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INFINITE-DIMENSIONAL FEEDBACK SYSTEMS: THE CIRCLE CRITERION AND INPUT-TO-STATE STABILITY*

BAYU JAYAWARDHANA[†], HARTMUT LOGEMANN[‡], AND EUGENE P RYAN[§]

Abstract. An input-to-state stability theory, which subsumes results of circle criterion type, is developed in the context of a class of infinite-dimensional systems. The generic system is of Lur'e type: a feedback interconnection of a well-posed infinite-dimensional linear system and a nonlinearity. The class of nonlinearities is subject to a (generalized) sector condition and contains, as particular subclasses, both static nonlinearities and hysteresis operators of Preisach type.

Keywords: Absolute stability, Circle criterion, Hysteresis, Input-to-state stability, Well-posed infinite-dimensional systems

1. Introduction. R.W. Brockett made major contributions to systems & control theory across a broad range of topics, including frequency-domain stability criteria for nonlinear feedback systems (see, for example, [6, 7]) and early contributions to the theory of infinite-dimensional linear systems (see [4, 5]). In this paper, we combine these two themes with the more recent concept of input-to-state stability (see [34] for a succinct survey of the latter area). In particular, the focus is on absolute stability and input-to-state stability of the feedback interconnection of an infinite-dimensional linear system Σ and a nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, where $\mathrm{dom}(\Phi)$ denotes the domain of Φ and U and Y (Hilbert spaces) denote the input and output spaces of Σ, respectively (see Figure 1, wherein v is an essentially bounded input signal). The system Σ is assumed to belong to the rather general class of well-posed systems (see, for example, [32, 35, 38]) and the nonlinearity is assumed to satisfy a (generalized) sector condition.

In the literature on the circle criterion for infinite-dimensional systems (see, for example, [12], [13]-[17], [24], [27], [37] and the references therein), the emphasis is usually on L^2- or L^∞-stability and global asymptotic or global exponential stability (or some variants thereof) of feedback systems of the type shown in Figure 1, with a static sector-bounded nonlinearity Φ in the feedback path. The new contribution of this paper as compared to the previous literature is twofold.

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[†]Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. Now with the Department of Discrete Technology and Production Automation, University of Groningen, 9747 AG Groningen, The Netherlands. E-mail: B.Jayawardhana@rug.nl

[‡]Corresponding author. Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. E-mail: hl@maths.bath.ac.uk

[§]Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK. E-mail: epr@maths.bath.ac.uk

(i) In addition to static nonlinearities, we include a class of dynamic nonlinearities which may exhibit bias, but still satisfy a generalized pointwise sector condition. As specific subclasses, the class of nonlinearities encompasses both static nonlinearities with "negative resistance" (typified, in a semiconductor context, by tunnel diodes, see e.g. [21]) and a wide range of hysteretic effects described by so-called Preisach operators (typified by mechanical systems with "hystereric spring" effects, see, e.g. [1], or by control systems with hysteretic actuation as arise in micro-positioning problems with piezo-electric actuators, see, e.g. [15]).

(ii) The main results of the paper formulate conditions which guarantee input-to-state-stability with "bias" (and "standard" input-to-state-stability if the nonlinearity is unbiased), thereby making contact with the important and rapidly developing input-to-state-stability theory in (finite-dimensional) nonlinear control.

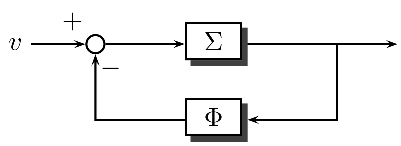

FIG. 1. *Feedback interconnection of linear system* Σ *and nonlinearity* Φ

As in the classical theory of absolute stability and circle criteria, the methodology involves a "symbiosis" of (generalized) sector data relating to the nonlinearity Φ and properties of the transfer function of the linear system Σ to conclude stability properties of the feedback interconnection.

We mention that the viewpoint of this paper is similar in spirit to that of [2]: however, the class of feedback systems considered here is very different to that in [2] as is the methodology adopted. Furthermore, whilst absolute stability problems for hysteretic feedback systems have been considered before, see [3, 19, 20], the results in those papers are restricted to finite-dimensional systems. We emphasize that the general infinite-dimensional setting considered in the present paper requires a fundamentally different analysis: the techniques used in [3, 19, 20] do not extend in a straightforward way to the infinite-dimensional case.

The paper is structured as follows. In Section 2, we assemble some preliminary technical lemmas. In Section 3, we describe the underlying class of well-posed infinite-dimensional linear systems Σ and highlight some of their fundamental properties. Sections 4 and 5 contain the novel contributions of the paper. First, in Section 4, we introduce a sector condition on the class of nonlinearities Φ: for purposes of illustration, we indicate, in Example 4.3, how classical sector-bounded static nonlinearities are embedded in our abstract setting. Theorem 4.5 contains the main result on input-to-state stability: its proof is followed by lemmas, corollaries and remarks pertaining to particular sub-cases and existing results in the literature. A generalized sector condition on Φ is introduced in Section 5 and Theorem 4.5 is generalized (in

Corollary 5.2) in a context of input-to-state stability with "bias". In Section 6, we describe a large class hysteresis operators Φ and show how these are incorporated within our general framework. Finally, in Section 7, we apply our results in two examples of systems modelled by partial differential equations. We complete this introduction with some remarks on notation and terminology.

Notation and terminology. For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$. If S is a non-empty subset of $\mathbb{C}$, then a set $R \subset S$ is said to be *discrete* in S, if, for every $s \in S$, there exists a neighbourhood N of s such that $N \cap R$ is finite. For Hilbert spaces U and Y, let $\mathcal{B}(U,Y)$ denote the space of all linear bounded operators mapping U to Y. We write $\mathcal{B}(U)$ for $\mathcal{B}(U,U)$. For $T \in \mathcal{B}(U)$, we define

$$\operatorname{Re} T := \frac{1}{2}(T + T^*) \in \mathcal{B}(U).$$

The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{B}(U,Y)$ is denoted by $H^\infty_\alpha(\mathcal{B}(U,Y))$. We write $H^\infty(\mathcal{B}(U,Y))$ for $H^\infty_0(\mathcal{B}(U,Y))$. Moreover, in the scalar case (that is $U = Y = \mathbb{C}$), we simply write H^∞_α, or, if $\alpha = 0$, H^∞ for $H^\infty_\alpha(\mathcal{B}(U,Y))$ and $H^\infty(\mathcal{B}(U,Y))$, respectively. For $\alpha \in \mathbb{R}$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\text{loc}}(\mathbb{R}_+, U) : f(\cdot)\exp(-\alpha\,\cdot) \in L^p(\mathbb{R}_+, U)\}$. The Laplace transform is denoted by $\mathfrak{L}$.

2. Some preliminary technical lemmas. In the following, let U and Y be separable (complex) Hilbert spaces.

Lemma 2.1. *Let $\Omega \subset \mathbb{C}_\alpha$ be open and such that $\mathbb{C}_\alpha \setminus \Omega$ is discrete in $\mathbb{C}_\alpha$, where $\alpha < 0$. Assume that $\mathbf{H} : \Omega \to \mathcal{B}(U,Y)$ is holomorphic, $F \in \mathcal{B}(Y,U)$ and $\mathbf{H}(I + F\mathbf{H})^{-1} \in H^\infty(\mathcal{B}(U,Y))$. Then, for $\theta > 0$,*

$$\|\mathbf{H}(I + F\mathbf{H})^{-1}\|_{H^\infty} \le \frac{1}{\sqrt{\theta}} \tag{1}$$

if and only if

$$\mathbf{H}^*(i\omega)\big(\theta I - F^*F\big)\mathbf{H}(i\omega) \le I + 2\operatorname{Re}\big(F\mathbf{H}(i\omega)\big), \quad \text{a.e. } \omega \in \mathbb{R}. \tag{2}$$

Proof. For every $\omega \in \mathbb{R}$ for which $\mathbf{H}(i\omega)$ and $(I + F\mathbf{H}(i\omega))^{-1}$ are defined, we argue as follows. The inequality

$$\|\mathbf{H}(i\omega)(I + F\mathbf{H}(i\omega))^{-1}\| \le \frac{1}{\sqrt{\theta}} \tag{3}$$

is satisfied if and only if

$$\big(I + \mathbf{H}^*(i\omega)F^*\big)^{-1}\mathbf{H}^*(i\omega)\mathbf{H}(i\omega)\big(I + F\mathbf{H}(i\omega)\big)^{-1} \le \frac{1}{\theta}I,$$

which in turn holds if and only if

$$\theta\mathbf{H}^*(i\omega)\mathbf{H}(i\omega) \le \big(I + \mathbf{H}^*(i\omega)F^*\big)\big(I + F\mathbf{H}(i\omega)\big).$$

The last inequality is equivalent to

$$\mathbf{H}^*(i\omega)\,(\theta I - F^*F)\,\mathbf{H}(i\omega) \le I + 2\,\mathrm{Re}\big(F\mathbf{H}(i\omega)\big). \tag{4}$$

Hence, for a.e. ω, the inequalities (3) and (4) are equivalent. Since $\mathbf{H}(I+F\mathbf{H})^{-1} \in H^\infty(\mathcal{B}(U,Y))$, we have that

$$\|\mathbf{H}(I+F\mathbf{H})^{-1}\|_{H^\infty} = \text{ess sup}\{\|\mathbf{H}(i\omega)(I+F\mathbf{H}(i\omega))^{-1}\| : \omega \in \mathbb{R}\}\,,$$

and the claim follows. □

REMARK 2.2. An argument similar to that used in the proof of Lemma 2.1 shows that, under the assumptions of Lemma 2.1, the frequency-domain condition

$$\mathbf{H}(i\omega)\big(\theta I - FF^*\big)\mathbf{H}^*(i\omega) \le I + 2\,\mathrm{Re}\big(\mathbf{H}(i\omega)F\big), \quad \text{a.e. } \omega \in \mathbb{R}.$$

is also necessary and sufficient for (1) to hold. ◇

LEMMA 2.3. *Let* $\mathbf{H} \in H^\infty(\mathcal{B}(U,Y))$*,* $F \in \mathcal{B}(Y,U)$ *and* $\theta > 0$*. If* (2) *is satisfied and there exists* $\rho < 1$ *such that*

$$\mathbf{H}^*(i\omega)\big(\theta I - F^*F\big)\mathbf{H}(i\omega) \ge -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}, \tag{5}$$

then $\mathbf{H}(I+F\mathbf{H})^{-1} \in H^\infty(\mathcal{B}(U,Y))$ *and* (1) *holds.*

Proof. Define $\mathbf{R} \in H^\infty(\mathcal{B}(U))$ by $\mathbf{R}(s) := I + F\mathbf{H}(s)$. Setting $\varepsilon := (1-\rho)/2 > 0$ and invoking (2) and (5) gives

$$\mathrm{Re}\,\mathbf{R}(i\omega) \ge \varepsilon I, \quad \text{a.e. } \omega \in \mathbb{R}.$$

Note that $\varepsilon > 0$ (by hypothesis on ρ). Let $w \in U$ with $\|w\| = 1$ and define $f \in H^\infty$ by $f(s) := \langle \mathbf{R}(s)w, w\rangle$. It follows that

$$\mathrm{Re} f(i\omega) \ge \varepsilon, \quad \text{a.e. } \omega \in \mathbb{R}.$$

Setting $g := \exp(-f) \in H^\infty$, we obtain

$$\exp(-\mathrm{Re} f(s)) = |g(s)| \le \text{ess sup}\{|g(i\omega)| : \omega \in \mathbb{R}\}, \quad \forall\, s \in \mathbb{C}_0.$$

Consequently,

$$\exp(-\mathrm{Re} f(s)) \le \text{ess sup}\{\exp(-\mathrm{Re} f(i\omega)) : \omega \in \mathbb{R}\} \le \exp(-\varepsilon), \quad \forall\, s \in \mathbb{C}_0.$$

