathbb{R}_t} \left[ \langle \mathcal{B}_2 \bar{v}, \bar{v} \rangle - \left\langle \mathcal{B}_1 \bar{v}, \frac{\partial \bar{v}}{\partial \nu} \right\rangle \right] - \int_{\mathbb{R}_t} (\mathbf{K}_3 \bar{z}, \bar{v}). \quad (49)$$

Multiply (49) by $\gamma$ and substitute into the energy identity (48):

$$\boxed{\begin{aligned} \gamma \int_{\mathbb{R}_t} \left[ \|\bar{v}_t\|^2 + a(\bar{v},\bar{v}) \right] + \gamma^3 \int_{\mathbb{R}_t} \|\bar{v}\|_{0,\Omega}^2 = \int_{\mathbb{R}_t} \left[ \langle \mathcal{B}_2 \bar{v}, \bar{v}_t - \gamma \bar{v} \rangle - \left\langle \mathcal{B}_1 \bar{v}, \frac{\partial \bar{v}_t}{\partial \nu} - \gamma \frac{\partial \bar{v}}{\partial \nu} \right\rangle \right] \\ - \int_{\mathbb{R}_t} (\mathbf{K}_3 \bar{z}, \bar{v}_t - \gamma \bar{v}). \end{aligned}} \quad (50)$$

The next step will be to obtain bounds on the RHS of the identity (50). We will exploit the strong ellipticity of positive operator $M_1$ (from Lemma 4.1) and the gain in smoothness provided by Lemma 4.2, which altogether yield, roughly speaking, one anisotropic unit of "hidden" regularity. *Formally* one could express it with the following table:

| Finite energy level (apply space trace first) | Monotone trace $M_1$ and smoothing of $P^+$ "+1" anisotropic unit on $z_t$ and $P^+z$ | Maximal corresponding boundary regularity (dual to $z_t \in H_a^{1/2}$) |
|---|---|---|
| $z_t \in H_a^{-1/2}(\Sigma^{(1)})$ | $z_t \in H_a^{1/2}$ | $A_1 h \in H_a^{-1/2} \Rightarrow h \in H_a^{1/2}$ |
| $\partial_\nu z_t \in H_a^{-3/2}$ | $\partial_\nu z_t \in H_a^{-\frac{1}{2}}$ | $A_0 g \in H_a^{-\frac{1}{2}} \Rightarrow g \in H_a^{-\frac{1}{2}}$ |
| $P^+ z \in H_a^{-\frac{3}{2}}(\Sigma^{(1)})$ | $P^+ z \in H_a^{-1/2}$ | |

#### 5.1.3 Step 3. Monotone branch: operator $M_1$

First, expand the boundary operators on the RHS of (50). From the condition (41) obtain

$$\langle \mathcal{B}_2 \bar{v}, \bar{v}_t - \gamma \bar{v} \rangle = \left\langle (k\gamma)(\partial_\nu \bar{v}) + \boldsymbol{\chi} \bar{g} + \mathbf{K}_2 \bar{z}, \bar{v}_t - \gamma \bar{v} \right\rangle - k \left\langle \frac{\partial \bar{v}_t}{\partial \nu}, \bar{v}_t - \gamma \bar{v} \right\rangle. \quad (51)$$

Apply the decomposition in Lemma 4.1, in which we need to

- replace "$h$" in Lemma 4.1 with $\boldsymbol{\chi} \bar{h} + \mathbf{K}_1 \bar{z}$,
- replace "$g$" in Lemma 4.1 with $\boldsymbol{\chi} \bar{g} + k\gamma \dfrac{\partial \bar{v}}{\partial \nu} + \mathbf{K}_2 \bar{z}$,

to conclude (omitting the lower order terms):

$$\frac{\partial \bar{v}_t}{\partial \nu} = M_1 \bar{v}_t + A_1 \left( \boldsymbol{\chi} \bar{h} + \mathbf{K}_1 \bar{z} \right) + A_0 P^+ \bar{v} - i A_0 \left( \boldsymbol{\chi} \bar{g} + k\gamma \frac{\partial \bar{v}}{\partial \nu} + \mathbf{K}_2 \bar{z} \right), \quad (52)$$

$$-k\left\langle \frac{\partial \bar v_t}{\partial \nu}, \bar v_t \right\rangle = -k\left\langle M_1 \bar v_t,\ \bar v_t \right\rangle + \left\langle \ldots,\ \bar v_t \right\rangle.$$

Likewise, from the boundary condition (40) and Lemma 4.1 get

$$\begin{aligned} -\left\langle \mathcal{B}_1 \bar v, \frac{\partial \bar v_t}{\partial \nu} \right\rangle = -\Big\langle & C^* \bar v + \boldsymbol{\chi} \bar h + \mathbf{K}_1 \bar z, \\ & M_1 \bar v_t + A_1 \left( \boldsymbol{\chi} \bar h + \mathbf{K}_1 \bar z \right) + A_0 P^+ \bar v - i A_0 \left( \boldsymbol{\chi} \bar g + k\gamma \frac{\partial \bar v}{\partial \nu} + \mathbf{K}_2 \bar z \right). \Big\rangle \end{aligned} \tag{53}$$

By Lemma 4.1 operator $M_1$ is positive strongly elliptic of (anisotropic) order 1. Use Gårding's inequality to claim that there exists a constant $c > 0$ for which

$$\boxed{c|\bar v_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} \ \leq \int_{\mathbb{R}_t} \Re\langle M_1 \bar v_t, \bar v_t \rangle + \text{l.o.t.}} \tag{54}$$

From the term (51) it follows that the coefficient of $M_1$ is *negative* ($= -k$) on the RHS (50). Thus, after extracting the real parts on each side of the identity, we can pass to an estimate which in the next step yields a bound on the energy *and* on $|\bar v_t|_{\frac{1}{2},a,\Sigma^{(1)}}$.

#### 5.1.4 Step 4. Fundamental inequality

Taking into the account the orders of the operators: $C^* \in OPS^2_a$, $M_1 \in OPS^1_a$, $A_i \in OPS^i_a$, $\mathbf{K}_1 \in OPS^1_a$ and $K_2 = \frac{\partial}{\partial \nu} \circ OPS^1_a$, we obtain from (50) - (53) and the monotone estimate (54), the following inequality (the parameter $\varepsilon$ will be specified later)

$$\begin{aligned} &\gamma \int\limits_{\mathbb{R}_t} \left[ \|\bar v_t\|^2 + a(\bar v, \bar v) \right] + \gamma^3 \int\limits_{\mathbb{R}_t} \|\bar v\|^2_{0,\Omega} + kc \int\limits_{\mathbb{R}_t} |\bar v_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} \leq \\ &\leq -\int_{\mathbb{R}_t} \Re\langle C^* \bar v, M_1 \bar v_t \rangle + \ldots \end{aligned} \tag{55}$$

$$\cdots + \Re \int_{\mathbb{R}_t} \left\{ \begin{array}{rl} k\gamma & \langle \partial_\nu \bar v, \bar v_t \rangle \\ + & \langle \boldsymbol{\chi} \bar g, \bar v_t \rangle \\ + & \langle \mathbf{K}_2 \bar z, \bar v_t \rangle \\ -k & \langle A_1 \boldsymbol{\chi} \bar h, \bar v_t \rangle \\ -k & \langle A_1 \mathbf{K}_1 \bar z, \bar v_t \rangle \\ -k & \langle A_0 P^+ \bar v, \bar v_t \rangle \\ +ik & \langle A_0 \boldsymbol{\chi} \bar g, \bar v_t \rangle \\ +ik^2\gamma & \langle A_0 \partial_\nu \bar v, \bar v_t \rangle \\ +ik & \langle A_0 \mathbf{K}_2 \bar z, \bar v_t \rangle \end{array} \right\} \lesssim O\left(\tfrac{1}{\varepsilon}\right) \begin{pmatrix} \gamma^2 \left| \frac{\partial \bar v}{\partial \nu} \right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \\ |\bar g|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \\ |\bar h|^2_{\frac{1}{2},a,\Sigma^{(1)}} \\ |A_0 P^- \bar v|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \\ \left| \frac{\partial \bar z}{\partial \nu} \right|^2_{\frac{1}{2},a,\Sigma^{(1)}} \\ |z|^2_{\frac{3}{2},a,\Sigma^{(1)}} \end{pmatrix} + \varepsilon |\bar v_t|^2_{\frac{1}{2},a,\Sigma^{(1)}}$$

