Communications in Information and Systems
Vol. 7, No. 1, pp. 21-30, 2007

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REGULARITY OF RENORMALIZED SELF-INTERSECTION LOCAL TIME FOR FRACTIONAL BROWNIAN MOTION*

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Abstract. Let B_t^H be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. We study the regularity, in the sense of the Malliavin calculus, of the renormalized self-intersection local time

$$\ell = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt - \mathbb{E}\left(\int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt \right),$$

where δ_0 is the Dirac delta function.

1. Introduction. The fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a d-dimensional Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with mean zero and covariance function given by

$$\mathbb{E}(B_t^{H,i} B_s^{H,j}) = \frac{\delta_{ij}}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

where $i, j = 1, \ldots, d$, and $s, t \geq 0$. We will assume that $d \geq 2$. The *self-intersection local time* of B^H is formally defined by

$$I = \int_0^T \int_0^t \delta_0(B_t^H - B_s^H) ds dt, \tag{1}$$

where $\delta_0(x)$ is the Dirac delta function. Using the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-d/2} e^{-\frac{|x|^2}{2\varepsilon}},$$

we approximate the self-intersection local time of B^H by

$$I_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t^H - B_s^H) ds dt. \tag{2}$$

The asymptotic behavior of I_ε as ε tends to zero is studied in [5], and the following results are proved.

i) If $H < \frac{1}{d}$, then I_ε converges in L^2 as ε tends to zero.

ii) If $\frac{1}{d} < H < \frac{3}{2d}$, then

$$I_\varepsilon - T C_{H,d} \varepsilon^{-\frac{d}{2} + \frac{1}{2H}},$$

converges in L^2 as ε tends to zero to a limit ℓ, where

$$C_{H,d} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_0^\infty \left(z^{\frac{1}{2H}} + 1 \right)^{-\frac{d}{2}} dz.$$

*Y. Hu is supported in part by the National Science Foundation under Grant No. DMS0504783; D. Nualart is supported in part by the National Science Foundation under Grant No. DMS0604207.

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iii) If $\frac{1}{d} = H < \frac{3}{2d}$, then

$$I_\varepsilon - \frac{T}{2H(2\pi)^{\frac{d}{2}}} \log \frac{1}{\varepsilon},$$

converges in L^2 as ε tends to zero.

iv) If $\frac{3}{2d} \le H < \frac{3}{4}$, then the random variables

$$\begin{cases} \frac{1}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}} \left(I_\varepsilon - \mathbb{E}(I_\varepsilon)\right) & \text{if} \quad H = \frac{3}{2d} \\ \varepsilon^{\frac{d}{2} - \frac{3}{4H}} \left(I_\varepsilon - \mathbb{E}(I_\varepsilon)\right) & \text{if} \quad H > \frac{3}{2d} \end{cases}$$

converge as ε tends to zero in distribution to a normal law $N(0, T\sigma^2)$, where σ^2 is a constant depending on d and H.

We denote by ℓ the limit introduced in ii) and iii). It turns out that ℓ is also equal to the limit in L^2 of $I_\varepsilon - \mathbb{E}(I_\varepsilon)$ as ε tends to zero. If $H < \frac{1}{d}$, then ℓ will be defined as the limit in L^2 of $I_\varepsilon - \mathbb{E}(I_\varepsilon)$ as ε tends to zero. The random variable ℓ is called the *renormalized self-intersection local time* of the fractional Brownian motion.

In this paper we shall study the regularity, in the sense of Malliavin calculus, of the renormalized self-intersection local time ℓ, assuming $H < \frac{3}{2d}$. We prove that, for any real $\alpha > 0$, ℓ belongs to the Sobolev space $\mathbb{D}^{\alpha,2}$, provided $H < \min(\frac{3}{2d}, \frac{2(\alpha\wedge 1)}{d+2\alpha})$. This result generalizes that obtained by Hu in [4] in the case $\alpha = 1$. The proof of this result is established via chaos expansions.