Therefore, $\mathrm{Re} f(s) \ge \varepsilon$ for all $s \in \mathbb{C}_0$ and thus

$$\mathrm{Re}\,\langle \mathbf{R}(s)w, w\rangle \ge \varepsilon, \quad \forall\, s \in \mathbb{C}_0.$$

The above argument is valid for every $w \in U$ with $\|w\| = 1$, showing that

$$\mathrm{Re}\,\mathbf{R}(s) \ge \varepsilon I, \quad \forall\, s \in \mathbb{C}_0.$$

As a straightforward consequence, we obtain

$$\|\mathbf{R}(s)w\| \geq \varepsilon\|w\|, \quad \|\mathbf{R}^*(s)w\| \geq \varepsilon\|w\|\,; \quad \forall\, w \in U,\ \ \forall\, s \in \mathbb{C}_0,$$

implying that $\mathbf{R}(s)$ is invertible for all $s \in \mathbb{C}_0$ (see [31, Proposition 3.2.6]) and, furthermore,

$$\|(I + F\mathbf{H}(s))^{-1}\| = \|\mathbf{R}^{-1}(s)\| \leq \frac{1}{\varepsilon}, \quad \forall\, s \in \mathbb{C}_0.$$

Hence, $\mathbf{H}(I + F\mathbf{H})^{-1} \in H^\infty(\mathcal{B}(U, Y))$. Finally, the argument used in the proof of Lemma 2.1 applies mutatis mutandis to conclude that (1) holds. □

LEMMA 2.4. *Let $\Omega \subset \mathbb{C}_0$ be open and such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$. Assume that $\mathbf{H} : \Omega \to \mathcal{B}(U, Y)$ is holomorphic, $F \in \mathcal{B}(Y, U)$ and $\theta > 0$.*

(1) *If $I + F\mathbf{H}(s)$ is invertible for all $s \in \Omega$, then* (1) *holds if and only if*

$$\mathbf{H}^*(s)\big(\theta I - F^*F\big)\mathbf{H}(s) \leq I + 2\,\mathrm{Re}\big(F\mathbf{H}(s)\big), \quad \forall\, s \in \Omega. \tag{6}$$

(2) *If $F\mathbf{H}(s)$ is compact for every $s \in \Omega$, then $I + F\mathbf{H}(s)$ is invertible for all $s \in \Omega$ and* (1) *holds if and only if* (6) *is satisfied.*

Proof. The proof of statement (1) is very similar to that of Lemma 2.1 and is therefore omitted. Statement (2) follows from statement (1), provided we can show that (6), together with the compactness of $F\mathbf{H}(s)$ for every $s \in \Omega$, implies the invertibility of $I + F\mathbf{H}(s)$ for every $s \in \Omega$. To this end, let $s \in \Omega$ and note that by compactness of $F\mathbf{H}(s)$, invertibility of $I + F\mathbf{H}(s)$ is equivalent to -1 not being an eigenvalue of $F\mathbf{H}(s)$. To prove the latter, let $w \in U$ and assume that $F\mathbf{H}(s)w = -w$ or, equivalently, that $(I + F\mathbf{H}(s))w = 0$. Now (6) is equivalent to

$$\theta\mathbf{H}^*(s)\mathbf{H}(s) \leq \big(I + \mathbf{H}^*(s)F^*\big)\big(I + F\mathbf{H}(s)\big) \quad \forall\, s \in \Omega,$$

and hence, $\mathbf{H}(s)w = 0$. We conclude that $w = 0$, showing that -1 is not an eigenvalue of $F\mathbf{H}(s)$. □

3. Well-posed linear systems with nonlinear feedback. In this section we provide some background on well-posed infinite-dimensional linear systems. There are a number of equivalent definitions of well-posed systems, see [32, 35, 36, 38, 39, 40]. We will be brief in the following and refer the reader to the above references for more details. Throughout, we shall be considering a well-posed system Σ with state-space X, input space U and output space Y, generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here X, U and Y are separable (complex) Hilbert spaces, A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X, $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$. The spaces X_1 and X_{-1}, respectively, are interpolation and extrapolation spaces associated with X: $X_1 = \mathrm{dom}(A)$ (the domain of A), endowed with the graph norm of A, whilst X_{-1} denotes the completion of X

with respect to the norm $\|x\|_{-1} = \|(\xi I - A)^{-1}x\|$, where $\xi \in \varrho(A)$, the resolvent set of A (different choices of ξ lead to equivalent norms) and $\|\cdot\|$ denotes the norm on X. Clearly, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. Moreover, the operator B is an *admissible control operator* for $\mathbf{T}$, i.e., for each $t \in \mathbb{R}_+$, there exists $\alpha_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-\tau} B u(\tau) d\tau \right\| \leq \alpha_t \|u\|_{L^2([0,t],U)}, \quad \forall\, u \in L^2([0,t],U)\,;$$

the operator C is an *admissible observation operator* for $\mathbf{T}$, i.e., for each $t \in \mathbb{R}_+$, there exists $\beta_t \geq 0$ such that

$$\left(\int_0^t \|C\mathbf{T}_\tau z\|^2 d\tau \right)^{1/2} \leq \beta_t \|z\|, \quad \forall\, z \in X_1.$$

The control operator B is said to be *bounded* if it is so as a map from the input space U to the state space X, otherwise is said to be *unbounded*; the observation operator C is said to be *bounded* if it can be extended continuously to X, otherwise, C is said to be *unbounded*.

The so-called Λ-*extension* C_Λ of C is defined by

$$C_\Lambda z = \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI - A)^{-1} z,$$

with $\mathrm{dom}(C_\Lambda)$ (the domain of C_Λ) consisting of all $z \in X$ for which the above limit exists. For every $z \in X$, $\mathbf{T}_t z \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ and, if $\omega > \omega(\mathbf{T})$, then $C_\Lambda \mathbf{T} z \in L^2_\omega(\mathbb{R}_+, Y)$, where

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathbf{T}_t\|$$

denotes the exponential growth constant of $\mathbf{T}$.

The transfer function $\mathbf{G}$ satisfies

$$(7) \quad \frac{1}{s - s_0}\left(\mathbf{G}(s) - \mathbf{G}(s_0)\right) = -C(sI - A)^{-1}(s_0 I - A)^{-1}B, \ \forall\, s, s_0 \in \mathbb{C}_{\omega(\mathbf{T})}, \ s \neq s_0,$$

and $\mathbf{G} \in H^\infty_\omega(\mathcal{B}(U,Y))$ for every $\omega > \omega(\mathbf{T})$. Moreover, the input-output operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ is continuous and shift-invariant; for every $\omega > \omega(\mathbf{T})$, $G \in \mathcal{B}(L^2_\omega(\mathbb{R}_+, U), L^2_\omega(\mathbb{R}_+, Y))$ and

$$(\mathfrak{L}(Gu))(s) = \mathbf{G}(s)(\mathfrak{L}(u))(s), \quad \forall\, s \in \mathbb{C}_\omega, \ \forall\, u \in L^2_\omega(\mathbb{R}_+, U).$$

Whilst, a priori, $\mathbf{G}$ is only defined on the half plane $\mathbb{C}_{\omega(\mathbf{T})}$, we say that $\mathbf{G}$ is holomorphic (meromorphic) on $\mathbb{C}_\alpha$ (where $\alpha < \omega(\mathbf{T})$) if there exists a holomorphic (meromorphic) function $\mathbb{C}_\alpha \to \mathcal{B}(U,Y)$ extending $\mathbf{G}$. This function (if it exists) will also be denoted by $\mathbf{G}$.

In the following, let $s_0 \in \mathbb{C}_{\omega(\mathbf{T})}$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, let x and y denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function u. Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)d\tau$ for all $t \in \mathbb{R}_+$, $x(t) - (s_0 I - A)^{-1}Bu(t) \in \operatorname{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ and

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}_+, \\ y(t) = C_\Lambda\left(x(t) - (s_0 I - A)^{-1}Bu(t)\right) + \mathbf{G}(s_0)u(t), \quad \text{a.e. } t \geq 0. \end{cases} \tag{8}$$

Of course, the differential equation in (8) has to be interpreted in X_{-1}. Note that the second equation in (8) yields the following formula for the input-output operator G

$$(Gu)(t) = C_\Lambda\Big[\int_0^t \mathbf{T}_{t-\tau} Bu(\tau)d\tau - (s_0 I - A)^{-1}Bu(t)\Big] + \mathbf{G}(s_0)u(t), \qquad \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U), \text{ a.e. } t \in \mathbb{R}_+. \tag{9}$$

In the following, we identify Σ and (8) and refer to (8) as a well-posed system.

We say that (8) is *exponentially stable* if $\omega(\mathbf{T}) < 0$ and we say that (8) is *input-output stable* if $\mathbf{G} \in H^\infty(\mathcal{B}(U,Y))$ or, equivalently, if $G \in \mathcal{B}(L^2(\mathbb{R}_+,U), L^2(\mathbb{R}_+,Y))$. Furthermore, (8) is said to be *optimizable*, if for every x^0, there exists $u \in L^2(\mathbb{R}_+,U)$ such that the function $t \mapsto \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)d\tau$ is in $L^2(\mathbb{R}_+,X)$. Writing $X^*_{-1} := (X^*)_{-1}$, we have that $X^*_{-1} = (X_1)^*$ and $C^* \in \mathcal{B}(Y, X^*_{-1})$ is an admissible control operator for the adjoint semigroup $\mathbf{T}^* = (\mathbf{T}^*_t)_{t\geq 0}$. We say that (8) is *estimatable* if for every x^0, there exists $u^* \in L^2(\mathbb{R}_+,Y)$ such the function $t \mapsto \mathbf{T}^*_t x^0 + \int_0^t \mathbf{T}^*_{t-\tau} C^* u^*(\tau)d\tau$ is in $L^2(\mathbb{R}_+,X)$.

The above formulas for the output, the input-output operator and the transfer function reduce to a more recognizable form for the subclass of regular systems. Recall that the well-posed system (8) is called *regular* if the following strong limit

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s)w = Dw, \quad \forall\, w \in U$$

exists. In this case, $x(t) \in \operatorname{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$, the output equation in (8) and the formula (9) for the input-output operator simplify to

$$y(t) = C_\Lambda x(t) + Du(t), \quad \text{a.e. } t \geq 0.$$

and

$$(Gu)(t) = C_\Lambda \int_0^t \mathbf{T}_{t-\tau} Bu(\tau)d\tau + Du(t), \quad \forall\, u \in L^2_{\mathrm{loc}}(\mathbb{R}_+,U), \text{ a.e. } t \in \mathbb{R}_+$$

respectively; moreover, $(sI - A)^{-1}BU \subset \operatorname{dom}(C_\Lambda)$ for all $s \in \varrho(A)$ and we have that

$$\mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D, \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}.$$

The operator $D \in \mathcal{B}(U,Y)$ is called the *feedthrough operator* of (8). It can be shown that, if B is a bounded control operator or if C is a bounded observation operator, then (8) is regular.

In the following, we will consider the closed-loop system obtained by applying the nonlinear feedback

$$u = v - \Phi(y) \tag{10}$$

to the well-posed linear system (8), where $v \in L^\infty(\mathbb{R}_+, U)$ and the nonlinear operator $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ is causal. To define the concept of a (local) solution of the feedback system given by (8) and (10), we first need to show that Φ can be "localized" in the sense that it can be "extended" to spaces of functions with a finite time horizon. To this end, let $0 < \sigma \leq \infty$ be arbitrary and set

$$\mathrm{dom}_\sigma(\Phi) := \left\{ w \in L^2_{\mathrm{loc}}([0,\sigma), Y) : \forall\, \tau \in (0,\sigma)\, \exists\, w_\tau \in \mathrm{dom}(\Phi) \text{ s.t. } w = w_\tau \text{ on } [0,\tau] \right\}.$$

Trivially, $\mathrm{dom}_\infty(\Phi) = \mathrm{dom}(\Phi)$. For $w \in \mathrm{dom}_\sigma(\Phi)$ with $\sigma < \infty$, we define $\Phi(w)$ by

$$(\Phi(w))(t) = (\Phi(w_\tau))(t), \quad 0 \leq t \leq \tau < \sigma\,,$$

where $w_\tau \in \mathrm{dom}(\Phi)$ such that $w = w_\tau$ on $[0,\tau]$. By causality of Φ, this definition does not depend on the choice of τ and thus $\Phi(w)$ is a well-defined element in $L^2_{\mathrm{loc}}([0,\sigma), U)$.