$$\cdots + \Re \int_{\mathbb{R}_t} \left\{ \begin{array}{ll} - & \left\langle \begin{array}{c} C^* \bar{v} \\ +\boldsymbol{\chi}\bar{h} \\ +\mathbf{K}_1 \bar{z} \end{array}, M_1 \bar{v}_t \right\rangle \\ & \\ - & \left\langle \ldots, A_1 \boldsymbol{\chi} \bar{h} \right\rangle \\ - & \left\langle \ldots, A_1 \mathbf{K}_1 \bar{z} \right\rangle \\ - & \left\langle \ldots, A_0 P^+ \bar{v} \right\rangle \\ +i & \left\langle \ldots, A_0 \boldsymbol{\chi} \bar{g} \right\rangle \\ +ik\gamma & \left\langle \ldots, A_0 \partial_\nu \bar{v} \right\rangle \\ +i & \left\langle \ldots, A_0 \mathbf{K}_2 \bar{z} \right\rangle \\ +\gamma & \left\langle \ldots, \partial_\nu \bar{v} \right\rangle \end{array} \right\} \lesssim O\left(\tfrac{1}{\varepsilon}\right) \begin{pmatrix} |\bar{h}|^2_{\frac{1}{2},a,\Sigma^{(1)}} \\ |\bar{g}|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \\ \left( |\bar{z}|^2_{\frac{1}{2},a,\Sigma^{(1)}} \right) \\ |\bar{z}|^2_{\frac{3}{2},a,\Sigma^{(1)}} \\ |A_0 P^+ \bar{v}|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \\ \gamma^2 \left| \frac{\partial \bar{v}}{\partial \nu} \right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \end{pmatrix} + \varepsilon \begin{pmatrix} |\bar{v}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} \\ \\ \overbrace{|C^* \bar{v}|_{\frac{1}{2},a,\Sigma^{(1)}}}^{\lesssim |\bar{v}|^2_{\frac{5}{2},a,\Sigma^{(1)}}} \end{pmatrix}$$

$$\cdots + \Re \int_{\mathbb{R}_t} \left\{ \begin{array}{c} -(\mathbf{K}_3 \bar{z}, \bar{v}_t) \\ +[\text{terms with } (\gamma \bar{v})] \end{array} \right\} \qquad \lesssim -\textstyle\int_{\mathbb{R}_t} \Re(\mathbf{K}_3 \bar{z}, \bar{v}_t) + O(\gamma)\text{l.o.t.}$$

The relation $\lesssim$ will henceforth indicate that the right-hand side is dominant when multiplied by a sufficiently large constant $C$ which is independent of the critical parameters (such as $\gamma$ or $\varepsilon$, for example). It may be easier to think of these estimates as being dual to the "hidden" regularity $\bar{v}_t \in H_a^{1/2}(\Sigma^{(1)})$ and the boundary data $\bar{g} \in H_a^{-1/2}(\Sigma^{(1)}), \bar{h} \in H_a^{1/2}(\Sigma^{(1)})$.

The product $\langle C^* \bar{v}, M_1 \bar{v}_t \rangle$ will require a special treatment, since if one formally replaces two space derivatives in $C^*$ by the anisotropic equivalent of one time derivative, at best this term will be of the order $|\bar{v}_t|^2_{\frac{1}{2},a,\Sigma}$. Since only a limited amount of the hidden regularity can be accommodated (as dictated by the monotone estimate (54)), the direct bound may be insufficient in general for this critical term. Instead take advantage of the fact that both $C^*$ and $M_1$ are strongly elliptic operators with positive principal parts.

Now further adjust (55):

- Note that from (52) it follows

$$\begin{aligned} \left| \frac{\partial \bar{v}_t}{\partial \nu} \right|_{-\frac{1}{2},a,\Sigma^{(1)}} \lesssim & |\bar{v}_t|_{\frac{1}{2},a,\Sigma^{(1)}} + |\bar{h}|_{\frac{1}{2},a,\Sigma^{(1)}} + |\bar{z}|_{\frac{3}{2},a,\Sigma^{(1)}} \\ & + |A_0 P^+ \bar{v}|_{-\frac{1}{2},a,\Sigma^{(1)}} + |\bar{g}|_{\frac{1}{2},a,\Sigma^{(1)}} \\ & + O(\gamma) \left| \frac{\partial \bar{v}}{\partial \nu} \right|_{-\frac{1}{2},a,\Sigma^{(1)}} + \left| \frac{\partial z}{\partial \nu} \right|_{\frac{1}{2},a,\Sigma^{(1)}} \end{aligned} \tag{56}$$

  where the norms on the RHS are of the same order as on the RHS of (55). Consequently we can add a small multiple (of order $O(\varepsilon)$) of inequality (56) to (55) without introducing new terms on the RHS of the latter, and still preserving order $O(\varepsilon)$ of the coefficient of the term $|\bar{v}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}}$ on the right-hand side.

- Recall that to verify the statement of Lemma 5.1 the norms $\left| \frac{\partial \bar{z}}{\partial \nu} \right|_{\frac{3}{2},a,\Sigma^{(1)}}$ and $|\bar{z}|_{\frac{5}{2},a,\Sigma^{(1)}}$ must be analyzed as well. To keep all the estimates together, let us add to each side of the inequality obtained in the last step the term $\varepsilon^2 \left|\frac{\partial \bar{z}}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}} + \varepsilon|\bar{v}|_{\frac{5}{2}}$. Jumping ahead we mention that the smaller coefficient $\varepsilon^2$ is chosen so that one could eventually absorb the resulting bounding terms into the term $|\partial_\nu \bar{v}_t|$ (which, in turn, has coefficient $O(\varepsilon)$).

As a result get an inequality of the form:

$$\begin{aligned}\gamma \int_{\mathbb{R}_t} \left[\|\bar{v}_t\|^2 + a(\bar{v},\bar{v})\right] + \gamma^3 \int_{\mathbb{R}_t} \|\bar{v}\|^2_{0,\Omega} + kc|\bar{v}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} + \\ +\varepsilon \left|\frac{\partial \bar{v}_t}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + \varepsilon|\bar{v}|^2_{\frac{5}{2},a,\Sigma^{(1)}} + \varepsilon^2 \left|\frac{\partial \bar{v}}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}} \leq \\ \leq \mathfrak{N}_{\gamma,\varepsilon} + \varepsilon|\bar{v}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} + O(\varepsilon^{-1})\left(|\bar{g}|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + |\bar{h}|^2_{\frac{1}{2},a,\Sigma^{(1)}}\right) + \text{l.o.t.}\end{aligned} \tag{57}$$

Where

$$\begin{aligned}\mathfrak{N}_{\gamma,\varepsilon} \approx\ & O(\varepsilon^{-1})|A_0 P^+ \bar{v}|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + \varepsilon|\bar{v}|^2_{\frac{5}{2},a,\Sigma^{(1)}} + O(\varepsilon^{-1})|\bar{z}|^2_{\frac{3}{2},a,\Sigma^{(1)}} \\ & + \varepsilon \left|\frac{\partial \bar{z}}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}} + O\left(\varepsilon^{-1}\gamma^2\right) \left|\frac{\partial \bar{v}}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + O(\varepsilon^{-1}) \left|\frac{\partial z}{\partial \nu}\right|^2_{\frac{1}{2},a,\Sigma^{(1)}} \\ & - \Re \int_{\mathbb{R}_t} \left[\langle C^* \bar{v}, \bar{v}_t\rangle + (\mathbf{K}_3 \bar{z}, \bar{v}_t)\right]\end{aligned}$$

The the cutoff $\boldsymbol{\chi}$ added to the solution (with $\bar{v} = \boldsymbol{\chi}\bar{z}$) permits to (micro) localize the of study of $\mathfrak{N}_{\gamma,\varepsilon}$. We will choose an open cover of the phase space and consider different values for operator $\boldsymbol{\chi}$ supported so that when combined they partition the identity:

$$\boldsymbol{\chi} := \boldsymbol{\chi}_\alpha, \qquad \sum_\alpha \boldsymbol{\chi}_\alpha = I$$

for $\alpha$ in some finite index set. In this setting it is possible to estimate terms involving $\bar{z}$ by separately analyzing $\boldsymbol{\chi}_\alpha \bar{z}$ for each value of $\chi$ (except the case when $\bar{z}$ is acted upon by a $\chi$-dependent commutator $\mathbf{K}_3$). Since the trace estimates pose the grater challenge, the subsequent argument will focus on $\boldsymbol{\chi}$ whose projected support corresponds to a $\Gamma_1$-collar of the domain, i.e. patches in some neighborhood of the boundary $\Gamma_1$.