In Section 2, we recall the chaos expansion of self-intersection local time obtained in [5]. In Section 3, we state and prove the main result of the paper.

2. Wiener chaos expansion of the self-intersection local time. In this section we recall the Wiener chaos expansion of the renormalized self-intersection local time ℓ obtained in [5].

Let $\mathcal{H}$ be the Hilbert space defined as the closure of set $\mathcal{E}$ of step functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ with respect to the scalar product

$$\left\langle \left(\mathbf{1}_{[0,t_1]}, \dots, \mathbf{1}_{[0,t_d]}\right), \left(\mathbf{1}_{[0,s_1]}, \dots, \mathbf{1}_{[0,s_d]}\right) \right\rangle_{\mathcal{H}} = \frac{1}{2^d} \prod_{i=1}^d \left(t_i^{2H} + s_i^{2H} - |t_i - s_i|^{2H}\right).$$

Then, the mapping $\mathbf{1}_{[0,t]} \to B_t^H$ is a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by B^H. For any $n \ge 1$ we denote by I_n the multiple stochastic integral which provides an isometry between the symmetric tensor product $(\mathcal{H})^{\otimes n}$ equipped with the norm $\sqrt{n!}\,\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the nth Wiener chaos of B^H.

Given a multi-index $\mathbf{i}_n = (i_1, \dots, i_n)$, $1 \le i_j \le d$, we set

$$\alpha(\mathbf{i}_n) = \mathbb{E}\left[X_{i_1} \cdots X_{i_n}\right],$$

where the X_i are independent $N(0,1)$ random variables. Notice that

$$\alpha(\mathbf{i}_{2m}) = \frac{(2m_1)! \cdots (2m_d)!}{(m_1)! \cdots (m_d)! 2^m},$$

if $n = 2m$ is even and for each $k = 1, \dots, d$, the number of components of $\mathbf{i}_{2m}$ equal to k, denoted by $2m_k$, is also even, and $\alpha(\mathbf{i}_n) = 0$, otherwise.

Proposition 1. *Assume $Hd < \frac{3}{2}$. Then, we have*

$$\ell = \sum_{m=1}^{\infty} I_{2m}(f_{2m}),$$

where f_{2m} is the element of $(\mathcal{H})^{\otimes 2m}$ given by

$$f_{2m}(\mathbf{i}_{2m}, r_1, \dots, r_{2m}) = \frac{(2\pi)^{-\frac{d}{2}} \alpha(\mathbf{i}_{2m})}{(2m)!} \times \int_0^T \int_0^t dsdt |t-s|^{-Hd-2Hm} \prod_{j=1}^{2m} \mathbf{1}_{[s,t]}(r_j). \tag{3}$$

Let us introduce the following notation.

$$\lambda = |t-s|^{2H}, \quad \rho = |t'-s'|^{2H}, \tag{4}$$

and

$$\mu = \frac{1}{2}\left[|s-t'|^{2H} + |s'-t|^{2H} - |t-t'|^{2H} - |s-s'|^{2H} \right]. \tag{5}$$

Notice that λ is the variance of $B_t^{H,1} - B_s^{H,1}$, ρ is the variance of $B_{t'}^{H,1} - B_{s'}^{H,1}$, and μ is the covariance between $B_t^{H,1} - B_s^{H,1}$ and $B_{t'}^{H,1} - B_{s'}^{H,1}$, where $B^{H,1}$ denotes a one-dimensional fractional Brownian motion with Hurst parameter H.