A *solution* on $[0,\sigma)$ (where $0 < \sigma \leq \infty$) of the feedback system given by (8) and (10) is a pair $(x,y) \in C([0,\sigma), X) \times \mathrm{dom}_\sigma(\Phi)$ such that, with u given by (10),

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau, \quad \forall\, t \in [0,\sigma) \tag{11}$$

and

$$y(t) = C_\Lambda \left(x(t) - (s_0 I - A)^{-1} B u(t) \right) + \mathbf{G}(s_0) u(t), \quad \text{a.e. } t \in [0,\sigma). \tag{12}$$

If $\sigma = \infty$, then we say that (x,y) is a *global solution*. Let $\mathcal{S}$ denote the set of all $(x^0, v) \in X \times L^\infty(\mathbb{R}_+, U)$ for which the feedback system given by (8) and (10) has at least one global solution. If $(x^0, v) \in \mathcal{S}$, then the notation $(x(\cdot\,; x^0, v), y(\cdot\,; x^0, v))$ is used to denote any global solution corresponding to the initial condition x^0 and the closed-loop input v. Furthermore, a routine argument based on Zorn's lemma shows that every solution (x,y) can be extended to a *maximal solution*, that is, to a maximally defined solution which cannot be extended any further. The interval on which a maximal solution is defined is called the *maximal interval of existence* of the solution. We say that the feedback system given by (8) and (10) has the *blow-up property* if, for every maximal solution (x,y) defined on a finite maximal interval of existence $[0,\sigma)$, the L^2-norm of y blows up, that is, $\|y\|_{L^2(0,\tau)} \to \infty$ as $\tau \uparrow \sigma$.

REMARK 3.1. **(a)** Assume that the feedback system given by (8) and (10) has the blow-up property and that (x,y) is a maximal solution defined on $[0,\sigma)$. If $\|y\|_{L^2(0,\sigma)} < \infty$, then $\sigma = \infty$, that, is (x,y) is a global solution.

(b) In this paper, we are mainly concerned with stability properties of the feedback system given by (8) and (10): whilst of fundamental importance, the question of existence of solutions is not the main concern here; this question requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration. Nevertheless, some general comments on the existence question are warranted. To establish existence of solutions, it is important to observe that, by formula (9) for the input-output operator G of (8), the output y has to satisfy the equation

$$y = C_\Lambda \mathbf{T} x^0 + G(v - \Phi(y)). \tag{13}$$

Once the existence of a solution $y \in \mathrm{dom}_\sigma(\Phi)$ of (13) has been established, the state component x can be obtained from (11) with u given by (10). Existence of solutions to (13) depends very much on the regularity properties of Φ and the "amount of feedthrough" contained in the feedback system. Results on the existence of solutions to equations of the form (13) can be found, for example, in [10, 18, 27, 28, 29, 41]. Finally, we mention one special situation in which existence (and uniqueness) of solutions (for every $(x^0, v) \in X \times L^\infty(\mathbb{R}_+, U)$) and the validity of the blow-up property are guaranteed: this is the case if C is bounded (implying in particular that (8) is regular with feedtrough D), $\mathrm{dom}(\Phi) = L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$, Φ satisfies a Lipschitz condition (in L^2) with Lipschitz constant λ and $\|D\|\lambda < 1$. $\diamond$

4. The sector condition and input-to-state stability. First, we introduce a sector condition on the class of nonlinearities (in due course, this condition will be weakened to a generalized sector condition).

DEFINITION 4.1. A nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ satisfies a *sector condition* if there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$ such that

$$\mathrm{Re}\,\langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t)\rangle \le 0, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+. \tag{14}$$

The following lemma gives a norm-based characterization of the above sector condition.

LEMMA 4.2. *The sector condition* (14) *holds if and only if*

$$\|(\Phi(w))(t) - \frac{1}{2}(K_1 + K_2) w(t)\| \le \frac{1}{2}\|(K_2 - K_1) w(t)\|, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+.$$

Proof. Setting

$$L := \frac{1}{2}(K_2 - K_1), \quad S(w,t) := \langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t)\rangle,$$

it follows from a routine calculation that, for $w \in \mathrm{dom}(\Phi)$ and $t \in \mathbb{R}_+$,

$$\|(\Phi(w))(t) - \frac{1}{2}(K_1 + K_2) w(t)\|^2 = S(w,t) + T(w,t) - \|L w(t)\|^2, \tag{15}$$

where

$$T(w,t) := i\, 2\,\mathrm{Im}\,\langle (\Phi(w))(t), L w(t)\rangle + \langle L w(t), K_2 w(t)\rangle - \langle K_1 w(t), L w(t)\rangle$$

Now

$$\langle Lw(t), K_2 w(t)\rangle = 2\|Lw(t)\|^2 + \langle Lw(t), K_1 w(t)\rangle,$$

so that

$$T(w,t) = i\,2\,\mathrm{Im}\big(\langle(\Phi(w))(t), Lw(t)\rangle + \langle Lw(t), K_1 w(t)\rangle\big) + 2\|Lw(t)\|^2$$

Inserting this into (15) and taking real parts, we obtain

$$\|(\Phi(w))(t) - \frac{1}{2}(K_1 + K_2)w(t)\|^2 = \mathrm{Re}\, S(w,t) + \|Lw(t)\|^2,$$

from which the claim follows. □

Example 4.3 (**Static nonlinearities**). Let $\varphi : Y \to U$ be continuous and assume that there exist $K_1, K_2 \in \mathcal{B}(Y,U)$ such that

$$\mathrm{Re}\langle \varphi(\xi) - K_1\xi\,,\, \varphi(\xi) - K_2\xi\rangle_U \le 0 \quad \forall\, \xi \in Y. \tag{16}$$

With φ we may associate the Němyckiĭ operator $\Phi : L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$, defined by $\Phi(w) := \varphi \circ w$. This operator satisfies the sector condition (14). Such operators provide a simple prototype class for the general nonlinearities considered in this section: at the simplest illustrative level, static sector-bounded scalar nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ of the type shown in Figure 2 (ubiquitous in the literature on the classical circle criterion) are subsumed by the formulation. This observation extends *mutatis mutandis* to encompass time-dependent static nonlinearities $\varphi \colon \mathbb{R}_+ \times Y \to U$. ◇

Anticipating Sections 5 and 6 below, we will also consider static nonlinearities for which the inequality in (16) is assumed to hold only outside some bounded set $E \subset Y$ (see Figure 3). To accommodate these and more general nonlinearities, in Section 5 we will introduce a generalized sector condition and remark here that the generalized formulation encompasses a large class of hysteresis operators, including hysteresis of Preisach and Prandtl type.

Let $K_1, K_2 \in \mathcal{B}(Y,U)$ and define

$$K := \frac{1}{2}\big(K_1 + K_2\big), \quad \kappa := \|K_2 - K_1\|^2. \tag{17}$$

We assemble the following hypotheses on the transfer function $\mathbf{G}$ of (8) which will be variously invoked in the theory developed below.

(H1) There exist $\alpha < 0$ and an open set $\Omega \subset \mathbb{C}_\alpha$ such that $\mathbb{C}_\alpha \setminus \Omega$ is discrete in $\mathbb{C}_\alpha$ and $\mathbf{G}$ is holomorphic on Ω, the frequency-domain condition

$$\mathbf{G}^*(i\omega)\big[\frac{\kappa+\delta}{4}I - K^*K\big]\mathbf{G}(i\omega) \le I + 2\,\mathrm{Re}\big(K\mathbf{G}(i\omega)\big), \quad \text{a.e. } \omega \in \mathbb{R}. \tag{18}$$

holds for some $\delta > 0$ and $\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U,Y))$.

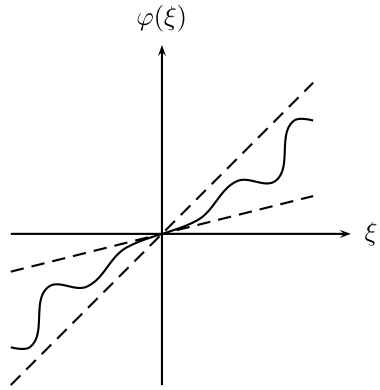

FIG. 2. *Sector-bounded static nonlinearity* φ

(H2) $\mathbf{G} \in H^\infty(\mathcal{B}(U,Y))$ and there exist $\delta > 0$ and $\rho < 1$ such that (18) holds and

$$\mathbf{G}^*(i\omega)\Big[\frac{\kappa+\delta}{4}I - K^*K\Big]\mathbf{G}(i\omega) \geq -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}. \tag{19}$$

(H3) There exists an open set $\Omega \subset \mathbb{C}_0$ such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$ and $\mathbf{G}$ is holomorphic on Ω, $I + K\mathbf{G}(s)$ is invertible for all $s \in \Omega$ and the frequency-domain condition

$$\mathbf{G}^*(s)\Big[\frac{\kappa+\delta}{4}I - K^*K\Big]\mathbf{G}(s) \leq I + 2\operatorname{Re}\big(K\mathbf{G}(s)\big), \quad \forall\, s \in \Omega \tag{20}$$

holds for some $\delta > 0$.

(H4) There exists an open set $\Omega \subset \mathbb{C}_0$ such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$ and $\mathbf{G}$ is holomorphic on Ω, $K\mathbf{G}(s)$ is compact for all $s \in \Omega$ and the frequency-domain condition (20) holds for some $\delta > 0$.

REMARK 4.4. **(a)** In the case of scalar “sector data”, that is $U = Y$ and there exist $k_1, k_2 \in \mathbb{C}$ such that $K_1 = k_1 I$ and $K_2 = k_2 I$, the term

$$\frac{\kappa+\delta}{4}I - K^*K$$

appearing on the left-hand sides of (18)-(20) simplifies to $(\delta/4 - \operatorname{Re}(\bar{k}_1 k_2))I$.

(b) If (8) is optimizable and estimatable, K is compact and $\mathbf{G}(I + K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U,Y))$, then it can be shown that there exists $\alpha < 0$ such that $\mathbf{G}$ is meromorphic on $\mathbb{C}_\alpha$. In particular, this means that there exists an open set $\Omega \subset \mathbb{C}_\alpha$ such that $\mathbb{C}_\alpha \setminus \Omega$ is discrete in $\mathbb{C}_\alpha$ and $\mathbf{G}$ is holomorphic on Ω (the existence of such a set Ω is imposed in (H1)).

(c) The inequality

$$\left(\frac{\kappa+\delta}{4} + \frac{\rho}{\|\mathbf{G}\|^2_{H^\infty}}\right) I \geq K^*K$$

is a sufficient condition for (19) to hold. In particular, if $K_1 = 0$ or $K_2 = 0$, then (19) holds for every $\rho \in [0,1)$ and every $\delta > 0$.

(d) Assume that one of the operators K_1 and K_2 is the zero operator and that the other is a scalar multiple of an isometry. Then it is not difficult to show that (H2) is satisfied, provided that $\mathbf{G} \in H^\infty(\mathcal{B}(U,Y))$ and the positive-real condition

$$\varepsilon I \leq I + 2\,\mathrm{Re}\big(K\mathbf{G}(i\omega)\big), \quad \text{a.e. } \omega \in \mathbb{R}$$

holds for some $\varepsilon > 0$.

(e) Trivially, the compactness assumption in (H4) is satisfied if K is compact or if at least one of the spaces U and Y is finite-dimensional. Without the compactness of $K\mathbf{G}(s)$, it is in general not true that invertibility of $I + K\mathbf{G}(s)$ is a consequence of (20) (a counterexample is given by $U = Y = l^2(\mathbb{N})$, $K_1 = K_2 = K = I$ and $G(s) \equiv R$, where R is the right-shift operator).