Summarizing: the goal so far is to obtain bounds on the term $\mathfrak{N}_{\gamma,\varepsilon}$, namely the norms and products:

$$\boxed{\begin{array}{l} |A_0 P^+ \bar{v}|_{-\frac{1}{2},a,\Sigma^{(1)}}, \quad |\bar{v}|_{\frac{3}{2},a,\Sigma^{(1)}}, \quad \varepsilon|\bar{v}|_{\frac{5}{2},a,\Sigma^{(1)}} \\ \varepsilon^2 \left|\frac{\partial \bar{v}}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}}, O\left(\gamma^2\right) \left|\frac{\partial \bar{v}}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}}, \quad \left|\frac{\partial \bar{v}}{\partial \nu}\right|_{\frac{1}{2},a,\Sigma^{(1)}}, \\ \Re \int_{\mathbb{R}_t} \langle C^* \bar{v}, M_1 \bar{v}_t\rangle, \qquad \Re \int_{\mathbb{R}_t} (\mathbf{K}_3 \bar{z}, \bar{v}_t) \end{array}} \tag{58}$$

for the cases $\boldsymbol{\chi} = \boldsymbol{\chi}_\alpha$ ( $\sum_\alpha \boldsymbol{\chi}_\alpha = I$ restricted to a neighborhood of $\Gamma_1$), via:

1. The finite energy (e.g. $\|\bar{v}\|_{2,a,Q}$ or $\|\bar{z}\|_{2,a,Q}$) scaled by a parameter $\ll \gamma$;

2. lower order terms, which we can always interpolate as a small fraction of the energy plus $C_\gamma \|\bar{v}\|_{0,Q}$ for sufficiently large $C_\gamma$. Here we will not outright take advantage of the coefficient $\gamma^3$ on the LHS of (57), and will need no restriction on the order of the coefficient of the lower order norms;

3. the hidden regularity due to the monotone part: $\|\bar{v}_t\|_{\frac{1}{2},a,\Sigma^{(1)}}$ with a coefficient proportional to parameters of smaller order than constant $c$ in (54);

4. The normal velocity trace $\left|\frac{\partial \bar{v}_t}{\partial \nu}\right|_{-\frac{1}{2},a,\Sigma^{(1)}}$ with a coefficient $\ll \varepsilon$;

5. The boundary data in the spaces dual to the order of the "hidden" regularity, namely the norms $|\bar{h}|_{\frac{1}{2},a,\Sigma^{(1)}}$ and $|\bar{g}|_{-\frac{1}{2},a,\Sigma^{(1)}}$.

#### 5.1.5 Step 5. Elliptic regularity

As already mentioned above, the estimates will be carried out for different values of the (tangential) cutoff operator $\boldsymbol{\chi}$. Denote the first case by $\boldsymbol{\chi}_{el}$. The projection (onto $Q$) of the support of $\boldsymbol{\chi}_{el}$ will fall into a neighborhood of the space time cylinder $\Sigma^{(1)}$, so whenever the point $(x,y) \in \Omega$ lies away from a collar of the boundary $\Gamma_1$ we can assume $\boldsymbol{\chi}_{el}(t,x,y) \equiv 0$. Near the boundary $\Sigma^{(1)}$ modulo a coordinate transformation $\boldsymbol{\chi}_{el}$ will be defined as an "elliptic" cutoff with the following symbol

$$\boldsymbol{\chi}_{el} \sim \chi_{el}(t,x,y;s,\eta) \in C^\infty\left(T^*_{(t;x,y)}\boldsymbol{\Sigma};[0,1]\right) \in S^0_a(T^*\boldsymbol{\Sigma})$$

With exception of a bounded neighborhood of the zero section of the cotangent bundle we define

$$\chi_{el}(s,\eta) := \begin{cases} 1 & |s| \leq \frac{1}{2}\varepsilon_0|\eta|^2 \\ 0 & |s| \geq \varepsilon_0|\eta|^2 \end{cases} \tag{59}$$

assuming $\chi_{el}$ is smooth near the zero section and in the transition region between the two parabolas in the fibers (see Figure 1).

On the support of the operator $\boldsymbol{\chi}_{el}$ with the symbol given by (59), the (principal) symbol representation (24) of the operator $\boxdot$ yields

$$p(x,y,\xi,\eta,s) = -a(x,y)s^2 + \left(\xi^2 + r(x,y,\eta)\right)^2 > -a(x,y)\varepsilon_0|\eta|^4 + (\xi^2 + r(x,y,\eta))^2$$

Since $r(x,y,\eta)$ is strongly elliptic there is a constant $r_0 > 0$ such that

$$r_0^2|\eta|^4 \leq r(x,y,\eta)^2$$

Consequently, if

$$\varepsilon_0^2 < \frac{r_0^2}{2\sup a(x,y)} \tag{60}$$

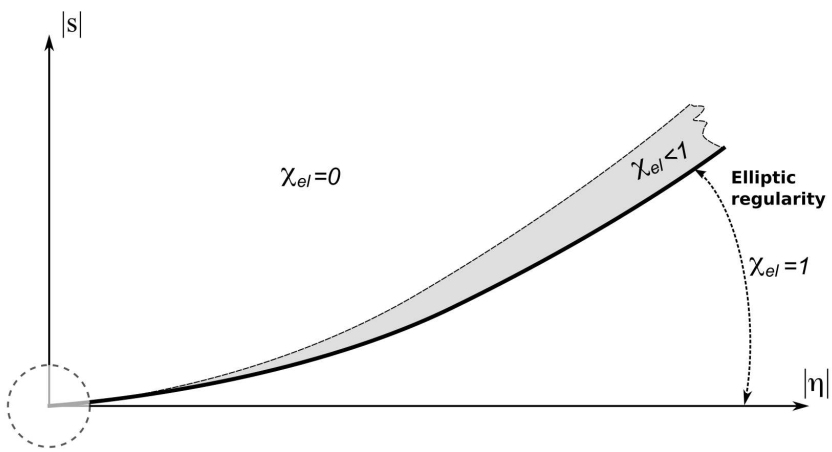


Figure 1: Microlocal partition of the cotangent bundle by the "elliptic" cutoff $\boldsymbol{\chi}_{el}$ (the definition can be altered in a bounded neighborhood of the zero section since such a change corresponds to a $C^\infty$ perturbation). It will be shown that on the regions of the phase space where $\boldsymbol{\chi}_{el}$ is strictly positive the solution $\bar{z}$ possesses (microlocal) $H^3$ regularity.

then for $(s,\eta) \in \operatorname{supp}\chi_{el}$ (for $\eta$ away from the origin)

$$\begin{aligned} p(x,y,\xi,\eta,s) =& \xi^4 + \left(\xi^2 r(x,y,\eta) + r(x,y,\eta)\xi^2\right) + r^2(x,y,\eta) - a(x,y)s^2 \\ >& \xi^4 + \left(\xi^2 r(x,y,\eta) + r(x,y,\eta)\xi^2\right) + \tfrac{1}{2}r^2(x,y,\eta) > \tfrac{1}{2}\left(\xi^2 + r(x,y,\eta)\right)^2 \end{aligned}$$

We obtain that for $\boldsymbol{\chi} = \boldsymbol{\chi}_{el}$, the corresponding solution $\bar{v} = \bar{v}^{el}$ satisfies the *elliptic* system (39) - (41)

$$P_0 \bar{v}^{el} = \operatorname{rhs}(\bar{v}^{el}, \bar{z})$$

where $\operatorname{Symb}(P_0) \sim (\xi^2 + r)^2$, i.e. $P_0 \sim \Delta^2$. Furthermore, now we can work with regular isotropic norms since since $H^r_a$ regularity on $\operatorname{supp}\boldsymbol{\chi}_{el}$ is equivalent to $r$ space derivatives.

Theorem 7.1 in the Appendix shows that this elliptic system system corresponding to the bi-laplacian and boundary operators $\{\mathcal{B}_1 - C^*, \mathcal{B}_2\}$ satisfies the Shapiro-Lopatinskii condition (e.g. see the L-condition in [Wl, p. 389]). Consequently the following elliptic estimate applies [LiMa, P. 188, Theorem 7.4]

$$\begin{aligned} \|\bar{v}^{el}\|_{m,Q} \lesssim & \|\mathbf{K}_3\bar{z} + 2\gamma\bar{v}^{el}_t - \gamma^2\bar{v}^{el}\|_{m-4,Q} \\ & + \left|\boldsymbol{\chi}_{el}\bar{h} + \mathbf{K}_1\bar{z}\right|_{m-2-\frac{1}{2},\Sigma^{(1)}} \\ & + \left|(k\gamma)\partial_\nu \bar{v}^{el} - k\partial_\nu \bar{v}^{el}_t + \boldsymbol{\chi}_{el}\bar{g} + \mathbf{K}_2\bar{z}\right|_{m-3-\frac{1}{2},\Sigma^{(1)}}, \end{aligned} \tag{61}$$

where we may take $m=3$ because the differential operators $\mathbf{K}_3$ and $\partial_t$ are tangential, whereas $\bar{v}^{el}$ has a priori regularity in $H^1_a$; hence the right-hand side in the interior has a well defined anisotropic derivative of order $-1$. Due to the support of the cutoff $\boldsymbol{\chi}_{el}$ (59) we can bound the time derivative by the tangential component:

$$|\bar{v}^{el}_t|_{\theta,\Sigma^{(1)}} \le O(\varepsilon_0)|\bar{v}^{el}|_{\theta+2,\Sigma^{(1)}} \le O(\varepsilon_0)\|\bar{v}^{el}\|_{\theta+\frac{5}{2},Q}, \tag{62}$$

$$\|\bar{v}_t^{el}\|_{\theta,Q} \leq O(\varepsilon_0)\|\bar{v}^{el}\|_{\theta+2,Q}. \tag{63}$$

Likewise:

$$\left|\frac{\partial \bar{v}^{el}}{\partial \nu}\right|_{\frac{3}{2},\Sigma^{(1)}} + \left|\frac{\partial \bar{v}_t^{el}}{\partial \nu}\right|_{-\frac{1}{2},\Sigma^{(1)}} \leq O(\varepsilon_0)\left|\frac{\partial \bar{v}^{el}}{\partial \nu}\right|_{\frac{3}{2},\Sigma^{(1)}} \leq O(\varepsilon_0)\|\bar{v}^{el}\|_{3,Q}. \tag{64}$$