The L^2-norm of the $2m$th Wiener chaos of ℓ can be computed as follows.

$$\begin{aligned} \mathbb{E}\left[(I_{2m}(f_{2m}))^2 \right] &= (2m)! \, \|f_{2m}\|^2_{\mathcal{H}^{\otimes(2m)}} \\ &= (2m)! \sum_{m_1+\cdots+m_d=m} \frac{(2m)!}{(2m_1)! \cdots (2m_d)!} \frac{(2\pi)^{-d}}{((2m)!)^2} \alpha(\mathbf{i}_{2m})^2 \\ &\quad \times \int_{\mathcal{T}} \lambda^{-\frac{d}{2}-m} \rho^{-\frac{d}{2}-m} \mu^{2m} \, dsdtds'dt' \\ &= \frac{\alpha_m}{(2\pi)^d 2^{2m}} \int_{\mathcal{T}} (\lambda\rho)^{-\frac{d}{2}-m} \, \mu^{2m} dsdtds'dt', \end{aligned} \tag{6}$$

where

$$\alpha_m = \sum_{m_1+\cdots+m_d=m} \frac{(2m_1)! \cdots (2m_d)!}{(m_1!)^2 \cdots (m_d!)^2},$$

and

$$\mathcal{T} = \{(s,t,s',t') : 0 < s < t < T, 0 < s' < t' < T\}.$$

The following lemma will be useful later.

LEMMA 2. *For any $z \in [0,1)$ we have*

$$\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 2^{2m}} z^m = \frac{1}{\sqrt{1-z}}.$$

Proof. This is a well-known result that can be checked, for instance, by noticing that

$$\sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2 2^{2m}} z^m = \sqrt{2\pi}\mathbb{E}\left(e^{z(Y/2)^2}\right),$$

where Y is a standard normal random variable. □

3. Regularity of the renormalized self-intersection local time. For any $\alpha > 0$, we denote by $\mathbb{D}^{\alpha,2}$ the class of "smooth" functionals of the fractional Brownian motion, in the sense of Meyer-Watanabe. That is,

$$\mathbb{D}^{\alpha,2} = \{F \in L^2 : \sum_{n=0}^{\infty} (n+1)^\alpha \mathbb{E}((J_n(F))^2) < \infty\},$$

where $J_n(F)$ is the n-th chaos of F, namely, $F = \sum_{n=0}^{\infty} J_n(F)$.

The following theorem is the main result of this paper.

THEOREM 3. *Fix $\alpha > 0$. Assume that $H < \min(\frac{3}{2d}, \frac{2(\alpha\wedge 1)}{d+2\alpha})$. Then the renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{\alpha,2}$.*

REMARK 4. *If $\alpha = 1$, we recover the result by Hu [4].*

The theorem is the direct consequence of the following two lemmas which are themselves interesting.

LEMMA 5. *a) The renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{N,2}$, where $N \geq 1$, is an integer, if and only if*

$$\int_{\mathcal{T}} \mu^{2N} \delta^{-\frac{d}{2}-N} ds dt ds' dt' < \infty.$$

b) The renormalized self-intersection local time ℓ belongs to $\mathbb{D}^{N+\beta,2}$, where $N \geq 0$, is an integer, and $0 < \beta < 1$, if for some $1 > \beta' > \beta$

$$\int_{\mathcal{T}} \mu^{2(N+\beta')} \delta^{-\frac{d}{2}-N-\beta'} ds dt ds' dt' < \infty. \tag{7}$$

Proof. From (6) we obtain that, for all $\alpha > 0$, a necessary and sufficient condition for ℓ to be in $\mathbb{D}^{\alpha,2}$ is

$$B := \sum_{m=1}^{\infty} \frac{m^\alpha \alpha_m}{2^{2m}} \int_{\mathcal{T}} \frac{\gamma^m}{(\lambda\rho)^{\frac{d}{2}}} ds dt ds' dt' < \infty, \tag{8}$$

where

$$\gamma = \frac{\mu^2}{\lambda\rho}.$$

Using Lemma 2 we deduce the following formula for all $z \in [0,1)$

$$\sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} z^m = (1-z)^{-\frac{d}{2}}. \tag{9}$$