(f) In certain situations – see part (d) of this remark and Corollary 4.7 below (together with its proof) – hypotheses (H2) and (H3) are implied by standard positive-real conditions. As is well-known, the latter are equivalent to (physically intuitive) passivity conditions in the time-domain. Moreover, in the single-input-single-output case, the hypotheses (H1)–(H4) can be replaced by graphical conditions familiar from the classic circle criterion, see (C1)–(C3) in Corollary 4.8. $\diamond$

We are now in the position to state and prove the main result of this section.

Theorem 4.5. *Assume that* (8) *is optimizable and estimatable and that there exist operators $K_1, K_2 \in \mathcal{B}(Y,U)$ such that Φ satisfies the sector condition* (14). *Let $K \in \mathcal{B}(Y,U)$ and $\kappa \geq 0$ be given by* (17). *If at least one of hypotheses* (H1)–(H4) *holds, then there exist positive constants Γ and γ, such that, for each $(x^0, v) \in \mathcal{S}$,*

$$\|x(t;x^0,v)\| \leq \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty}\right), \quad \forall\, t \in \mathbb{R}_+. \tag{21}$$

For the above theorem to be non-vacuous, $\mathcal{S}$ should be non-empty: thus, there is a tacit assumption of global existence of solutions. However, if the feedback system given by (8) and (10) has the blow-up property, then the assumptions of Theorem 4.5 imply that every (local) solution can be extended to a global solution, see part (a) of Remark 4.6 below. Furthermore, we emphasize that (21) implies in particular that the feedback system is input-to-state stable in the sense of Sontag (see [34] for a recent survey of the theory of input-to-state stability).

Proof of Theorem 4.5. Let $(x^0, v) \in \mathcal{S}$ and define (x, y) by setting

$$(x(t), y(t)) := (x(t;x^0,v), y(t;x^0,v)), \quad \forall\ t \in \mathbb{R}_+.$$

Then (11) and (12) hold, with $u = v - \Phi(y)$. Invoking formula (9) for the input-output operator G of the well-posed system (8), we conclude that y satisfies

$$y = C_\Lambda \mathbf{T} x^0 + G(v - \Phi(y)).$$

Since

$$\mathbf{G}(I + K\mathbf{G})^{-1} = (I + \mathbf{G}K)^{-1}\mathbf{G} \in H^\infty(\mathcal{B}(U,Y)) \tag{22}$$

(this is part of hypothesis (H1), whilst it is a consequence of each of the hypotheses (H2)-(H4), as follows from Lemma 2.3 in the case of (H2) and from Lemma 2.4 in the case of (H3) and (H4)), we conclude that $(I + \mathbf{G}K)^{-1} \in H^\infty(\mathcal{B}(Y))$. Consequently, $(I + GK)^{-1} \in \mathcal{B}(L^2(\mathbb{R}_+, Y))$. Setting

$$f := (I + GK)^{-1} C_\Lambda \mathbf{T} x^0, \tag{23}$$

it follows that

$$y = f + (I + GK)^{-1} G(v - \Phi(y) + Ky) = f + G(I + KG)^{-1}(v - \Phi(y) + Ky). \tag{24}$$

The fact that $\mathbf{G}(I+K\mathbf{G})^{-1} \in H^\infty(\mathcal{B}(U, Y))$ means in particular that K is a so-called admissible feedback operator (see [39]). Hence, application of linear output feedback of the form $u = w - Ky$ (where $w \in L^2_{\text{loc}}(\mathbb{R}_+, U)$) to the well-posed system (8) results in a well-posed feedback system (with input w) which will be denoted by Σ^K (see [39] for details). The semigroup and the generating operators of Σ^K are denoted by $\mathbf{T}^K = (\mathbf{T}^K_t)_{t\geq 0}$ and (A^K, B^K, C^K), respectively, where A^K is the generator of $\mathbf{T}^K$, B^K is the control operator of Σ^K and C^K is the observation operator of Σ^K. The transfer function of Σ^K is $\mathbf{G}(I + K\mathbf{G})^{-1}$ and therefore, by (22), Σ^K is input-output stable. This, together with the assumptions of optimizability and estimatability, guarantees that Σ^K is exponentially stable (that is, the exponential growth constant $\omega(\mathbf{T}^K)$ of $(\mathbf{T}^K)$ is negative) as follows from results in [40]. Consequently, there exists $\alpha < 0$ such that $\mathbf{G}(I + K\mathbf{G})^{-1}$ extends to a bounded holomorphic function defined on $\mathbb{C}_\alpha$, that is, $\mathbf{G}(I+K\mathbf{G})^{-1} \in H^\infty_\alpha(\mathcal{B}(U, Y))$. This implies that $\mathbf{G}(I+K\mathbf{G})^{-1}$ is uniformly continuous on any vertical strip of the form $\beta_1 \leq \operatorname{Re} s \leq \beta_2$, where $\alpha < \beta_1 < \beta_2$ (see [11, Theorem 3.7]). Next we observe that

$$\|\mathbf{G}(I + K\mathbf{G})^{-1}\|_{H^\infty} \leq \frac{2}{\sqrt{\kappa + \delta}}.$$

This follows from (18) and Lemma 2.1 if (H1) or (H2) hold and from Lemma 2.4 if (H3) or (H4) hold. Thus, by the uniform continuity property, there exists a constant β with

$$\max(\omega(\mathbf{T}^K), \alpha) < \beta < 0 \tag{25}$$

and such that

$$\sup_{s\in\mathbb{C}_\beta} \|\mathbf{G}(s)(I + K\mathbf{G}(s))^{-1}\| \leq \frac{2}{\sqrt{\kappa + \delta/2}}. \tag{26}$$

We conclude that the operator H defined by

$$Hw = \exp(-\beta\,\cdot) G(I + KG)^{-1}(\exp(\beta\,\cdot) w), \quad \forall\, w \in L^2(\mathbb{R}_+, U)$$

is a shift-invariant bounded operator from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$ with transfer function $\mathbf{H}$ given by

$$\mathbf{H}(s) = \mathbf{G}(s - \beta)(I + K\mathbf{G}(s - \beta))^{-1}, \quad \forall\, s \in \mathbb{C}_0.$$

Hence, by (26)

$$\|H\| = \|\mathbf{H}\|_{H^\infty} \le \frac{2}{\sqrt{\kappa+\delta/2}}. \tag{27}$$

Furthermore, we define a causal operator $\Psi : \mathrm{dom}(\Psi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ by

$$\Psi(w) = \exp(-\beta\,\cdot)\big(\Phi(\exp(\beta\,\cdot)w) - K(\exp(\beta\,\cdot)w)\big), \quad \forall\, w \in \mathrm{dom}(\Psi),$$

where $\mathrm{dom}(\Psi) = \{\exp(-\beta\,\cdot)w : w \in \mathrm{dom}(\Phi)\}$. It follows from the sector condition (14) via Lemma 4.2 that

$$\|(\Psi(w))(t)\| \le \frac{\sqrt{\kappa}}{2}\|w(t)\|, \quad \forall\, w \in \mathrm{dom}(\Psi),\ \text{a.e. } t \in \mathbb{R}_+.$$

Hence,

$$\|\Psi(w)\|_{L^2(0,t)} \le \frac{\sqrt{\kappa}}{2}\|w\|_{L^2(0,t)}, \quad \forall\, w \in \mathrm{dom}(\Psi),\ \forall\, t \in \mathbb{R}_+. \tag{28}$$

Combining (27) and (28), we obtain

$$\|H\Psi(w)\|_{L^2(0,t)} \le \rho\|w\|_{L^2(0,t)}, \quad \forall\, w \in \mathrm{dom}(\Psi),\ \forall\, t \in \mathbb{R}_+, \tag{29}$$

where

$$\rho := \sqrt{\frac{\kappa}{\kappa+\delta/2}} < 1. \tag{30}$$

Defining exponentially weighted functions f_β, v_β and y_β by

$$f_\beta(t) := \exp(-\beta t)f(t),\ v_\beta(t) := \exp(-\beta t)v(t),\ y_\beta(t) := \exp(-\beta t)y(t);\ \forall\, t \in \mathbb{R}_+,$$

we derive from (24)

$$y_\beta = f_\beta + H(v_\beta - \Psi(y_\beta)). \tag{31}$$

Therefore, invoking (29) and (30), we conclude that

$$\|y_\beta\|_{L^2(0,t)} \le \Gamma_0\big(\|f_\beta\|_{L^2(0,t)} + \|v_\beta\|_{L^2(0,t)}\big), \quad \forall\, t \in \mathbb{R}_+, \tag{32}$$

where $\Gamma_0 := \max(1, \|H\|)/(1-\rho)$. By [39, Remark 6.3], the function f defined in (23) satisfies

$$f = C^K_\Lambda \mathbf{T}^K x^0,$$

where C^K_Λ denotes the Λ-extension of C^K. Combined with (25) this shows that there exists a constant $\Gamma_1 > 0$ such that

$$\|f_\beta\|_{L^2} \le \Gamma_1\|x^0\|, \quad \forall\, x^0 \in X.$$

Moreover, a routine calculation shows that

$$\|v_\beta\|_{L^2(0,t)} \le \Gamma_2 \exp(-\beta t)\|v\|_{L^\infty}, \quad \forall\, t \in \mathbb{R}_+, \tag{33}$$

where $\Gamma_2 := 1/\sqrt{2|\beta|}$. The last two inequalities, combined with (32), show that there exists $\Gamma_3 > 0$ such that

$$\|y_\beta\|_{L^2(0,t)} \le \Gamma_3\big(\|x^0\| + \exp(-\beta t)\|v\|_{L^\infty}\big), \quad \forall\, t \in \mathbb{R}_+. \tag{34}$$

By [39, Remark 6.2], the state trajectory x satisfies

$$x(t) = \mathbf{T}^K(t)x^0 + \int_0^t \mathbf{T}^K(t-\tau)B^K\big(v(\tau) - (\Phi(y))(\tau) + Ky(\tau)\big)d\tau,\ \forall\, t \in \mathbb{R}_+.$$

Now $x(t)$ can be re-written in the form

$$x(t) = \mathbf{T}^K(t)x^0 + J(t),\ \forall\, t \in \mathbb{R}_+, \tag{35}$$

where

$$J(t) := \exp(\beta t)\int_0^t \mathbf{T}^K(t-\tau)\exp(-\beta(t-\tau))B^K\big(v_\beta(\tau) - (\Psi(y_\beta))(\tau)\big)d\tau,\ \forall\, t \in \mathbb{R}_+.$$

By the admissibility of B^K, the exponential stability of $\mathbf{T}^K$ and the fact that $\beta > \omega(\mathbf{T}^K)$, there exists $\Gamma_4 > 0$ such that

$$\Big\|\int_0^t \mathbf{T}^K(t-\tau)B^K\exp(-\beta(t-\tau))w(\tau)d\tau\Big\| \le \Gamma_4\|w\|_{L^2(0,t)},\ \forall\, t \in \mathbb{R}_+,$$
$$\forall\, w \in L^2_{\mathrm{loc}}(\mathbb{R}_+, U).$$

Consequently, we obtain from (35)

$$\|x(t)\| \le \Gamma_5 \exp(\beta t)\|x^0\| + \Gamma_4 \exp(\beta t)\|v_\beta - \Psi(y_\beta)\|_{L^2(0,t)}, \quad \forall\, t \in \mathbb{R}_+,$$

where $\Gamma_5 > 0$ is a suitable constant (the existence of which is guaranteed by the fact $\beta > \omega(\mathbf{T}^K)$). Invoking (28), (33) and (34), it follows that

$$\|x(t)\| \le \Gamma_5 \exp(\beta t)\|x^0\| + \Gamma_4\left[\Gamma_2\|v\|_{L^\infty} + \frac{\Gamma_3\sqrt{\kappa}}{2}\big(\exp(\beta t)\|x^0\| + \|v\|_{L^\infty}\big)\right],\ \forall\, t \in \mathbb{R}_+.$$

Setting $\gamma := -\beta > 0$, this shows that there exists $\Gamma > 0$ such that

$$\|x(t)\| \le \Gamma\big(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty}\big), \quad \forall\, t \in \mathbb{R}_+.$$

Finally, since the constants Γ_j and β do not depend on $(x^0, v) \in \mathcal{S}$, the same applies to Γ and γ, completing the proof. □

REMARK 4.6. **(a)** Under the additional assumption that the feedback system given by (8) and (10) has the blow-up property, the hypotheses of Theorem 4.5 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Theorem 4.5 apply). Indeed, if, in the proof of Theorem 4.5, (x, y) is a maximal solution, defined on the maximal interval of existence $[0, \sigma)$, then, by using an argument identical to that leading to (32), we may conclude that

$$\|y_\beta\|_{L^2(0,t)} \leq \Gamma_0\big(\|f_\beta\|_{L^2(0,t)} + \|v_\beta\|_{L^2(0,t)}\big), \quad \forall\, t \in [0, \sigma)\,.$$

Combined with the blow-up property and part (a) of Remark 3.1, this shows that $\sigma = \infty$.