In addition, since commutator $\mathbf{K}_3$ is a tangential operator, then

$$\|\mathbf{K}_3\bar{z}\|_{-1,Q} \lesssim \|\bar{z}\|_{2,Q}. \tag{65}$$

Similarly, taking into account $\mathbf{K}_2 = \frac{\partial}{\partial \nu} \circ \left[OPS^1(\Sigma^{(1)})\right]$

$$\|\mathbf{K}_2\bar{z}\|_{-\frac{1}{2},\Sigma^{(1)}} \lesssim \left|\frac{\partial \bar{z}}{\partial \nu}\right|_{\frac{1}{2},\Sigma^{(1)}} \lesssim \|\bar{z}\|_{2,Q}. \tag{66}$$

Now setting $m = 3$ in (61) along with (62)-(66) produces

$$\begin{aligned}\|\bar{v}^{el}\|_{3,Q} \lesssim\ & \|\bar{z}\|_{2,Q} + O(\gamma)\|\bar{v}^{el}\|_{1,Q} + O(\gamma^2)\|\bar{v}^{el}\|_{-1,Q} \\ & + |\bar{h}|_{\frac{1}{2},\Sigma^{(1)}} + |\bar{z}|_{\frac{3}{2},\Sigma^{(1)}} + |\bar{g}|_{-\frac{1}{2},\Sigma^{(1)}} \\ & + O(\gamma)\|\bar{v}^{el}\|_{1,Q} + \|\bar{z}\|_{2,Q}\end{aligned}$$

Interpolate the lower norms (below $H^3(Q)$) of $\bar{v}^{el}$ to obtain the *elliptic regularity estimate*

$$\boxed{\|\bar{v}^{el}\|_{3,Q} \lesssim \|\bar{z}\|_{2,Q} + O(\gamma^2)\|\bar{v}^{el}\|_{0,Q} + |\bar{h}|_{\frac{1}{2},\Sigma^{(1)}} + |\bar{g}|_{-\frac{1}{2},\Sigma^{(1)}}} \tag{67}$$

### 5.1.6 Step 6. Cutoff functions for the energy estimates

The cutoff $\boldsymbol{\chi}_{el}$, however, will not be directly used to carry out the energy estimates for the problem. Having established (67) it is now more convenient to change the partition. We shall consider cases denoted

$$\boldsymbol{\chi} = \boldsymbol{\chi}_I \quad \text{and} \quad \boldsymbol{\chi} = \boldsymbol{\chi}_{II}, \quad \text{where} \quad \boldsymbol{\chi}_{II} := I - \boldsymbol{\chi}_I$$

with the respective solutions named $\bar{v}^I$, $\bar{v}^{II}$. Similarly, use superscripts $I$ and $II$ to label the commutators $\mathbf{K}_j$ corresponding to each cutoff.

Let $\varepsilon_0$ be as defined for the elliptic cutoff $\boldsymbol{\chi}_{el}$ and now let the symbol $\chi_I$ satisfy (away from a bounded neighborhood of the zero section)

$$\chi_I := \begin{cases} 1 & |s| \leq \frac{1}{8}\varepsilon_0|\eta|^2 \\ 0 & |s| \geq \frac{1}{4}|\eta|^2 \end{cases} \tag{68}$$

Essentially $\boldsymbol{\chi}_I$ and $\boldsymbol{\chi}_{II}$ form the same decomposition of the domain into the "elliptic" and complementary "hyperbolic" parts, only now the *transition region* between the two falls into the set where the elliptic regularity holds according to (67), as illustrated in Figure 2. Such a "redundant" construction will help bound the commutators in the region when it comes to the trace estimates for the equation.

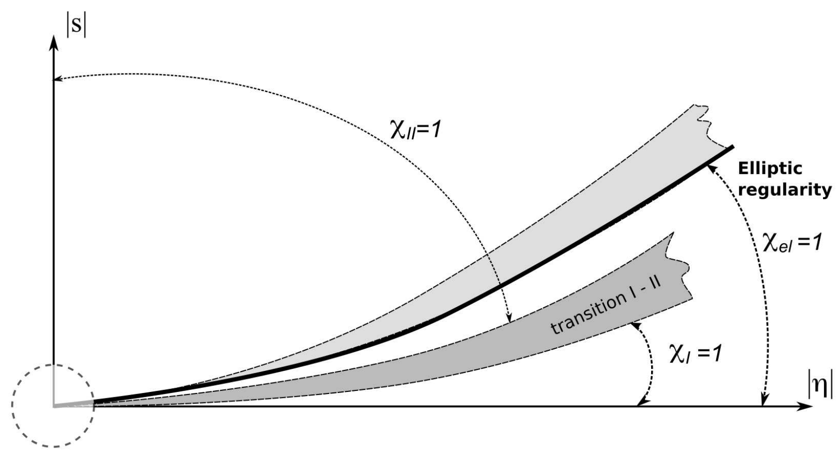


Figure 2: Partition of the phase space by the microlocal cutoffs $\boldsymbol{\chi}_I$ and $\boldsymbol{\chi}_{II}$.

The cutoff $\boldsymbol{\chi}_I$ has the properties analogous to $\boldsymbol{\chi}_{el}$, in particular, we can restate (62) - (67) for $\bar{v}^I$, obtaining:

$$
\begin{aligned}
|\bar{v}^I_t|_{\theta,\Sigma^{(1)}} + \|\bar{v}^I_t\|_{\theta,Q} &\leq O(\varepsilon_0)\|\bar{v}^I\|_{\theta+2,Q} && (69)\\
\left|\frac{\partial \bar{v}^I}{\partial \nu}\right|_{\frac{3}{2},\Sigma^{(1)}} &\leq O(\varepsilon_0)\|\bar{v}^I\|_{3,Q} && (70)\\
\|\bar{v}^I\|_{3,Q} &\lesssim \|\bar{z}\|_{2,Q} + O(\gamma^2)\|\bar{v}^I\|_{0,Q} + |\bar{h}|_{\frac{1}{2},\Sigma^{(1)}} + |\bar{g}|_{-\frac{1}{2},\Sigma^{(1)}}. && (71)
\end{aligned}
$$

Furthermore, on the *entire support* of $\boldsymbol{\chi}_I$ it is now possible to take advantage of the elliptic regularity estimate (67). Whereas on the complementing supp$(\boldsymbol{\chi}_{II})$ (just as on the complement of $\boldsymbol{\chi}_{el}$ or on any region "parabolically" bounded away from the line $|s| = 0$ for that matter) the dynamics is time-like. Reversing the relations in (68) gives the frequency relation for the hyperbolic sector:

$$|\eta|^2 \leq \frac{8}{\varepsilon_0}|s| \tag{72}$$

Thus the tangential gradient of the function $\bar{v}^{II} = \boldsymbol{\chi}_{II}\bar{z}$ can be estimated by the time-derivative. Consequently when the elliptic regularity does not hold it will be possible to "trade" the hidden regularity $\bar{v}_t \in H^{1/2}_a(\Sigma)$ for tangential derivatives in the space variable.

## 5.2 Proving Lemma 5.1 - Part II: The estimates

### 5.2.1 Step 1. Smoothing from the interior dynamics

In this subsection there is no need to specialize to a particular cutoff, since the argument works for both $\boldsymbol{\chi}_I$ and $\boldsymbol{\chi}_{II}$. Invoke Lemma 4.2, with $\mathrm{rhs}(z) := \mathbf{K}_3\bar{z} + 2\gamma\bar{v}_t - \gamma^2\bar{v}$ (without loss of generality ignoring the lower order terms)

$$|A_0P^+\bar{v}|_{-\frac{1}{2},a,\Sigma} \lesssim \|\bar{v}_t\|_{0,Q} + \|\Delta\bar{v}\|_{0,Q} + \left\|\mathbf{K}_3\bar{z} + 2\gamma\bar{v}_t - \gamma^2\bar{v}\right\|_{H^{0,-1}_a(Q)}$$

Since microlocally $\mathbf{K}_3$ is of order zero in the normal direction then

$$\left\|\mathbf{K}_3\bar{z}+2\gamma\bar{v}_t-\gamma^2\bar{v}\right\|_{H_a^{0,-1}(Q)} \lesssim \|\bar{z}\|_{2,a,Q}+\gamma\|\bar{v}_t\|_{H_a^{0,-1}(Q)}+\gamma^2\|\bar{v}\|_{0,Q}$$

The term $\gamma\|\bar{v}_t\|_{H_a^{0,-1}(Q)}$ needs a more careful attention since it's factor is $O(\gamma)$ hence cannot be directly bounded by the finite energy, which is of the same order in (50). Instead, expand the norm and apply interpolation in space and time

$$\begin{aligned}\gamma\|\bar{v}\|^2_{H_a^{0,-1}(Q)} =&\gamma\int_{\mathbb{R}_x^+}|\bar{v}_t(x)|^2_{H_a^{-1}(\Sigma)}dx \leq \gamma\int_{\mathbb{R}_x^+}|\bar{v}(x)|^2_{H_a^1(\Sigma)}dx\\ &\lesssim \|\bar{v}\|^2_{2,a,Q}+O(\gamma^2)\|\bar{v}\|^2_{0,Q}\end{aligned}$$

Combining the above three inequalities conclude

$$\boxed{|A_0P^+\bar{v}|_{-\frac{1}{2},a,\Sigma} \leq \|\bar{z}\|_{2,a,Q}+O(\gamma^2)\|\bar{z}\|_{0,Q}}$$

Here the $L^2$ norm of $\bar{v}$ was estimated by a multiple of the $L^2$ norm of $\bar{z}$.