Suppose first that $\alpha = N$ is an integer. In this case, differentiating both sides of (9) N times with respect to z yields

$$\sum_{m=N}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1)\cdots(m-N+1) z^{m-N} = C(1-z)^{-\frac{d}{2}-N},$$

where $C = \frac{d}{2}\left(\frac{d}{2}+1\right)\cdots\left(\frac{d}{2}+N-1\right)$. Hence,

$$\sum_{m=N}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1)\cdots(m-N+1) z^{m} = Cz^N(1-z)^{-\frac{d}{2}-N},$$

and we get that (8) is equivalent to

$$\int_{\mathcal{T}} \frac{\gamma^N (1-\gamma)^{-\frac{d}{2}-N}}{(\lambda\rho)^{\frac{d}{2}}} ds dt ds' dt' = \int_{\mathcal{T}} \mu^{2N} \delta^{-\frac{d}{2}-N} ds dt ds' dt' < \infty,$$

where

$$\delta = \lambda\rho - \mu^2.$$

This proves part a) of the lemma.

Suppose now that $k = N + \beta$, with $0 < \beta < 1$, and $N \geq 0$. Multiplying both members of Equation (9) by $(y-z)^{-\beta}$ and integrating in the variable z from 0 to y, we obtain

$$\sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} y^{m-\beta+1} = \int_0^y (1-z)^{-\frac{d}{2}} (y-z)^{-\beta} dz.$$

Hence,

$$\sum_{m=0}^{\infty} \frac{\alpha_m}{2^{2m}} \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} y^{m} = \int_0^1 (1-yt)^{-\frac{d}{2}} (1-t)^{-\beta} dt.$$

Differentiating this identity $N+1$ times with respect to z yields

$$\sum_{m=N+1}^{\infty} \frac{\alpha_m}{2^{2m}} m(m-1)\cdots(m-N-2) \frac{\Gamma(1-\beta)\Gamma(m)}{\Gamma(1-\beta+m)} z^{m-N-1}$$
$$= C \int_0^1 (1-zt)^{-\frac{d}{2}-N-1} t^{N+1} (1-t)^{-\beta} dt,$$

where $C = \frac{d}{2}\left(\frac{d}{2}+1\right)\cdots\left(\frac{d}{2}+N\right)$. Hence, (8) is equivalent to

$$\int_{\mathcal{T}} (\lambda\rho)^{-\frac{d}{2}}\gamma^{N+1}\left(\int_0^1 (1-\gamma y)^{-\frac{d}{2}-N-1} y^{N+1}(1-y)^{-\beta}dy\right) dsdtds'dt' < \infty. \tag{10}$$

We claim that for all $\beta' > \beta$,

$$\int_0^1 (1-y)^{-\beta}(1-\gamma y)^{-\frac{d}{2}-N-1}dy \le k(1-\gamma)^{-\frac{d}{2}-N-\beta'}.$$

In fact, $(1-\gamma y)^{-\frac{d}{2}-N-1} \le (1-y)^{\beta'-1}(1-\gamma)^{-\frac{d}{2}-N-\beta'}$. Thus,

$$\begin{aligned}\int_0^1 (1-y)^{-\beta}(1-\gamma y)^{-\frac{d}{2}-N-1}dy &\le (1-\gamma)^{-\frac{d}{2}-N-\beta'}\int_0^1 (1-y)^{-\beta+\beta'-1}dy \\ &\le \frac{1}{\beta'-\beta}(1-\gamma)^{-\frac{d}{2}-N-\beta'}.\end{aligned}$$

Hence, (10) holds if

$$\begin{aligned}&\int_{\mathcal{T}} (\lambda\rho)^{-\frac{d}{2}}\gamma^{N+1}(1-\gamma)^{-\frac{d}{2}-N-\beta'}dsdtds'dt' \\ &= \int_{\mathcal{T}} (\lambda\rho)^{\beta'-1}\mu^{2(N+1)}\delta^{-\frac{d}{2}-N-\beta'}dsdtds'dt' < \infty,\end{aligned}$$

and (7) holds because $\mu^2 \le \lambda\rho$. □

Lemma 6. *Fix a positive real number $\alpha > 0$. Suppose that $H < \min\left(\frac{3}{2d}, \frac{2(\alpha\wedge 1)}{d+2\alpha}\right)$. Then*

$$\int_{\mathcal{T}} \mu^{2\alpha}\delta^{-\frac{d}{2}-\alpha}dsdtds'dt' < \infty.$$