(b) Theorem 4.5 can be considered as a generalization and refinement of the circle criterion (see, for example, [14, 21, 37]): in particular, it shows that, under the standard assumptions of the circle criterion (see also Corollaries 4.7 and 4.8 below), input-to-state stability is guaranteed. The exponential weighting technique used in the proof of Theorem 4.5 is well-known and has been used to prove stability results of input-output type (see [14, Section V.3] and the references therein). The application of this technique in an input-to-state stability context seems to be new (even in the finite-dimensional case). In particular, whilst the standard text-book version of the circle criterion for finite-dimensional state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [21, Theorem 7.1] or [37, p. 227]), the above proof of Theorem 4.5 provides an alternative, more elementary, approach. $\diamond$

The following corollary considers the case of scalar "sector data".

Corollary 4.7. *Assume that* (8) *is optimizable and estimatable,* $U = Y$ *and that there exists an open set* $\Omega \subset \mathbb{C}_0$ *such that* $\mathbb{C}_0 \setminus \Omega$ *is discrete in* $\mathbb{C}_0$ *and* $\mathbf{G}$ *is holomorphic on* Ω. *Furthermore, assume that there exist* $k_1, k_2 \in \mathbb{C}$ *and* $\varepsilon > 0$ *such that* Φ *satisfies* (14) *with* $K_1 = k_1 I$ *and* $K_2 = k_2 I$, $I + k_1\mathbf{G}(s)$ *is invertible for every* $s \in \Omega$ *and*

$$\mathrm{Re}\big[\big(I + k_2\mathbf{G}(s)\big)\big(I + k_1\mathbf{G}(s)\big)^{-1}\big] \geq \varepsilon I, \quad \forall\, s \in \Omega. \tag{36}$$

Then there exist positive constants Γ *and* γ, *such that, for each* $(x^0, v) \in \mathcal{S}$, (21) *holds.*

Note that (36) is a positive-real condition.

Proof of Corollary 4.7. By Theorem 4.5, it is sufficient to show that (H3) is satisfied. To this end, we re-write (36) in the form

$$(I + k_2\mathbf{G})(I + k_1\mathbf{G})^{-1} + (I + \bar{k}_1\mathbf{G}^*)^{-1}(I + \bar{k}_2\mathbf{G}^*) \geq 2\varepsilon I, \tag{37}$$

to obtain

$$(I + \bar{k}_1\mathbf{G}^*)^{-1}\big[(I + \bar{k}_1\mathbf{G}^*)(I + k_2\mathbf{G}) + (I + \bar{k}_2\mathbf{G}^*)(I + k_1\mathbf{G})\big](I + k_1\mathbf{G})^{-1} \geq 2\varepsilon I. \tag{38}$$

Since, for every $s \in \Omega$, $I + k_1\mathbf{G}(s)$ is invertible, there exists a function $a : \Omega \to (0, \infty)$ such that

$$\varepsilon I \geq a(I + \bar{k}_1\mathbf{G}^*)^{-1}\mathbf{G}^*\mathbf{G}(I + k_1\mathbf{G})^{-1} \tag{39}$$

and

$$\varepsilon I \geq a(I + k_1\mathbf{G})^{-1}\mathbf{G}\mathbf{G}^*(I + \bar{k}_1\mathbf{G}^*)^{-1}. \tag{40}$$

Combining (39) with (38) gives

$$(I + \bar{k}_1\mathbf{G}^*)(I + k_2\mathbf{G}) + (I + \bar{k}_2\mathbf{G}^*)(I + k_1\mathbf{G}) \geq 2a\mathbf{G}^*\mathbf{G},$$

implying that

$$\big(a(s) - \mathrm{Re}(k_1\bar{k}_2)\big)\mathbf{G}^*(s)\mathbf{G}(s) \leq I + \mathrm{Re}\big((k_1 + k_2)\mathbf{G}(s)\big), \quad \forall\, s \in \Omega. \tag{41}$$

Note that by part (a) of Remark 4.4, the frequency-domain condition (41) is the same as (20) (with $K_1 = k_1 I$, $K_2 = k_2 I$ and $\delta = 4a(s)$). Therefore, in order to show that (H3) holds, it is sufficient to show that $I + k\mathbf{G}(s)$ is invertible for every $s \in \Omega$, where $k := (k_1 + k_2)/2$. To prove this, we re-write the left-hand side of (37) to obtain

$$(I + k_1\mathbf{G})^{-1}(I + k_2\mathbf{G}) + (I + \bar{k}_2\mathbf{G}^*)(I + \bar{k}_1\mathbf{G}^*)^{-1} \geq 2\varepsilon I.$$

An argument similar to that leading to (41) (invoking (40) instead of (39)) gives

$$\big(a(s) - \mathrm{Re}(k_1\bar{k}_2)\big)\mathbf{G}(s)\mathbf{G}^*(s) \leq I + \mathrm{Re}\big((k_1 + k_2)\mathbf{G}(s)\big), \quad \forall\, s \in \Omega. \tag{42}$$

Setting $b(s) := a(s) + |k_2 - k_1|^2/4 > 0$, (41) and (42) yield

$$b\mathbf{G}^*\mathbf{G} \leq (I + \bar{k}\mathbf{G}^*)(I + k\mathbf{G}) \tag{43}$$

and

$$b\mathbf{G}\mathbf{G}^* \leq (I + k\mathbf{G})(I + \bar{k}\mathbf{G}^*). \tag{44}$$

It follows from (43) that

$$\sqrt{b(s)}\|\mathbf{G}(s)w\| \leq \|(I + k\mathbf{G}(s))w\|, \quad \forall\, w \in U,\ \forall\, s \in \Omega. \tag{45}$$

We claim that, for every $s \in \Omega$, the operator $I + k\mathbf{G}(s)$ is bounded away from zero. To this end, let $w \in U$ and $s \in \Omega$. If $\|\mathbf{G}(s)w\| \geq \|w\|/(2|k|)$, then, by (45),

$$\frac{\sqrt{b(s)}}{2|k|}\|w\| \leq \|(I + k\mathbf{G}(s))w\|.$$

Furthermore, if $\|\mathbf{G}(s)w\| < \|w\|/(2|k|)$, then

$$\frac{1}{2}\|w\| \leq \|w\| - |k|\|\mathbf{G}(s)w\| \leq \|(I + k\mathbf{G}(s))w\|.$$

We conclude that

$$\|(I + k\mathbf{G}(s))w\| \geq c(s)\|w\|, \quad \forall\, w \in U,\ \forall\, s \in \Omega, \tag{46}$$

where $c(s) := \min(1/2, \sqrt{b(s)}/(2|k|)) > 0$. Invoking (44), a very similar argument shows that

$$\|(I + \bar{k}\mathbf{G}^*(s))w\| \geq c(s)\|w\|, \quad \forall\, w \in U,\ \forall\, s \in \Omega. \tag{47}$$

The inequalities (46) and (47) show that, for every $s \in \Omega$, $I + k\mathbf{G}(s)$ and $(I + k\mathbf{G}(s))^*$ are bounded away from zero, implying that $I + k\mathbf{G}(s)$ is invertible (see [31, Proposition 3.2.6]). □

For non-zero real numbers k_1 and k_2, we define

$$\Delta(k_1, k_2) := \text{open disk in } \mathbb{C} \text{ with centre in } \mathbb{R} \text{ and } -\frac{1}{k_1} \text{ and } -\frac{1}{k_2} \text{ in its boundary.}$$

The next corollary focuses on the single-input-single-output case. In particular, the classical circle criterion is recovered.

Corollary 4.8. *Assume that* (8) *is optimizable and estimatable,* $U = Y = \mathbb{R}$ *and there exist real numbers* $k_1 < k_2$ *such that*

$$\big((\Phi(w))(t) - k_1 w(t)\big)\big((\Phi(w))(t) - k_2 w(t)\big) \leq 0, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+. \tag{48}$$

Then there exist positive constants Γ *and* γ*, such that, for each* $(x^0, v) \in \mathcal{S}$*,* (21) *holds, provided that one of the following conditions is satisfied:*

(C1) $0 < k_1 < k_2$, $\mathbf{G}/\big(1 + [(k_1 + k_2)/2]\mathbf{G}\big) \in H^\infty$, $\mathbf{G}(i\omega)$ *is bounded away from* $\Delta(k_1, k_2)$ *for all* $\omega \in \mathbb{R}$ *for which* $i\omega$ *is not a pole of* $\mathbf{G}$*;*

(C2) $0 = k_1 < k_2$, $\mathbf{G} \in H^\infty$ *and there exists* $\delta > 0$ *such that* $1 + k_2 \mathrm{Re}\, \mathbf{G}(i\omega) \geq \delta$ *for all* $\omega \in \mathbb{R}$*;*

(C3) $k_1 < 0 < k_2$, $\mathbf{G} \in H^\infty$, $\mathbf{G}(i\omega) \in \Delta(k_1, k_2)$ *for all* $\omega \in \mathbb{R}$ *and* $\mathbf{G}(i\omega)$ *is bounded away from* $\partial\Delta(k_1, k_2)$ *for all* $\omega \in \mathbb{R}$*.*

Observe that, in this single-input-single-output setting, the sector condition (48) can be expressed in the equivalent form:

$$k_1 w^2(t) \leq (\Phi(w))(t)w(t) \leq k_2 w^2(t), \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+. \tag{49}$$

In many situations, the input-output stability condition $\mathbf{G}/\big(1 + [(k_1 + k_2)/2]\mathbf{G}\big) \in H^\infty$ (imposed in (C1)) is satisfied, provided that the number of anticlockwise encirclements of $\Delta(k_1, k_2)$ by the Nyquist diagram of $\mathbf{G}$ is equal to the number of poles of $\mathbf{G}$ in $\mathbb{C}_0$, see, for example, [14], [30], [37].

The following elementary lemma (the proof of which is left to the reader) is crucial for the proof of Corollary 4.8.