### 5.2.2 Step 2. Analysis on the support of $\boldsymbol{\chi}_I$ - elliptic sector.

With the help of elliptic estimates (69) - (71) obtain bounds on the target terms (58):

- Operator $C^*$ is in $OPS^2(\Sigma^{(1)})$ so bound it directly (taking into account $M_1 \in OPS_a^1(\Sigma^{(1)})$): $\left|\int_{\mathbb{R}_t}\langle C^*\bar{v}^I, M_1\bar{v}_t^I\rangle\right| \leq |C^*\bar{v}^I|_{\frac{1}{2},\Sigma^{(1)}}|\bar{v}^I|_{\frac{1}{2},\Sigma^{(1)}}$ and invoke

$$|C^*\bar{v}^I|_{\frac{1}{2},\Sigma^{(1)}}+|\bar{v}^I|_{\frac{5}{2},\Sigma^{(1)}} \lesssim \|\bar{v}^I\|_{3,Q} \tag{73}$$

- For norms with large coefficient interpolate:

$$\gamma^2\left|\frac{\partial\bar{v}^I}{\partial\nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \lesssim \gamma^2|\bar{v}^I|^2_{1,Q} \lesssim |\bar{v}^I|^2_{2,a,Q}+O(\gamma^4)|\bar{v}^I|^2_{0,Q} \tag{74}$$

- Finally, for the commutator in the interior will exploit the fact that by construction (possibly excluding a bounded neighborhood of the zero section) $\operatorname{supp}\mathbf{K}_3^I \subset \operatorname{supp}\boldsymbol{\chi}_I \subset \{\boldsymbol{\chi}_{el}=1\}$. Consequently on the support of $\mathbf{K}_3^I$ the functions $\bar{v}^I=\boldsymbol{\chi}_I\bar{z}$, $\bar{v}^{el}=\boldsymbol{\chi}_{el}\bar{z}$ and $\bar{z}$ possess the same microlocal regularity. From (67) we have

$$\begin{aligned}\|\mathbf{K}_3^I\bar{v}^I\|_{0,Q} \lesssim& \|\mathbf{K}_3^I\bar{v}^{el}\|_{0,Q} \lesssim \|\bar{v}^{el}\|_{3,Q}\\ &\lesssim \|\bar{z}\|_{2,Q}+O(\gamma^2)\|\bar{z}\|_{0,Q}+|\bar{h}|_{\frac{1}{2},\Sigma^{(1)}}+|\bar{g}|_{-\frac{1}{2},\Sigma^{(1)}}\end{aligned} \tag{75}$$

Combine (69) - (71) with the latter bounds (73) - (75), square each side, and switch back to the anisotropic norm notation (equivalent to the regular one in the elliptic sector):

$$\boxed{\mathfrak{N}^I_{\gamma,\varepsilon} \lesssim \|\bar{z}\|^2_{2,a,Q}+O(\gamma^4)\|\bar{z}\|^2_{0,Q}+|\bar{h}|^2_{\frac{1}{2},a,\Sigma^{(1)}}+|\bar{g}|^2_{-\frac{1}{2},a,\Sigma^{(1)}}} \tag{76}$$

which completes the analysis for the region described by $\boldsymbol{\chi}=\boldsymbol{\chi}_I$.

### 5.2.3 Step 3. Analysis in the "hyperbolic" sector: $\chi_{II}$

Due to the microlocal relation (72) in this sector two spatial derivatives can be bounded by their anisotropic equivalent of one time derivative:

$$|\bar{v}|_{r,a,\Sigma^{(1)}} \lesssim O(\varepsilon_0^{-1})|\bar{v}|_{r-2,a,\Sigma^{(1)}} \quad \text{and} \quad \left|\frac{\partial\bar{v}}{\partial\nu}\right|_{r,a,\Sigma^{(1)}} \lesssim O(\varepsilon_0^{-1})\left|\frac{\partial\bar{v}}{\partial\nu}\right|_{r-2,a,\Sigma^{(1)}}. \tag{77}$$

Hence it is possible can take advantage of the hidden regularity arising due to the monotone part of the trace, namely the equation (54), which shows control of the trace norm $kc|\bar{v}_t|_{\frac{1}{2},a,\Sigma^{(1)}}$ for some $c > 0$. Now this gain in regularity is equivalent to being able to control small multiples of $|\bar{v}|_{\frac{5}{2},a,\Sigma^{(1)}}$.

- Using (77) obtain
$$\varepsilon|\bar{v}^{II}|^2_{\frac{5}{2},a,\Sigma^{(1)}} \lesssim O(\varepsilon\varepsilon_0^{-1})|\bar{v}^{II}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}}. \tag{78}$$
Now simply choosing $\varepsilon \sim \varepsilon_0^2$ for small enough $\varepsilon_0$ will permit to absorb the RHS of this estimate into the hidden regularity term on the LHS of (57).

- Similarly,
$$\varepsilon^2\left|\frac{\partial\bar{v}^{II}}{\partial\nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}} \lesssim O(\varepsilon^2\varepsilon_0^{-1})\left|\frac{\partial\bar{v}^{II}_t}{\partial\nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}}. \tag{79}$$
If we choose, as in the previous step $\varepsilon \sim \varepsilon_0^2$, then $\varepsilon^2\varepsilon_0^{-1} \sim \varepsilon^{3/2}$ which, for small $\varepsilon$, will permit us to absorb the RHS of the last inequality into the term $\varepsilon|\frac{\partial\bar{v}_t}{\partial\nu}|^2_{-\frac{1}{2},a,\Sigma^{(1)}}$ on the LHS of (57).

- Since the term $|\bar{v}^{II}|^2_{\frac{3}{2},a,\Sigma^{(1)}}$ has no small coefficient in front, but represents a norm of a smaller order (than $5/2$), use interpolation:
$$\begin{aligned}|\bar{v}^{II}|^2_{\frac{3}{2},a,\Sigma^{(1)}} &\lesssim C_\varepsilon|\bar{v}^{II}|^2_{0,a,\Sigma^{(1)}} + \varepsilon^2|\bar{v}^{II}|^2_{2,a,\Sigma^{(1)}}\\ &\lesssim C_\varepsilon\|\bar{v}^{II}\|^2_{2,a,\Omega} + O(\varepsilon^2\varepsilon_0^{-1})|\bar{v}^{II}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}}.\end{aligned} \tag{80}$$

- Next,
$$O(\gamma^2)\left|\frac{\partial\bar{v}^{II}}{\partial\nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} \lesssim O(\gamma^2)\|\bar{v}^{II}\|^2_{L^2\left(\mathbb{R}_t;H^{3/2+\delta}(\Omega)\right)} \lesssim \|\bar{v}^{II}\|^2_{2,a,Q} + C_\gamma\|\bar{v}^{II}\|^2_{0,Q}, \tag{81}$$
$$\begin{aligned}\left|\frac{\partial\bar{v}^{II}}{\partial\nu}\right|^2_{\frac{1}{2},a,\Sigma^{(1)}} &\lesssim C_\varepsilon\left|\frac{\partial\bar{v}^{II}}{\partial\nu}\right|^2_{0,\Sigma^{(1)}} + \varepsilon^2\left|\frac{\partial\bar{v}^{II}}{\partial\nu}\right|_{1,a,\Sigma^{(1)}}\\ &\lesssim C_\varepsilon\|\bar{v}^{II}\|^2_{2,a,Q} + O(\varepsilon^2\varepsilon_0^{-1})\left|\frac{\partial\bar{v}^{II}_t}{\partial\nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}}.\end{aligned} \tag{82}$$
Note that $C_\varepsilon$ is independent of $\gamma$ (which is the coefficient of the energy norms on the LHS of (57).

- Since the supports of commutators $\mathbf{K}_3^{I}$ and $\mathbf{K}_3^{II}$ coincide, the estimate on the interior product $\int_{\mathbb{R}_t}(\mathbf{K}_3^{II}\bar{z},\bar{v}^{II})$ is analogous to that in the previous case (for $\boldsymbol{\chi}_I$), and simply takes advantage of the elliptic regularity estimate (67) for $\bar{v}^{el}$.