Proof. Denote

$$\mathcal{T}\cap\{s<s'\} = \mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_3, \tag{11}$$

where

$$\begin{aligned}\mathcal{T}_1 &= \{(t,s,t',s') : 0<s<s'<t<t'<T\},\\ \mathcal{T}_2 &= \{(t,s,t',s') : 0<s<s'<t'<t<T\},\\ \mathcal{T}_3 &= \{(t,s,t',s') : 0<s<t<s'<t'<T\}.\end{aligned}$$

We will make use of the notation:

i) If $(t,s,t',s')\in\mathcal{T}_1$, we put $a = s'-s$, $b = t-s'$ and $c = t'-t$. On this region, the functions λ, ρ and μ defined in (4) and (5) take the following values

$$\lambda = \lambda_1 := \lambda_1(a,b,c) := (a+b)^{2H}, \rho = \rho_1 := (b+c)^{2H}, \tag{12}$$

$$\mu = \mu_1 := \mu_1(a,b,c) := \frac{1}{2}\left[(a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}\right]. \tag{13}$$

ii) If $(t,s,t',s') \in \mathcal{T}_2$, we put $a = s'-s$, $b = t'-s'$ and $c = t-t'$. On this region we will have

$$\lambda = \lambda_2 := b^{2H}, \rho = \rho_2 := (a+b+c)^{2H}, \tag{14}$$

$$\mu = \mu_2 := \frac{1}{2}\left[(b+c)^{2H} + (a+b)^{2H} - c^{2H} - a^{2H}\right]. \tag{15}$$

iii) If $(t,s,t',s') \in \mathcal{T}_3$, we put $a = t-s$, $b = s'-t$ and $c = t'-s'$. On this region we will have

$$\lambda = \lambda_3 := a^{2H}, \rho = \rho_3 := c^{2H}, \tag{16}$$

$$\mu = \mu_3 := \frac{1}{2}\left[(a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}\right]. \tag{17}$$

For $i = 1,2,3$ we set

$$\delta_i = \lambda_i \rho_i - \mu_i^2.$$

Note that λ_i, ρ_i, μ_i and so on, $i = 1,2,3$, are functions of a, b, and c.

In the sequel we will denote by k a generic constant that may depend on H and d.

The following lower bounds were obtained by Hu in [4] using the local nondeterminism property of the fractional Brownian motion (see Berman [2]).

$$\delta_1 \geq k\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right], \tag{18}$$

$$\delta_i \geq k\lambda_i\rho_i,\ i = 2,3. \tag{19}$$

Using the above decomposition of the region $\mathcal{T}$, it suffices to show that $A_i < \infty$, for $i = 1,2,3$, where

$$A_i := \int_{[0,T]^3} \mu_i^{2N}\delta_i^{-\frac{d}{2}-N} dadbdc.$$

Then the proof of the lemma will de done in three steps:

Step 1. We claim that

$$A_1 < \infty.$$

We have

$$\begin{aligned}
\mu_1 &= \frac{1}{2}\left((a+b+c)^{2H} + b^{2H} - a^{2H} - c^{2H}\right) \\
&= \frac{1}{2}\left((a^2+b^2+c^2+2ab+2ac+2bc)^H + b^{2H} - a^{2H} - c^{2H}\right) \\
&\leq b^{2H} + 2^{H-1}a^Hb^H + 2^{H-1}a^Hc^H + 2^{H-1}b^Hc^H.
\end{aligned}$$

The, using (18) yields

$$\begin{aligned}\mu_1^{2\alpha} &\le k\left(b^{4\alpha H} + \left(a^{2\alpha H}b^{2\alpha H} + a^{2\alpha H}c^{2\alpha H} + b^{2\alpha H}c^{2\alpha H}\right)\right)\\ &\le 3k\left(b^{2\alpha H} + \delta^{\alpha}\right).\end{aligned}$$