Lemma 4.9. *Let* k_1 *and* k_2 *be real numbers.*

(a) *If* $0 < k_1 < k_2$ *and* $S \subset \mathbb{C}$ *is such that* $\mathrm{dist}(S, \Delta(k_1, k_2)) > 0$*, then there exists* $\eta > 0$ *such that*

$$-(k_1 k_2 - \eta)|s|^2 \leq 1 + (k_1 + k_2)\mathrm{Re}\, s, \quad \forall\, s \in S. \tag{50}$$

(b) *If $k_1 < 0 < k_2$ and $S \subset \Delta(k_1, k_2)$ is such that* $\mathrm{dist}(S, \partial\Delta(k_1, k_2)) > 0$*, then there exists $\eta > 0$ such that* (50) *is satisfied.*

Proof of Corollary 4.8. First assume that (C1) holds. Applying part (a) of Lemma 4.9 with S given by

$$S := \{\mathbf{G}(i\omega) : \omega \in \mathbb{R} \text{ such that } i\omega \text{ is not a pole of } \mathbf{G}\}$$

and invoking part (b) of Remark 4.4, shows that (H1) holds and the claim follows from Theorem 4.5. If (C3) holds, then a similar argument (based on part (b) of Lemma 4.9) shows that (H2) is satisfied. Again, the claim then follows from Theorem 4.5. Finally, if (C2) holds, then part (c) of Remark 4.4 shows that (H2) is satisfied and Theorem 4.5 yields the claim. □

REMARK 4.10. If the the feedback system given by (8) and (10) has the blow-up property, then the hypotheses of Corollaries 4.7 and 4.8 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollaries 4.7 and 4.8, respectively, apply), cf. part (a) of Remark 4.6. ◇

5. Generalized sector condition and input-to-state stability with bias. Next, we seek to relax the condition (14) to a *generalized sector condition*. Loosely speaking, we wish to impose the (pointwise) inequality in (14) only when $t \in \mathbb{R}_+$ and $w \in \mathrm{dom}(\Phi)$ are such that $w(t) \in Y \backslash E$, where E (the exceptional set) is some bounded subset of Y. A prototype to bear in mind is the case wherein Φ is the Nĕmyckiĭ operator, given by $\Phi(w) := \varphi \circ w$, associated with a static nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$, of the form shown in Figure 3 (a nonlinearity with negative resistance), satisfying a sector condition outside the interval $E = [-1, 1]$.

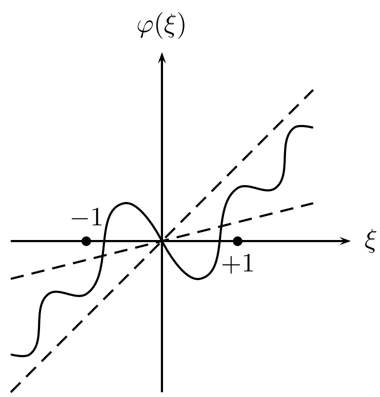

FIG. 3. *Static nonlinearity φ satisfying a generalized sector condition*

Extrapolating this prototype to our abstract setting requires care. The issue is to circumvent the technical difficulty engendered by the fact that the general operator Φ has domain $\mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y)$ and so Φ acts on equivalence classes of functions

$\mathbb{R}_+ \to Y$. Let $w \in L^2_{\rm loc}(\mathbb{R}_+, Y)$ and $Z \subset Y$ be arbitrary. Let $w_r : \mathbb{R}_+ \to Y$ be any representative of w and denote the preimage of Z under w_r by $w_r^{-1}(Z) := \{t \in \mathbb{R}_+ : w_r(t) \in Z\}$. Let $\mathbb{I}_{w_r^{-1}(Z)}$ be the indicator or characteristic function of the set $w_r^{-1}(Z)$ and define $\chi_Z(w) \in L^2_{\rm loc}(\mathbb{R}_+, Y)$ to be the equivalence class of this function, that is,

$$\chi_Z(w) := \left[\mathbb{I}_{w_r^{-1}(Z)}\right].$$

Every choice of representative w_r of w yields the same equivalence class $\left[\mathbb{I}_{w_r^{-1}(Z)}\right]$ and so $\chi_Z(w)$ is a well-defined element of $L^2_{\rm loc}(\mathbb{R}_+, Y)$ for all $w \in L^2_{\rm loc}(\mathbb{R}_+, Y)$. We are now in a position to define the requisite generalized sector condition.

Definition 5.1. A nonlinearity $\Phi : \mathrm{dom}(\Phi) \subset L^2_{\rm loc}(\mathbb{R}_+, Y) \to L^2_{\rm loc}(\mathbb{R}_+, U)$ satisfies a *generalized sector condition* if there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$, a bounded set $E \subset Y$ and a constant $b \geq 0$ such that

$$\begin{cases} \mathrm{Re}\,\langle (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t)\rangle (\chi_{Y\setminus E}(w))(t) \leq 0, \\ \hfill \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+ \end{cases} \tag{51}$$

and

$$\|(\Phi(w))(t)\|(\chi_E(w))(t) \leq b, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+ . \tag{52}$$

The following result generalizes Theorem 4.5.

Corollary 5.2. *Assume that* (8) *is optimizable and estimatable and that there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$, $b \geq 0$ and a bounded set $E \subset Y$ such that Φ satisfies* (51) *and* (52)*. Let $K \in \mathcal{B}(Y, U)$ and $\kappa \geq 0$ be given by* (17)*. If at least one of hypotheses* (H1)–(H4) *holds, then there exist positive constants Γ and γ such that, for each $(x^0, v) \in \mathcal{S}$,*

$$\|x(t; x^0, v)\| \leq \Gamma\left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty} + \beta\right), \quad \forall\, t \in \mathbb{R}_+, \tag{53}$$

where

$$\beta := \sup\left\{\|(\Phi(w) - Kw)\chi_E(w)\|_{L^\infty} : w \in \mathrm{dom}(\Phi)\right\} \leq b + \sup_{\xi \in E}\|K\xi\|, \tag{54}$$

In particular, (53) provides an input-to-state stability estimate with bias β (input-to-state stability with bias β).

Proof. We define a new nonlinearity $\tilde{\Phi}$ by setting $\mathrm{dom}(\tilde{\Phi}) := \mathrm{dom}(\Phi)$ and

$$\tilde{\Phi}(w) := \Phi(w)\chi_{Y\setminus E}(w) + Kw\,\chi_E(w) \quad \forall\ w \in \mathrm{dom}(\Phi).$$

In view of (51), $\tilde{\Phi}$ satisfies the sector condition

$$\mathrm{Re}\,\langle (\tilde{\Phi}(w))(t) - K_1 w(t), (\tilde{\Phi}(w))(t) - K_2 w(t)\rangle \leq 0, \quad \forall\, w \in \mathrm{dom}(\tilde{\Phi}),\ \text{a.e. } t \in \mathbb{R}_+. \tag{55}$$

Let $\tilde{\mathcal{S}}$ denote the set of all $(x^0, \tilde{v}) \in X \times L^\infty(\mathbb{R}_+, U)$ for which the feedback system given by (8) and

$$u = \tilde{v} - \tilde{\Phi}(y) \tag{56}$$

has at least one global solution. If $(x^0, \tilde{v}) \in \tilde{\mathcal{S}}$, then the notation $\tilde{x}(\cdot\,; x^0, \tilde{v})$ is used to denote the state component of any global solution corresponding to the initial condition x^0 and the closed-loop input $\tilde{v}$. By Theorem 4.5, there exist positive constants Γ and γ, such that, for each $(x^0, \tilde{v}) \in \tilde{\mathcal{S}}$,

$$\|\tilde{x}(t; x^0, \tilde{v})\| \le \Gamma \left(\exp(-\gamma t)\|x^0\| + \|\tilde{v}\|_{L^\infty} \right), \quad \forall\, t \in \mathbb{R}_+. \tag{57}$$

Now let $(x^0, v) \in \mathcal{S}$ and define (x, y) by setting $(x(t), y(t)) := (x(t; x^0, v), y(t; x^0, v))$ for all $t \in \mathbb{R}_+$, so that (x, y) is a global solution of the feedback systems given by (8) and (10). Setting $\tilde{v} := v + \tilde{\Phi}(y) - \Phi(y)$, it follows from (52) that $\tilde{v} \in L^\infty(\mathbb{R}_+, U)$. Since $\tilde{v} - \tilde{\Phi}(y) = v - \Phi(y)$, we conclude that (x, y) is also a global solution of the feedback system given by (8) and (56), so that, in particular, $(x^0, \tilde{v}) \in \tilde{\mathcal{S}}$. Thus, it follows from (57) that

$$\|x(t)\| \le \Gamma \left(\exp(-\gamma t)\|x^0\| + \|\tilde{v}\|_{L^\infty} \right), \quad \forall\, t \in \mathbb{R}_+.$$

Combining this inequality with

$$\|\tilde{v}\|_{L^\infty} \le \|v\|_{L^\infty} + \|\tilde{\Phi}(y) - \Phi(y)\|_{L^\infty} = \|v\|_{L^\infty} + \|(Ky - \Phi(y))\chi_E(w)\|_{L^\infty} \le \|v\|_{L^\infty} + \beta,$$

yields the claim. □

Remark 5.3. Under the additional assumption that the feedback system given by (8) and (10) has the blow-up property, the hypotheses of Corollary 5.2 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollary 5.2 apply). To see this, assume that in the proof of Corollary 5.2, (x, y) is a maximal solution of the feedback system given by (8) and (10) with maximal interval of existence $[0, \sigma)$. As in the proof Corollary 5.2, we conclude that (x, y) is also a (not necessarily maximal) solution of the feedback system given by (8) and (56) with $\tilde{v} \in L^\infty(\mathbb{R}_+, U)$ defined by $\tilde{v} := v + \tilde{\Phi}(y) - \Phi(y)$ on $[0, \sigma)$ and $\tilde{v} := 0$ on $\mathbb{R}_+ \setminus [0, \sigma)$. Since $\tilde{\Phi}$ satisfies the sector condition (55), we may invoke the argument (with $\beta = 0$) used in the proof of Theorem 4.5 and leading to (32) to conclude that there exist a positive constant Γ_0 such that

$$\|y\|_{L^2(0,t)} \le \Gamma_0 \big(\|f\|_{L^2(0,t)} + \|\tilde{v}\|_{L^2(0,t)} \big), \quad \forall\, t \in [0, \sigma)\,.$$

Combined with the blow-up property and part (a) of Remark 3.1, this shows that $\sigma = \infty$. ◇

Arguments identical to those in the proof of Corollary 5.2 can be used to obtain the following generalizations of Corollaries 4.7 and 4.8 which apply to nonlinearities satisfying (51) and (52).

COROLLARY 5.4. *Assume that* (8) *is optimizable and estimatable,* $U = Y$ *and that there exists an open set* $\Omega \subset \mathbb{C}_0$ *such that* $\mathbb{C}_0 \setminus \Omega$ *is discrete in* $\mathbb{C}_0$ *and* $\mathbf{G}$ *is holomorphic on* Ω*. Furthermore, assume that there exist* $k_1, k_2 \in \mathbb{C}$*, a bounded set* $E \subset Y$ *and constants* $b \geq 0$ *and* $\varepsilon > 0$ *such that* Φ *satisfies* (51) *and* (52) (*with* $K_1 = k_1 I$ *and* $K_2 = k_2 I$), $I + k_1\mathbf{G}(s)$ *is invertible for every* $s \in \Omega$ *and the positive-real condition*

$$\mathrm{Re}\big[\big(I + k_2\mathbf{G}(s)\big)\big(I + k_1\mathbf{G}(s)\big)^{-1}\big] \geq \varepsilon I, \quad \forall\, s \in \Omega$$

holds. Then there exist constants $\Gamma > 0$ *and* $\gamma > 0$ *such that, for each* $(x^0, v) \in \mathcal{S}$, (53) *holds, where* $\beta \geq 0$ *is given by* (54).

COROLLARY 5.5. *Assume that* (8) *is optimizable and estimatable,* $U = Y = \mathbb{R}$ *and there exist real numbers* $k_1 < k_2$*, a bounded set* $E \subset \mathbb{R}$ *and* $b \geq 0$ *such that*

$$\begin{cases} \big((\Phi(w))(t) - k_1 w(t)\big)\big((\Phi(w))(t) - k_2 w(t)\big)(\chi_{Y\setminus E}(w))(t) \leq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+ \end{cases} \tag{58}$$

and

$$|(\Phi(w))(t)|(\chi_E(w))(t) \leq b, \quad \forall\, w \in \mathrm{dom}(\Phi),\ \text{a.e. } t \in \mathbb{R}_+\,. \tag{59}$$

If at least one of the conditions (C1)–(C3) *of* Corollary 4.8 *is satisfied, then there exist* $\Gamma > 0$ *and* $\gamma > 0$ *such that, for each* $(x^0, v) \in \mathcal{S}$, (53) *holds, where*

$$\begin{aligned} \beta :&= \sup\big\{\|(\Phi(w) - (k_1 + k_2)w/2)\chi_E(w)\|_{L^\infty} : w \in \mathrm{dom}(\Phi)\big\} \\ &\leq b + |k_1 + k_2| \sup_{\xi \in E} |\xi|/2, \end{aligned} \tag{60}$$

Finally, we mention that Remark 5.3 (with obvious modifications) also applies to Corollaries 5.4 and 5.5.