- Finally, it remains to estimate the product $\Re\langle C^*\bar{v}^{II},M_1\bar{v}_t^{II}\rangle_{\Sigma^{(1)}}$ which is critical with respect to the level of the hidden regularity, and cannot be interpolated by a mere adjustment of weights. Operators $C^*\in OPS_a^2(\Sigma^{(1)})$ and $M_1\in OPS_a^1(\Sigma^{(1)})$ have real positive principal symbols, hence modulo lower order terms are self-adjoint, and

$$\begin{aligned}\langle C^*\bar{v}^{II},M_1\bar{v}_t^{II}\rangle_{\Sigma^{(1)}} &= \left\langle (C^*M_1)^{\frac12}\bar{v}^{II},\,(C^*M_1)^{\frac12}\bar{v}_t^{II}\right\rangle_{\Sigma^{(1)}}\\ &+\left\langle (C^*)^{\frac12}[\![M_1^{\frac12},(C^*)^{\frac12}]\!]\bar{v}^{II}+[\![M_1^{\frac12},(C^*)^{\frac12}]\!](C^*)^{\frac12}\bar{v}^{II},\,M_1^{\frac12}\bar{v}_t^{II}\right\rangle_{\Sigma^{(1)}}.\\ &+\text{l.o.t.}\end{aligned}\tag{83}$$

The higher order term cancels due to the decay in time:

$$\begin{aligned}\int_{\mathbb{R}_t}\left\langle (C^*M_1)^{\frac12}\bar{v}^{II},\,(C^*M_1)^{\frac12}\bar{v}_t^{II}\right\rangle &= \overbrace{\int_{\mathbb{R}_t}\frac12\partial_t\left|(C^*M_1)^{1/2}\bar{v}^{II}\right|_{0,\Gamma_1}}^{0}\\ &+\left\langle (C^*M_1)^{\frac12}\bar{v}^{II},\,\left[\!\left[(C^*M_1)^{\frac12},\partial_t\right]\!\right]\bar{v}^{II}\right\rangle_{\Sigma^{(1)}}.\end{aligned}$$

As for the commutator, since $\left[\!\left[(C^*M_1)^{\frac12},\partial_t\right]\!\right]\in OPS_a^{5/2}(\Sigma)$ and $(C^*M_1)^{1/2}\in OPS_a^{3/2}(\Sigma)$, it only remains to assess

$$O(\varepsilon^{-1})|\bar{v}^{II}|_{\frac32,a,\Sigma^{(1)}}+\varepsilon|\bar{v}^{II}|_{\frac52,a,\Sigma^{(1)}}$$

which was already done in (78) and (80).

The other two products on the RHS of (83) are handled analogously: apply $[\![M_1^{\frac12},(C^*)^{\frac12}]\!]\in OPS_a^{1/2}(\Sigma)$ to derive

$$\begin{aligned}&\left|(C^*)^{\frac12}\left[\!\left[M_1^{\frac12},(C^*)^{\frac12}\right]\!\right]\bar{v}^{II}+\left[\!\left[M_1^{\frac12},(C^*)^{\frac12}\right]\!\right](C^*)^{\frac12}\bar{v}^{II}\right|_{0,\Sigma}\times\left|M_1^{\frac12}\bar{v}_t^{II}\right|_{0,\Sigma}\\ &\qquad\lesssim O(\varepsilon)|\bar{v}^{II}|_{\frac32,a,\Sigma^{(1)}}+\varepsilon|\bar{v}_t^{II}|_{\frac12,a,\Sigma^{(1)}}\end{aligned}$$

Summarizing, we conclude

$$\left|\Re\int_{\mathbb{R}_t}\langle C^*\bar{v}^{II},M_1\bar{v}_t^{II}\rangle\right|\le O(\varepsilon+\varepsilon\varepsilon_0^{-1})|\bar{v}_t^{II}|^2_{\frac12,a,\Sigma^{(1)}}+C_\varepsilon\|\bar{v}^{II}\|^2_{2,a,Q}\tag{84}$$

Combining the estimates (78) -(84), and interpolating the lower order norms yields

$$\boxed{\begin{aligned}\mathfrak{N}^{II}_{\gamma,\varepsilon}\lesssim\,&O(\varepsilon\varepsilon_0^{-1})|\bar{v}_t^{II}|^2_{\frac12,a,\Sigma^{(1)}}+O(\varepsilon^2\varepsilon_0^{-1})\left|\frac{\partial\bar{v}_t^{II}}{\partial\nu}\right|_{-\frac12,a,\Sigma^{(1)}}\\ &+C_\varepsilon\|\bar{z}\|^2_{2,a,Q}+C_\gamma\|\bar{z}\|^2_{0,Q}+|\bar{h}|^2_{\frac12,a,\Sigma^{(1)}}+|\bar{g}|^2_{-\frac12,a,\Sigma^{(1)}}\end{aligned}}\tag{85}$$

Now set $\varepsilon = \varepsilon_0^2$. Then there is a sufficiently small value for $\varepsilon_0$ (one of the requirements is to satisfy the elliptic condition (60)) and a large enough value for $\gamma$ so that the energy terms on the RHS of (76) and (85) can be absorbed into the terms on the left of (57). Combine these three inequalities and use equivalence of $a(\cdot,\cdot)$ with the $H^2(\Omega)$ norm to conclude (after renormalization)

$$\begin{aligned} &|\bar{z}_t|^2_{\frac{1}{2},a,\Sigma^{(1)}} + |\bar{z}|^2_{\frac{5}{2},a,\Sigma^{(1)}} + \left|\frac{\partial \bar{z}_t}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + \left|\frac{\partial \bar{z}}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}} \\ &\qquad \le C_\gamma \left( |\bar{h}|^2_{\frac{1}{2},a,\Sigma^{(1)}} + |\bar{g}|^2_{-\frac{1}{2},a,\Sigma^{(1)}} + \|\bar{z}\|^2_{0,Q} \right). \end{aligned} \tag{86}$$

Here all other lower order terms have interpolated and absorbed into the finite energy along with $c(\gamma)\|\bar{z}\|^2_{0,Q}$.

#### 5.2.4 Step 4. Absorbing the lower order terms

Since $h$ and $g$ are arbitrary, and the measured regularity in time does not exceed $H^{1/4}$ ($= H_a^{1/2}$) we can truncate $h$ and $g$ in time and claim that (86) holds on every interval $[t,T]$. On a finite interval simply bound the exponential weight $e^{\gamma t}$ above and below by time dependent constants. Then (86) is equivalent to

$$|z_t|^2_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + |z|^2_{\frac{5}{2},a,\Sigma^{(1)}_{t,T}} + \left|\frac{\partial z_t}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + \left|\frac{\partial z}{\partial \nu}\right|^2_{\frac{3}{2},a,\Sigma^{(1)}_{t,T}}$$

$$\le C_{t,T} \left( |h|^2_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + |g|^2_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + C_{t,T}\|z\|^2_{0,Q_{t,T}} \right)$$

which is the first statement (36) of the Lemma 5.1, but perturbed by $\|z\|^2_{0,Q_{t,T}}$. Next, from the boundary conditions (21), (22) (or their dual versions (21*), (22*)) and the previous inequality, conclude that the second statement (37) of the lemma holds as well modulo the $L^2$ norm of the solution

$$|\mathcal{B}_1 z|_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + |\mathcal{B}_2 z|_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}} \le \mathbf{bt}_{t,T} + C_{t,T}\|z\|^2_{0,Q_{t,T}}.$$

Next, integrate the energy identity (47) on $[0,T]$ and let $\bar{v} = \bar{z}$ (i.e. $\boldsymbol{\chi} \equiv 1$ cutoff and consequently $\mathbf{K}_3 \equiv 0$). Then in combination with the last two inequalities it yields

$$|E_z(T) - E_z(t)| = E_z(t) \le \mathbf{bt}_{t,T} + C_{t,T}\|z\|^2_{0,Q_{t,T}}. \tag{87}$$

Thus we have all three statements of Lemma 5.1. It remains to eliminate the norm $\|z\|^2_{0,Q_{t,T}} = \int_t^T \|z\|^2_{0,\Omega}$. Use

$$\int_t^T \|z\|^2_{0,\Omega} \lesssim \int_t^T E_z(s)ds,$$

which holds due to the fact that the segment $\Gamma_0$ of the boundary is non-empty, whence the form $a(\cdot,\cdot)$ controls the $L^2$ norm as well. Consequently, Gronwall's inequality applied to

(87) permits to conclude that the energy is pointwise dominated (for a readjusted constant $C_{t,T}$) by the boundary data alone

$$E_z(t) \leq C_{t,T}\left(|h|^2_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + |g|^2_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}}\right)$$

Thus

$$\int_t^T \|z\|^2_{0,\Omega} \lesssim (T-t)C_{t,T}\left(|h|^2_{\frac{1}{2},a,\Sigma^{(1)}_{t,T}} + |g|^2_{-\frac{1}{2},a,\Sigma^{(1)}_{t,T}}\right),$$

which precisely eliminates the lower order terms in the preceding estimates. Relabeling the leading time-dependent constant by $C_{t,T}$ we recover the statements of Lemma 5.1.