As a consequence,

$$\mu_1^{2\alpha}\delta_1^{-\frac{d}{2}-\alpha} \le k\left(\delta_1^{-\frac{d}{2}} + b^{4\alpha H}\delta_1^{-\frac{d}{2}-\alpha}\right). \tag{20}$$

Using again (18) we obtain

$$\begin{aligned}\delta_1^{-\frac{d}{2}} &\le k\left[(a+b)^H(b+c)^H a^H c^H\right]^{-\frac{d}{2}}\\ &\le k\,(abc)^{-\frac{2}{3}Hd},\end{aligned}$$

where $-\frac{2}{3}Hd > -1$.

In order to treat the second term of (20) we consider two different cases. Assume first that $d \le 6\alpha$ .Then

$$\begin{aligned}b^{2\alpha H}\delta_1^{-\frac{d}{2}-\alpha} &\le k\left[(a+b)^{2H}c^{2H} + (b+c)^{2H}a^{2H}\right]^{-\frac{d}{2}-\alpha} b^{4\alpha H}\\ &\le k\left[(bc)^{2H} \ + (ba)^{2H}\right]^{-\frac{d}{2}-\alpha} b^{4\alpha H}\\ &\le k\,(ac)^{-H\left(\frac{d}{2}+\alpha\right)}\, b^{H(2\alpha-d)},\end{aligned}$$

and both exponents are larger than -1, because $H < \frac{2}{d+2\alpha} \le \frac{1}{d-2\alpha}$.

For $d > 6\alpha$, we make use of the estimate

$$\begin{aligned}b^{2\alpha H}\delta_1^{-\frac{d}{2}-\alpha} &\le k\left[(a+b)^H c^H (b+c)^H a^H\right]^{-\frac{d}{2}-\alpha} b^{4\alpha H}\\ &\le k\,(ac)^{-(\beta_1+1)\left(\frac{d}{2}+\alpha\right)H}\, b^{4\alpha H-\beta_2(d+2\alpha)H},\end{aligned}$$

where β_1, $\beta_2 \ge 0$, and $\beta_1 + \beta_2 = 1$. Taking

$$\beta_1 = \frac{d-6\alpha}{3(d+2\alpha)}, \beta_2 = \frac{2d+12\alpha}{3(d+2\alpha)}$$

we obtain

$$b^{2\alpha H}\delta_2^{-\frac{d}{2}-\alpha} \le k\,(abc)^{-\frac{2dH}{3}}.$$

Step 2. We claim that

$$A_2 < \infty.$$

If $H \ge \frac{1}{2}$ we have

$$\begin{aligned}\mu_2 &= \frac{1}{2}\left((b+c)^{2H} + (a+b)^{2H} - a^{2H} - c^{2H}\right)\\ &= Hb\int_0^1 \left[(a+bu)^{2H-1} + (c+bu)^{2H-1}\right] du\\ &\le kb(a+b+c)^{2H-1}.\end{aligned}$$

Therefore, using (19)

$$\mu_2^{2\alpha}\delta_2^{-\frac{d}{2}-\alpha} \le b^{-H(d+2\alpha)+2\alpha}(a+b+c)^{H(2\alpha-d)-2\alpha}.$$

Using the inequality $a+b+c \ge Ca^{\beta}c^{\beta}b^{1-2\beta}$, with $\beta = \frac{2Hd}{3Hd+6\alpha-6\alpha H}$, we obtain

$$\mu_2^{2\alpha}\delta_2^{-\frac{d}{2}-\alpha} \le k\,(abc)^{-\frac{2dH}{3}}.$$

Notice that $\beta \in (0,\frac{1}{2}]$, because $H < \frac{2\alpha}{d+2\alpha}$.