6. Hysteretic feedback systems. Consider again the feedback interconnection of Figure 1, but now in a single-input ($U = \mathbb{R}$), single-output ($Y = \mathbb{R}$) setting and with a hysteresis operator Φ in the feedback path. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that $\Phi(w \circ \zeta) = \Phi(w) \circ \zeta$ for every $w \in C(\mathbb{R}_+)$ and every time transformation ζ, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if it is continuous, non-decreasing and surjective. For simplicity of presentation, henceforth we restrict attention to the class of Preisach hysteresis operators.

Preisach and Prandtl hysteresis. The Preisach operator described in this section encompasses both backlash and Prandtl operators. It can model complex hysteresis effects: for example, nested loops in input-output characteristics. A basic building block for these operators is the *backlash* operator. A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references, see for example [8], [22] and [25]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\big\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\big\}$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\,\xi}(w)(t) = \begin{cases} b_\sigma(w(0), \xi) & \text{for } t = 0\,, \\ b_\sigma(w(t), (\mathcal{B}_{\sigma,\,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \dots\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{n\to\infty} t_n = \infty$ and w is monotone on each interval $[t_i, t_{i+1}]$. We remark that ξ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition (t_i). Figure 4 illustrates how $\mathcal{B}_{\sigma,\,\xi}$ acts. It is well-known that $\mathcal{B}_{\sigma,\,\xi}$ extends to a Lipschitz continuous

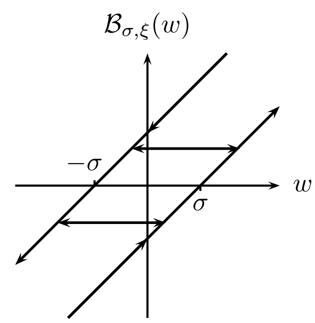

FIG. 4. *Backlash hysteresis*

hysteresis operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{\sigma,\,\xi}$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a regular signed Borel measure on $\mathbb{R}_+$. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $f_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\text{(61)} \qquad \begin{cases} (\mathcal{P}_\xi(w))(t) = \displaystyle\int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(w))(t)} f(s,\sigma)\mu_L(ds)\mu(d\sigma) + f_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, w \in C(\mathbb{R}_+)\,,\ \ \forall\, t \in \mathbb{R}_+\,, \end{cases}$$

is called a *Preisach* operator, cf. [8, p. 55]. It is well-known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \ge$

0). Under the assumption that the measure μ is finite and f is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|(\mathbb{R}_+)\|f\|_\infty$ (see [25]) in the sense that

$$\sup_{t\in\mathbb{R}_+} |\mathcal{P}_\xi(w_1)(t) - \mathcal{P}_\xi(w_2)(t)| \le L \sup_{t\in\mathbb{R}_+} |w_1(t) - w_2(t)| \quad \forall\, w_1, w_2 \in C(\mathbb{R}_+).$$

Setting $f(\cdot,\cdot) = 1$ and $f_0 = 0$ in (61), we obtain the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\mathcal{P}_\xi(w)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(w))(t)\mu(d\sigma) \quad \forall\, w \in C(\mathbb{R}_+)\,,\ \forall\, t \in \mathbb{R}_+\,. \tag{62}$$

For $\xi(\cdot) = 0$ and μ given by $\mu(S) = \int_S \mathbb{I}_{[0,5]}(\sigma)d\sigma$ (where $\mathbb{I}_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is illustrated in Figure 5.

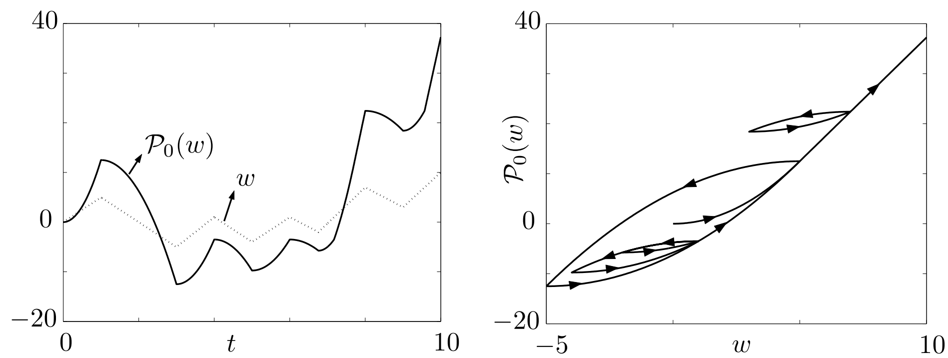

FIG. 5. *Example of Prandtl hysteresis*

The next proposition identifies (rather “mild”) conditions under which the Preisach operator (61) satisfies a generalized sector bound and hence fits into the theory developed in Section 5. For simplicity, we assume that the measure μ and the function f are non-negative (an important case in applications), although the proposition can be extended to signed measures μ and sign-indefinite functions f.

PROPOSITION 6.1. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in* (61)*. Assume that the measure μ is non-negative, $a_1 := \mu(\mathbb{R}_+) < \infty$ and $a_2 := \int_0^\infty \sigma\mu(d\sigma) < \infty$. Furthermore, assume that*

$$b_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} f(s,\sigma) \ge 0\,, \quad b_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} f(s,\sigma) < \infty$$

and set

$$a_{\mathcal{P}} := a_1 b_1, \quad b_{\mathcal{P}} := a_1 b_2\,, \quad c_{\mathcal{P}} := a_2 b_2 + |f_0|\,. \tag{63}$$

Then, for all $w \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$w(t) \ge 0 \implies a_{\mathcal{P}} w(t) - c_{\mathcal{P}} \le (\mathcal{P}_\xi(w))(t) \le b_{\mathcal{P}} w(t) + c_{\mathcal{P}}\,, \tag{64}$$

$$w(t) \le 0 \implies b_{\mathcal{P}} w(t) - c_{\mathcal{P}} \le (\mathcal{P}_\xi(w))(t) \le a_{\mathcal{P}} w(t) + c_{\mathcal{P}}\,, \tag{65}$$

and, furthermore, for every $\eta > 0$*,*

$$|w(t)| \ge c_{\mathcal{P}}/\eta \implies \big((\mathcal{P}_\xi(w))(t) - (a_{\mathcal{P}} - \eta) w(t)\big)\big((\mathcal{P}_\xi(w))(t) - (b_{\mathcal{P}} + \eta) w(t)\big) \le 0. \tag{66}$$

In particular, for every $\eta > 0$*, the generalized sector conditions* (58) *and* (59) *hold with* $U = \mathbb{R} = Y$*,* $E = [-c_{\mathcal{P}}/\eta\,,\, c_{\mathcal{P}}/\eta]$*,* $k_1 = a_{\mathcal{P}} - \eta$*,* $k_2 = b_{\mathcal{P}} + \eta$*, and* $b = (b_{\mathcal{P}}/\eta + 1)c_{\mathcal{P}}$*.*

Proof. Arguments very similar to those in the proof of [29, Proposition 2.5] show that (64) and (65) hold. The inequality (66) is a straightforward consequence of (64) and (65). □

7. Examples. We illustrate the results in Sections 4 and 5 with two examples.

EXAMPLE 7.1. For $z \in (0,1)$ and $t > 0$, we consider the following system:

$$\begin{cases} w_{tt}(z,t) - w_{zz}(z,t) + 2a w_t(z,t) + a^2 w(z,t) = 0, \\ w(0,t) = 0, \quad w_z(1,t) = u(t), \\ y(t) = w_t(1,t)\,, \end{cases} \tag{67}$$

where we assume that the viscous damping parameter a is non-negative.

A straightforward computation shows that the transfer function $\mathbf{G}_a$ of (67) is given by

$$\mathbf{G}_a(s) = \frac{s}{s+a}\left(\frac{1 - e^{-2(s+a)}}{1 + e^{-2(s+a)}}\right) = \frac{s}{s+a}\left(\frac{\sinh(s+a)}{\cosh(s+a)}\right).$$

Trivially, $\mathbf{G}_a \in H^\infty_\alpha$ for every $\alpha > -a$ and $\mathbf{G}_0$ has poles at $s = i(2m+1)\pi/2$ for $m \in \mathbb{Z}$. It is known that, for every $k > 0$, the feedback control $u = -ky$ leads to an exponentially stable closed-loop system in the sense that there exist $\alpha > 0$ and $c \ge 1$ such that, if w is a solution, then

$$V(t) \le c e^{-\alpha t} V(0) \ \ \forall\, t \ge 0, \quad \text{where} \ \ V(t) := \int_0^1 \big(w_t^2(z,t) + w_z^2(z,t) + a w^2(z,t)\big) dz,$$

see, for instance, [9]. Hence, it follows in particular that, for every $k > 0$, $\mathbf{G}_a/(1 + k\mathbf{G}_a) \in H^\infty$. Elementary calculations show that $\mathbf{G}_a$ is positive real for every $a \ge 0$. Furthermore, it follows from [26, Example 9.1] that

$$\|\mathbf{G}_a\|_{H^\infty} = \frac{e^{2a}+1}{e^{2a}-1} = \frac{\cosh(a)}{\sinh(a)} =: g_a, \quad \text{for every } a > 0.$$

Consider the closed-loop system obtained by applying the nonlinear feedback $u = v - \Phi(y)$ to (67). Invoking the above observations and Corollary 4.8 (Corollary 5.5), we see that the conclusions of Corollary 4.8 (Corollary 5.5) hold, provided that at least one of the following assumptions are satisfied:

(i) $a = 0$ and Φ satisfies a sector condition (generalized sector condition) with $0 < k_1 < k_2$;

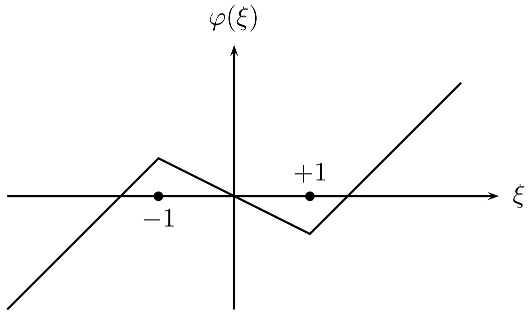

FIG. 6. *Static nonlinearity adopted in simulations (S2) and (S3)*

(ii) $a > 0$ and Φ satisfies a sector condition (generalized sector condition) with $0 = k_1 < k_2$;

(iii) $a > 0$ and Φ satisfies a sector condition (generalized sector condition) with $-1/g_a < k_1 < 0$ and $0 < k_2 < 1/(|k_1|g_a^2)$.

The rationale underpinning (iii) is as follows: since, for all $\omega \in \mathbb{R}$, $\mathbf{G}_a(i\omega)$ is bounded by g_a and $\operatorname{Re}\mathbf{G}_a(i\omega) \geq 0$, it follows that $\mathbf{G}_a(i\omega)$ lies in the half-disk $H := \{s \in \overline{\mathbb{C}}_0 : |s| \leq g_a\}$. If $k_1 < 0 < k_2$, then $H \subset \Delta(k_1, k_2)$ if and only if g_a and ig_a are in $\Delta(k_1, k_2)$, which in turn leads to the inequalties $-1/g_a < k_1$ and $k_2 < 1/(|k_1|g_a^2)$.