□

# 6 PROOFS III: Energy estimates for the original problem

Finally by duality we will transfer the estimates established in Lemma 5.1 for the adjoint problem to the original system (1), (2), with $(10, 11)$ or $(10^*, 11^*)$ . From equations (1) and (19) we have

$$\begin{aligned}
\int_t^T (f, z_t) &= \int_t^T [(\boxdot w, z_t) + (w_t, \boxdot z)] \\
&= \int_t^T \partial_t (w_t, z_t) + \int_t^T (\Delta^2 w, z_t) + (\Delta^2 z, w_t) \\
&= \int_t^T \partial_t [(w_t, z_t) + a(w, z)] \\
&\quad + \int_t^T \left[\langle \mathcal{B}_2 w, z_t\rangle - \left\langle \mathcal{B}_1 z, \frac{\partial w_t}{\partial \nu}\right\rangle + \langle \mathcal{B}_2 z, w_t\rangle - \left\langle \mathcal{B}_1 w, \frac{\partial z_t}{\partial \nu}\right\rangle\right] = \dots
\end{aligned} \tag{88}$$

Apply boundary conditions (10) - (11) and corresponding conditions of the backward adjoint $z$ problem (21)-(22); alternatively use the dual versions ($10^*$) - ($11^*$) and ($21^*$)-($22^*$). In either case obtain the same expression which, after cancelation, continues the chain of identities (88) as

$$\begin{aligned}
\dots = &- (w_t(t), z_t(t)) - a(w(t), z(t)) \\
&+ \int_t^T \left[\langle Cw, z_t\rangle - \left\langle C^* z, \frac{\partial w_t}{\partial \nu}\right\rangle - \left\langle h, \frac{\partial w_t}{\partial \nu}\right\rangle + \langle g, w_t\rangle\right].
\end{aligned} \tag{89}$$

Since $\Gamma_1$ is closed, the operator $\partial^2_\tau$ is self-adjoint, so

$$\begin{aligned}
\int_t^T \langle Cw, z_t\rangle - \left\langle C^* z, \frac{\partial w_t}{\partial \nu}\right\rangle &= \int_t^T c_{\mu\delta}\left\langle \frac{\partial^2}{\partial \tau^2}\frac{\partial w}{\partial \nu}, z_t\right\rangle + c_{\mu\delta}\left\langle \frac{\partial^2 z}{\partial \tau^2}, \frac{\partial w_t}{\partial \nu}\right\rangle \\
&= \int_t^T \partial_t c_{\mu\delta}\left\langle \frac{\partial^2 z}{\partial \tau^2}, \frac{\partial w}{\partial \nu}\right\rangle = -c_{\mu\delta}\left\langle \frac{\partial^2 z}{\partial \tau^2}(t), \frac{\partial w}{\partial \nu}(t)\right\rangle.
\end{aligned}$$

Substituting into the preceding derivation obtain

$$\int_t^T (f,z_t) = -(w_t(t),z_t(t)) - a(w(t),z(t)) + \int_t^T \left[ \langle g, w_t\rangle - \left\langle h, \frac{\partial w_t}{\partial \nu}\right\rangle \right] - c_{\mu\delta}\left\langle \frac{\partial^2 z}{\partial \tau^2}(t), \frac{\partial w}{\partial \nu}(t)\right\rangle. \tag{90}$$

Thus

$$\int_t^T \left[ \langle g, w_t\rangle - \left\langle h, \frac{\partial w_t}{\partial \nu}\right\rangle \right] = (w_t(t),z_t(t)) + a(w(t),z(t)) + c_{\mu\delta}\left\langle \frac{\partial^2 z}{\partial \tau^2}(t), \frac{\partial w}{\partial \nu}(t)\right\rangle + \int_t^T (f,z_t).$$

Furthermore, for $-\partial_\tau^2 = R$ we have (for any $t$)

$$\left|\left\langle \frac{\partial^2 z}{\partial \tau^2}, \frac{\partial w}{\partial \nu}\right\rangle\right| = \left|\left\langle R^{3/4}z, R^{1/4}\frac{\partial w}{\partial \nu}\right\rangle\right| \le C|z|_{\frac{3}{2},\Gamma_1}\cdot\left|\frac{\partial w}{\partial \nu}\right|_{\frac{1}{2},\Gamma_1} \le C'\|z\|_{0,\Omega}\|\Delta w\|_{0,\Omega}.$$

Setting $t=0$, conclude

$$\left|\int_0^T \langle g, w_t\rangle - \left\langle h, \frac{\partial w_t}{\partial \nu}\right\rangle\right| \le \|w_1\|_{0,\Omega}\|z_t(0)\|_{0,\Omega} + C\|\Delta w_0\|_{0,\Omega}\|\Delta z(0)\|_{0,\Omega} + \left|\int_0^T (f,z_t)\right|.$$

Now apply Lemma 5.1 with $t=0$ to estimate the RHS

$$\left|\int_0^T \langle g, w_t\rangle - \left\langle h, \frac{\partial w_t}{\partial \nu}\right\rangle\right| \le C_T\left(|g|_{-\frac{1}{2},a\Sigma_T^{(1)}} + |h|_{\frac{1}{2},a,\Sigma_T^{(1)}}\right)\times \times\left(\|w_1\|_{0,\Omega} + \|\Delta w_0\|_{0,\Omega} + \|f\|_{0,Q_T}\right). \tag{91}$$

Since $g$ and $h$ are arbitrary we infer

$$\boxed{|w_t|^2_{\frac{1}{2},a,\Sigma_T^{(1)}} + \left|\frac{\partial w_t}{\partial \nu}\right|^2_{-\frac{1}{2},a,\Sigma_T^{(1)}} \le C_T\left(E_w(0) + \|f\|^2_{0,Q_T}\right)} \tag{92}$$

Repeat the duality pairing, as in (88), but now with $Rz$ , where $R$ is locally given by the symbol $r(x,y,\eta)$ i.e. a strongly elliptic self-adjoint tangential operator of order 2, which could be smoothly extended to the interior of domain $\Omega$:

$$\begin{aligned}\int_t^T (f,Rz) = &\int_t^T [(\boxdot w, Rz) - (Rw, \boxdot z)]\\ =&[(w_t,Rz) - (Rw,z_t)]\Big|_t^T\\ &+\int_t^T \left[ \langle \mathcal{B}_2 w, Rz\rangle + \left\langle \mathcal{B}_1 z, \frac{\partial}{\partial \nu}Rw\right\rangle - \langle \mathcal{B}_2 z, Rw\rangle - \left\langle \mathcal{B}_1 w, \frac{\partial}{\partial \nu}Rz\right\rangle \right]\\ =&[(w_t,Rz) - (Rw,z_t)]\Big|_t^T\\ &+\int_t^T \left[ \left\langle h, \frac{\partial}{\partial \nu}Rw\right\rangle - \langle g, Rw\rangle \right] + \int_t^T k\partial_t\left\langle \frac{\partial z}{\partial \nu}, Rw\right\rangle.\end{aligned}$$

Thus, after setting $t = 0$

$$\int_0^T \left[ \langle g, Rw \rangle - \left\langle h, \frac{\partial Rw}{\partial \nu} \right\rangle \right] = (Rw_0, z_t(0)) - (w_1, Rz(0)) - k \left\langle \frac{\partial z(0)}{\partial \nu}, Rw_0 \right\rangle - \int_0^T (f, Rz).$$

As before, via Lemma 5.1, and using

$$\left\langle \frac{\partial z(0)}{\partial \nu}, Rw_0 \right\rangle \leq \left| \frac{\partial z(0)}{\partial \nu} \right|_{\frac{1}{2},\Gamma_1} |Rw_0|_{-\frac{1}{2},\Gamma_1} \leq \|z(0)\|_{2,\Omega} \|w_0\|_{2,\Omega}.$$

Conclude:

$$\left| \int_0^T \langle g, Rw \rangle - \left\langle h, \frac{\partial Rw}{\partial \nu} \right\rangle \right| \leq C_T \left( |g|_{-\frac{1}{2}, a\Sigma_T^{(1)}} + |h|_{\frac{1}{2}, a, \Sigma_T^{(1)}} \right) \times \\ \times \left( \|w_1\|_{0,\Omega} + \|\Delta w_0\|_{0,\Omega} + \|f\|_{0,Q_T} \right),$$

obtaining by duality

$$\boxed{|w|^2_{\frac{5}{2}, a, \Sigma_T^{(1)}} + \left| \frac{\partial w}{\partial \nu} \right|_{\frac{3}{2}, a, \Sigma_T^{(1)}} \leq C_T \left( E_w(0) + \|f\|^2_{0,Q_T} \right)} \tag{93}$$

Finally, substitute (92) and (93) into the energy identity (16) for the original $w$-problem

$$E(T) \leq C_T \left( E(0) + \int_0^T \|f\|^2_{0,\Omega} + \int_0^T E(t) dt \right)$$

Then Gronwall's lemma helps establish an a priori bound

$$E(t) \leq C_t \left( E(0) + \int_0^t \|f\|^2_{0,\Omega} \right)$$

where $C_t$ is continuous increasing in $t$. Since the system is linear, such an priori estimate implies that (1), (2), with $(10, 11)$ or $(10^*, 11^*)$ generates a strongly continuous semigroup on the energy space $H^2_\Gamma(\Omega) \times L^2(\Omega)$ (e.g. see the proof of Theorem 3 in [Mi]). Here $H^2_\Gamma$ denotes the $H^2$ closure of smooth functions satisfying the homogeneous free boundary conditions on $\Gamma_1$, and clamped condition on $\Gamma_1$.