Suppose now that $H < \frac{1}{2}$. In this case we have

$$\mu_2 \le kb\left(a^{\beta(2H-1)}b^{(1-\beta)(2H-1)} + c^{\beta(2H-1)}b^{(1-\beta)(2H-1)}\right),$$

for all $\beta \in [0,1]$. Hence,

$$\begin{aligned}\mu_2^{2\alpha}\delta_2^{-\frac{d}{2}-\alpha} &\le ka^{\beta(2H-1)2\alpha}b^{(1-\beta)(2H-1)2\alpha+2\alpha}\delta_2^{-\frac{d}{2}-\alpha}\\ &\quad +kc^{\beta(2H-1)2\alpha}b^{(1-\beta)(2H-1)2\alpha+2\alpha}\delta_2^{-\frac{d}{2}-\alpha}\\ &=: I_1 + I_2.\end{aligned}$$

By symmetry it suffices to treat the term I_1. We have

$$I_1 \le ka^{\beta(2H-1)2\alpha}b^{2(1-2\beta)\alpha H+2\alpha\beta-dH}(a+b+c)^{-dH-2\alpha H}.$$

Now we make use of the lower bound

$$(a+b+c)^{-1} \ge ka^{\gamma_1}b^{\gamma_2}c^{\gamma_3},$$

where $\gamma_1+\gamma_2+\gamma_3 = 1$, and γ_1, γ_2, $\gamma_3 \ge 0$. In this way we obtain

$$I_1 \le ka^{\beta_1}b^{\beta_2}c^{\beta_3},$$

where

$$\begin{aligned}\beta_1 &= \beta(2H-1)2\alpha - \gamma_1 H(d+2\alpha)\\ \beta_2 &= 2(1-2\beta)\alpha H + 2\alpha\beta - dH - \gamma_2 H(d+2\alpha)\\ \beta_3 &= -\gamma_3 H(d+2\alpha).\end{aligned}$$

If $d \le 6\alpha$, we choose $\beta = 0$, $\gamma_1 = \gamma_3 = \frac{1}{2}$, and $\gamma_2 = 0$, and we obtain the exponents

$$\begin{aligned}\beta_1 = \beta_3 &= -\frac{H(d+2\alpha)}{2} > -1\\ \beta_2 &= H(2\alpha-d) > -1.\end{aligned}$$

If $d > 6\alpha$, we choose

$$\beta = \frac{H(d-6\alpha)}{6(1-2H)\alpha}, \gamma_1 = \frac{d+6\alpha}{3(d+2\alpha)}, \gamma_2 = 0, \gamma_3 = \frac{2d}{3(d+2\alpha)},$$

and we obtain the exponents

$$\beta_1 = \beta_2 = \beta_3 = -\frac{2dH}{3} > -1.$$

Step 3.- We claim that

$$A_3 < \infty.$$

In this case, (17) and the inequality

$$b + vc + ua \geq k(vcua)^{\beta} b^{1-2\beta},$$

with $\beta \in [0,1]$, yield

$$\mu_3 \leq k\,(ac)^{1+\beta(2H-2)}\, b^{(1-2\beta)(2H-2)},$$

provided $\beta < \frac{1}{2(1-H)}$. As a consequence,

$$\mu_3^{2\alpha} \delta_3^{-\frac{d}{2}-\alpha} \leq k\,(ac)^{[1+\beta(2H-2)]2\alpha - dH - 2H\alpha}\, b^{(1-2\beta)(2H-2)2\alpha}.$$

Choosing $\beta = \frac{6\alpha - 6H\alpha - Hd}{12\alpha(1-H)}$, we obtain

$$\mu_3^{2\alpha} \delta_3^{-\frac{d}{2}-\alpha} \leq k\,(ac)^{-\frac{2dH}{3}}.$$

Notice that $\beta > 0$ because $H < \frac{2\alpha}{2\alpha+d} < \frac{6\alpha}{6\alpha+d}$, and also $\beta < \frac{1}{2(1-H)}$. □

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