The findings in the above statements (i) and (iii) are illustrated by three simulations referred to as (S1)-(S3) and shown in Figures 7-9. These simulations were performed using MATLAB in conjunction with [33]. In all simulations, the initial conditions are

$$w(z,0) = \sin(2\pi z) \quad \text{and} \quad w_t(z,0) = \cos(2\pi z); \qquad \forall\, z \in (0,1)\,.$$

Specifically, the time evolution of the L^2-norms of the functions $w(\cdot\,,t)$, $w_z(\cdot\,,t)$ and $w_t(\cdot\,,t)$ (where w is the solution of the closed-loop system obtained by applying the nonlinear feedback $u = -\Phi(y)$ to (67)) are shown:

(S1) in Figure 7, with $a = 0$ and $\Phi = \mathcal{B}_{1,0}$ (i.e. backlash in the input channel);

(S2) in Figure 8, with $a = 0$ and Φ is the Němyckiĭ operator associated with the continuous piecewise linear static nonlinearity $\xi \mapsto \varphi(\xi)$, where $\varphi'(\xi) = -1/2$ if $|\xi| < 1$ and $\varphi'(\xi) = 1$ if $|\xi| \geq 1$, shown in Figure 6 (i.e. a component with "negative resistance" effect);

(S3) in Figure 9, with $a = 2$ and Φ is the same as in (S2).

In (S1), the nonlinearity $\Phi = \mathcal{B}_{1,0}$ satisfies a generalized sector conditions with $k_1 = 1 - \delta$, $k_2 = 1 + \delta$ and $E = [-1/\delta, 1/\delta]$ for every $\delta \in (0,1)$, so that (i) holds. It is not difficult to show that the bias β defined by (60) is equal to 1 (independent of δ). In (S2) and (S3), the nonlinearity φ exhibits "negative resistance". It satisfies a generalized sector conditions with $k_1 = 1 - \delta$, $k_2 = 1$ and $E = [-3/(2\delta), 3/(2\delta)]$ for every $\delta \in (0,1)$. In particular, if $a = 0$, then (i) holds. The bias β defined by (60)

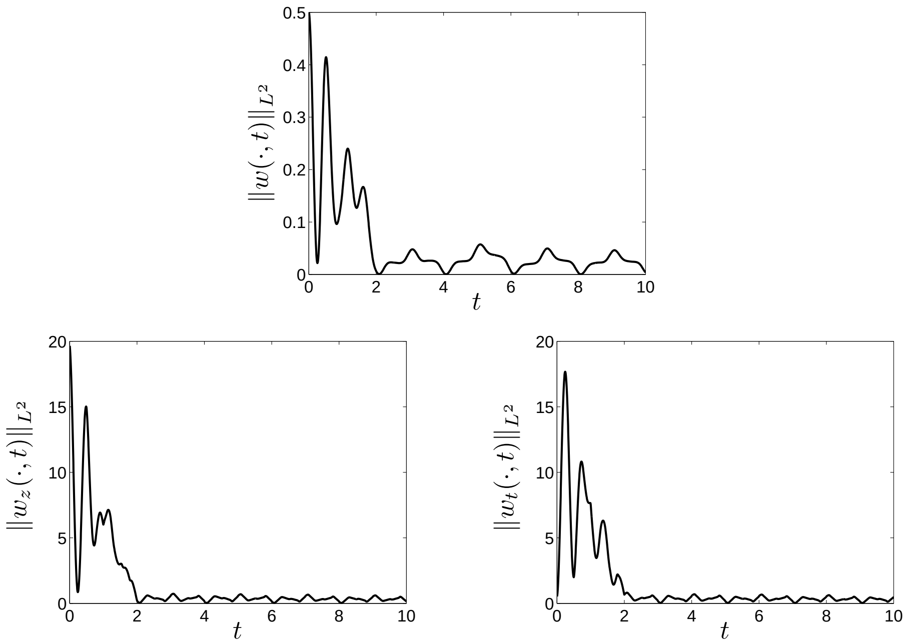

Fig. 7. *Simulation S1*

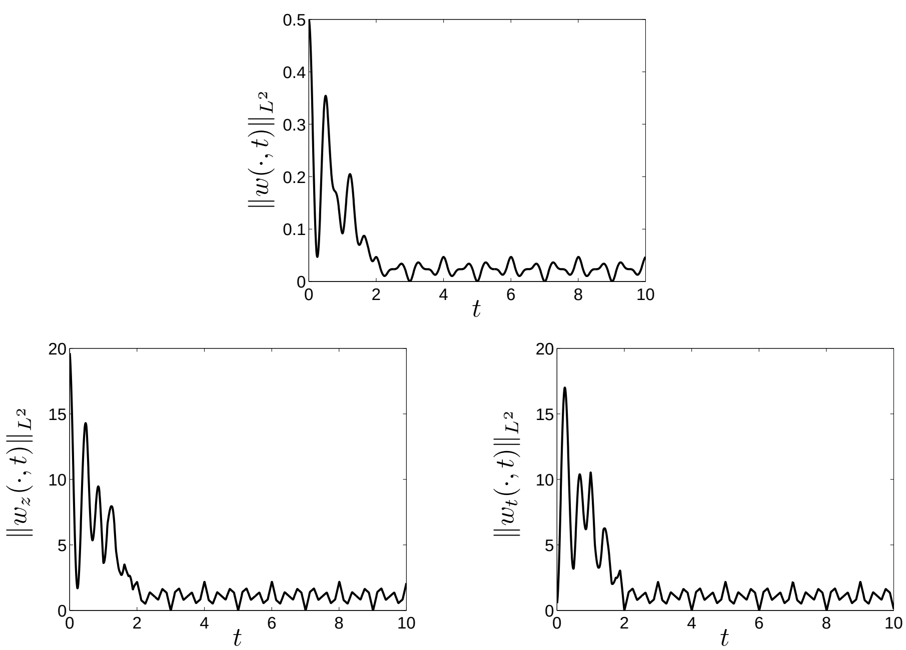

Fig. 8. *Simulation S2*

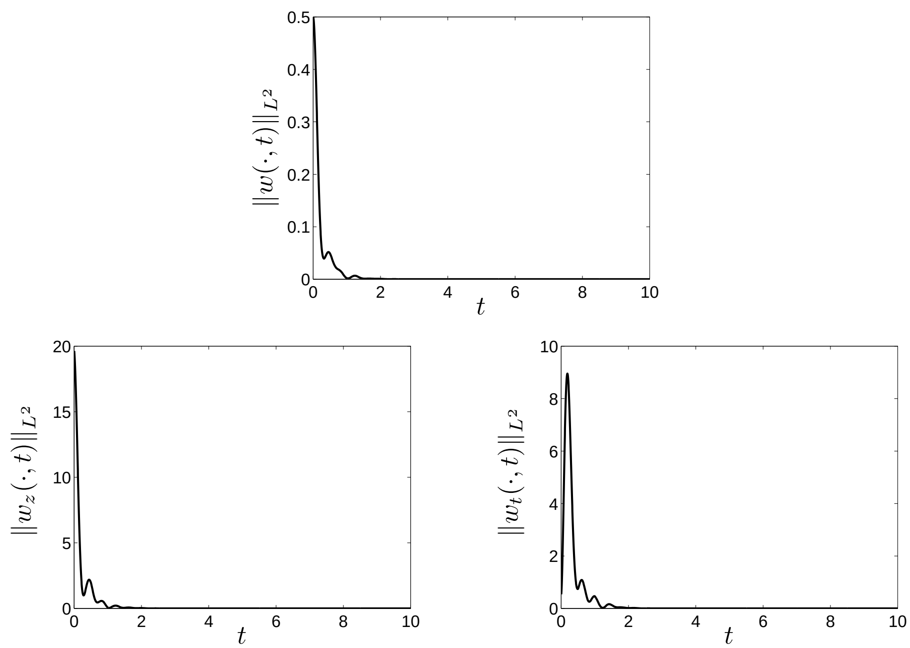

FIG. 9. *Simulation S3*

is in this case equal to $(3-\delta)/2$. On the other hand, φ satisfies a "proper" sector condition with $k_1 = -1/2$ and $k_2 = 1$ and routine calculations show that, if $a = 2$, then (iii) holds.

EXAMPLE 7.2. In this example, the input and output spaces are $\mathbb{R}^2$. We consider two coupled vibrating undamped strings, one with spatial extent $0 \le z \le 1$ and the other with spatial extent $1 \le z \le 2$.

$$\begin{cases} w_{tt}(z,t) - w_{zz}(z,t) = 0, \quad z \in (0,1) \cup (1,2); \\ w(1^-,t) = w(1^+,t), \quad w(2,t) = 0; \quad t > 0; \\ w_z(1^-,t) - w_z(1^+,t) = u_1(t), \quad w_z(0,t) = u_2(t); \quad t > 0; \\ y_1(t) = w_t(1,t), \quad y_2(t) = -w_t(0,t); \quad t > 0. \end{cases} \tag{68}$$

In the above system, the displacement is continuous at the linkage, the right endpoint is fixed, the discontinuity of the vertical tension force is equal to the control variable u_1 and, at the left endpoint, the vertical tension force is equal to the control variable u_2.

The transfer function matrix $\mathbf{G}$ for this system is given by

$$\mathbf{G}(s) = \begin{pmatrix} \frac{1}{2}\frac{1-e^{-4s}}{1+e^{-4s}} & \frac{e^{-s}(e^{-2s}-1)}{1+e^{-4s}} \\ \frac{e^{-s}(e^{-2s}-1)}{1+e^{-4s}} & \frac{1-e^{-4s}}{1+e^{-4s}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{\sinh(2s)}{\cosh(2s)} & \frac{\sinh(s)}{\cosh(2s)} \\ \frac{\sinh(s)}{\cosh(2s)} & \frac{\sinh(2s)}{\cosh(2s)} \end{pmatrix}.$$

Note that $\mathbf{G}(s)$ has poles at $s = i(2m+1)\pi/2$ for every $m \in \mathbb{Z}$. It is not hard to show that for any $k > 0$, the closed-loop transfer function matrix $\mathbf{G}(I + k\mathbf{G})^{-1}$ is in $H^\infty(\mathbb{C}^2, \mathbb{C}^2)$: in fact, as is proved in [23], the closed-loop system given by (68) and the feedback law $y = -ku$ (where $y = (y_1, y_2)^T$ and $u = (u_1, u_2)^T$) is exponentially stable in the sense that there exist $\alpha > 0$ and $c \geq 1$ such that, if w is a solution, then

$$V(t) \leq ce^{-\alpha t}V(0) \;\; \forall\, t \geq 0, \quad \text{where} \;\; V(t) := \int_0^2 \big(w_t^2(z,t) + w_z^2(z,t)\big)dz.$$

A straightforward calculation shows that $\mathbf{G}(i\omega)$ is skew-Hermitian for all $\omega \in \mathbb{R} \setminus \{(2m+1)\pi/2 : m \in \mathbb{Z}\}$. Consequently, invoking part (a) of Remark 4.4, we conclude that, if $K_1 = k_1 I$ and $K_2 = k_2 I$ with $0 < k_1 < k_2$, then there exists $\delta > 0$ such that hypothesis (H1) is satisfied.

Now, consider the closed-loop system obtained by applying the nonlinear feedback $u = v - \Phi(y)$ to (68). Invoking the above observations and Theorem 4.5 (Corollary 5.2), we see that the conclusions of Theorem 4.5 (Corollary 5.2) hold, provided Φ satisfies a sector condition (generalized sector condition) with $K_1 = k_1 I$ and $K_2 = k_2 I$, where $0 < k_1 < k_2$.

8. Concluding remarks. An input-to-state stability theory, which subsumes results of circle criterion type, has been developed in the context of feedback interconnections with a linear system Σ in the forward path and a nonlinear causal operator Φ in the feedback path. The approach combines ideas from absolute stability theory with the more recent concept of input-to-state stability. Distinguishing features of the paper are: (1) infinite-dimensionality of the linear component Σ, which is required only to belong to the broad class of well-posed systems; (2) the breadth of the class of operators $\Phi\colon \mathrm{dom}(\Phi) \subset L^2_{\mathrm{loc}}(\mathbb{R}_+, Y) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, U)$ (U and Y Hilbert spaces), which are assumed only to satisfy a generalized sector condition – thereby enlarging the class of admissible nonlinearies and, in particular, encompassing hysteretic components. The main results formulate conditions under which input-to-state stability with bias is guaranteed (reducing to input-to-state stability in the special case of nonlinearities satisfying a standard sector condition).

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