# 7 APPENDIX

**Theorem 7.1** (The L-condition)**.** *For any sufficiently small $\delta > 0$ the elliptic system*

$$\left\{ \begin{array}{rcll} \Delta^2 v &=& 0 & in \quad \Omega \\ (\mathcal{B}_1 v + c_{\mu\delta} \partial_\tau^2) v &=& g & on \quad \Gamma_1 \\ \mathcal{B}_2 v &=& h & on \quad \Gamma_1 \\ v = \frac{\partial v}{\partial \nu} &=& 0 & on \quad \Gamma_0 \end{array} \right.$$

*with $c_{\mu\delta} = 1 - \mu + \delta$, satisfies the Shapiro-Lopatinskii condition (the L-condition).*

*Proof.* It suffices to prove the result for the half space problem, with $\Omega = \mathbf{\Omega} := \mathbb{R}_x^+ \times \mathbb{R}_y$, $\mathbf{\Gamma} := \mathbb{R}_y$ ([Ho, Wl]). Following [Wl, Def. 9.28] we consider the system

$$\begin{aligned} \left(-\left(\frac{1}{i}\partial_x\right)^2 - |\eta|^2\right)^2 v &= 0 \quad \text{in} \quad \mathbf{\Omega} \qquad (\eta \neq 0) && \text{(a-94)} \\ \left[\partial_x^2 - (1+\delta)|\eta|^2\right] v &= 0 \quad \text{in} \quad \mathbf{\Gamma} \qquad (x = 0) && \text{(a-95)} \\ \left[-\partial_x^3 + (2-\mu)|\eta|^2\partial_x\right] v &= 0 \quad \text{in} \quad \mathbf{\Gamma} && \text{(a-96)} \end{aligned}$$

For a fixed $\eta$ the general solution to (a-94) has the form

$$v(x) = (a_0 + a_1 x)\, e^{|\eta| x} + (b_0 + b_1 x)\, e^{-|\eta| x}$$

Let $\mathfrak{M}^+$ denote all solutions $v(x)$ of (a-94) which vanish as $x \to \infty$. Then

$$\mathfrak{M}^+(\Delta^2) = \left\{ v(x) = e^{-|\eta| x}\,(c_1 + c_2 x), \quad c_1, c_2 \in \mathbb{C} \right\}$$

To verify the statement of the theorem it remains to check that the homogeneous boundary conditions (a-95), (a-96) uniquely determine the constants $c_1$, $c_2$. For $v(x) \in \mathfrak{M}^+$ we have

$$\begin{aligned} v_x(x) &= e^{-|\eta| x}\,\left(-|\eta| c_1 - |\eta| c_2 x + c_2\right) \\ v_{xx}(x) &= e^{-|\eta| x}\,\left(\ |\eta|^2 c_1 + |\eta|^2 c_2 x - 2|\eta| c_2\right) \\ v_{xxx}(x) &= e^{-|\eta| x}\,\left(-|\eta|^3 c_1 - |\eta|^3 c_2 x + 3|\eta|^2 c_2\right) \end{aligned}$$

$$\begin{aligned} v(0) &= c_1 \\ v_x(0) &= -|\eta|\, c_1 + c_2 \\ v_{xx}(0) &= \ |\eta|^2 c_1 - 2|\eta| c_2 \\ v_{xxx}(0) &= -|\eta|^3 c_1 + 3|\eta|^2 c_2 \end{aligned}$$

Substitute these values into (a-95) and (a-96):

$$\begin{cases} |\eta|^2 c_1 - 2|\eta| c_2 - (1+\delta)|\eta|^2 c_1 = 0 \\ |\eta|^3 c_1 - 3|\eta|^2 c_2 + (2-\mu)|\eta|^2(-|\eta| c_1 + c_2) = 0 \end{cases}$$

Simplify

$$\begin{cases} \delta|\eta| c_1 + 2c_2 = 0 \\ (1-\mu)|\eta| c_1 + (1+\mu) c_2 = 0 \end{cases}$$

Computing the determinant implies that $c_1 = 0$, $c_2 = 0$ is a unique solution to the system provided $\delta \neq \frac{2(1-\mu)}{1+\mu}$. To account for coordinate transformations when going back to the original system on $\Omega$, it may be necessary to make $\delta$ sufficiently small (or, instead, sufficiently large) in order to guarantee that it stays away from the "singular" value regardless of the changes in the curvature of $\partial\Omega$. □

## References

[Av] G. Avalos, *Null controllability of von Kármán thermoelastic plates under the clamped or free mechanical boundary conditions*, J. Math. Anal. Appl., **318** (2006), 410-432.

[BLR] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim., **30** (1992), 1024–1065.

[BeLa] B. Belinskiy and I. Lasiecka, *Gevrey's and trace regularity of a semigroup associated with beam equation and non-monotone boundary conditions*, J. Math. Anal. Appl., **332** (2007), 137–154.

[ChTr2] S. Chen and R. Triggiani, *Characterization of domains of fractional powers of certain operators arising in elastic systems*, Journal Diff. Eq., **88** (1990), 279–293.

[ChTr1] S. Chen and R. Triggiani, *Gevrey class semigroups arising from elastic systems with gentle perturbation* Proceedings Amer. Math. Society., **110** (1990), 401–415

[GWY] Bao-Zhu Guo, Jun-min Wang and Siu-Pang Yung, *On the $C_0$-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam*, Systems Contr. Letters, **54** (2005), 557–574.

[Ho] L. Hörmander, “The Analysis of Linear Partial Differential Operators I-IV”, Springer, 2003.

[HoLi] Mary Ann Horn and W. Littman, *Local smoothing properties of a Schrödinger equation with nonconstant principal part*, in “Modeling and optimization of distributed parameter systems (Warsaw, 1995),” 104–110, Chapman & Hall, New York, 1996.

[Is] V. Isakov, *Carleman type estimates in anisotropic case and applications*, J. Diff. Equations, **105** (1993), 217–238.

[Ko] V. Komornik, “Exact Controllability and Stabilization -Multipliers Method”, Masson, 1989.

[KoLo] V. Komornik and P. Loreti, “Fourier Series in Control Theory”, Springer, 2005.

[Lag] J. E. Lagnese, “Boundary Stabilization of Thin Plates”, SIAM, 1989.

[LagLeu] J. E. Lagnese, G. Leugering, *Modelling of dynamic networks of thin elastic plates*, Math. Methods Appl. Sci., **16** (1993), 379–407.

[LLS] J. E. Lagnese, G. Leugering, E.J.P.G. Schmidt, “Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures”, Birkhäuser, Boston, 1994.

[LagLio] J. E. Lagnese and J. L. Lions, “Modeling analysis and control of thin plates”, Masson, Paris, 1988.

[Las1] I. Lasiecka, *Boundary stabilization of a 3-dimensional structural acoustic model*, J. Math. Pure Appl., **78** (1999), 203–232.

[Las2] I. Lasiecka, “Mathematical Control Theory of Coupled PDE’s,” CBMS-SIAM Lecture Notes, SIAM, Philadelphia, 2002.

[LLT] I. Lasiecka, J. L. Lions and R. Triggiani, *Nonhomogenuous boundary value problems for second order hyperbolic equations*, J. Math. Pure Appl., **65** (1986), 149–192.

[LaTrV1] I. Lasiecka and R. Triggiani, “Control Theory for Partial Differential Equations. Vol. 1” Cambridge University Press, 2000.

[LuGu] Zheng-Hua Luo and Bao-Zhu Guo, *Shear force feedback control of a single-link flexible robot with a revolute joint*, IEEE Trans. Automatic Control, **42** (1997), 53–65.

[LiMa] J.-L. Lions and E. Magenes, “Problèmes Aux Limites Non Homogènes et Applications,” Vol. 1, Dunod, Paris, 1968.

[Mi] S. Miyatake, *Mixed problems for hyperbolic equation of second order*, J. Math. Kyoto Univ., **13** (1973), 435 – 487.

[Sa] R. Sakamoto, *Mixed problems for hyperbolic equations I, II*, J. Math. Kyoto Univ., **10-2** (1970) 343–373, and **10-3** (1970) 403–417.

[Sh] M. A. Shubov, *Exact controllability of damped coupled EulerBernoulli and Timoshenko beam model*, IMA J. Math. Control and Information, **23** (2006), 279-300.

[Sy] W. W. Symes, *A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources*, Math. Methods Appl. Sci., **5** (1983), 131–152.

[Tat1] D. Tataru, *A priori estimates of Carleman’s type in domains with boundary*, J. Math. Pure Appl., **73** (1994), 353 – 387.

[Tat2] D. Tataru, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pure Appl., **75** (1996), 367–408.

[Tay1] M. E. Taylor, “Pseudodifferential operators”, Princeton University Press, Princeton, 1981.

[Tay2] M. Taylor, “Pseudodifferential Operators and Applications to Nonlinear PDEs,” Birkhäuser, Boston, 1991.

[Wl] J.T. Wloka, B. Lawruk and B. Rowley, “Boundary Value Problems for Elliptic Systems” Cambridge University Press, 1995.