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A CHARACTERIZATION OF A CLASS OF DISCRETE NONLINEAR FEEDBACK SYSTEMS

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Abstract. A class of discrete dynamical systems with nonlinear feedback is considered. These systems generalize various maps arising in connection with chaotic dynamical systems, topological dynamics, and linear systems theory. We give a complete characterization of this class of systems.

1. Introduction. Let $\mathbf{F}$ be the finite field $\{0, 1\}$ and let B^n be the collection of binary n-tuples. In this paper we consider the discrete dynamical system with control

$$\begin{aligned} \vec{x}_{k+1} &= f(\vec{x}_k) + u_k g(\vec{x}_k) \\ y_k &= h(\vec{x}_k) + u_k \end{aligned} \tag{1}$$

as well as the more general system

$$\begin{aligned} \vec{x}_{k+1} &= f(\vec{x}_k) + u_k g(\vec{x}_k) \\ y_k &= h(\vec{x}_k) + u_k p(\vec{x}_k)\ , \end{aligned} \tag{2}$$

where $\vec{x}_k \in B^n$, $u_k \in \mathbf{F}$, and $f, g : B^n \to B^n$, and $h, p : B^n \to \mathbf{F}$. The functions f, g, h and p are necessarily polynomials [2, Theorem 19.1] so that, for example, f is of the form

$$f(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix},$$

with each $x_k \in \mathbf{F}$ and the f_i's are polynomials from B^n into $\mathbf{F}$.

We refer to the systems (1) and (2) as *nonlinear feedback systems*. These systems arise in various contexts. For example, taking $u_k = 0$ for all k we have

$$\begin{aligned} \vec{x}_{k+1} &= f(\vec{x}_k) \\ y_k &= h(\vec{x}_k). \end{aligned}$$

This particular system appears in [3] and [4] where it is used to analyze pseudo-random binary sequence generators.

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Let B^∞ be the collection of infinite sequences over $\mathbf{F}$. In [5] the system

$$\begin{aligned} \vec{x}_{k+1} &= \sigma(\vec{x}_k) \\ y_k &= h(\vec{x}_k) \end{aligned} \tag{3}$$

is considered, where σ is the left-shift, i.e., $\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$, and h is a polynomial in the first n elements of $\vec{x}$. In [5] it is shown that (3) is equivalent to the "dyadic system," which is defined by $f : [0,1) \to [0,1)$ where

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 \le x < 1 \,, \end{cases}$$

and the observation is given by the characteristic function of a union of certain half-open intervals. The dyadic map arises in the study of chaotic dynamical systems; see, for example, the article by Di Masi and Gombani [1].

If we let $f(\vec{x}) = (x_2, x_3, \ldots, x_n, 0)$, $g(\vec{x}) = (0, 0, \ldots, 0, 1)$, $p(\vec{x}) \equiv 0$, and $\vec{u} = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$, then (2) is essentially (3). The system (3) is also equivalent to certain maps studied in topological dynamics [2].

We now define a metric on the space B^∞ by

$$d(\vec{x}, \vec{y}) = \begin{cases} 0 & \text{if } \vec{x} = \vec{y} \\ \dfrac{1}{k+1} & \text{otherwise} \,, \end{cases}$$

where k is the first index such that $x_k \neq y_k$. With this topology, B^∞ is a compact, totally disconnected, perfect, metric space and hence homeomorphic to the Cantor set.

For any input sequence (or control) $(u_0, u_1, u_2, \ldots)$, a nonlinear feedback system produces an output sequence $(y_0, y_1, y_2, \ldots)$ and hence any such systems defines a maps $T : B^\infty \to B^\infty$.

Lemma 1.1. *The map $T : B^\infty \to B^\infty$ defined by (1) is a homeomorphism of B^∞ onto B^∞ for any choice of f, g, h, and $\vec{x}_0$.*

Proof. The proof that T is one-to-one and onto is trivial. To show that T is continuous, let $\varepsilon > 0$ be given and choose k such that $1/k < \varepsilon$. Now let $\delta = 1/(k+n)$ and suppose we are given $\vec{u}, \vec{v} \in B^\infty$ with $d(\vec{u}, \vec{v}) < \delta$. Let $\vec{y} = T(\vec{u})$ and $\vec{x} = T(\vec{v})$. Then $u_j = v_j$ for $j = 0, 1, \ldots, k+n$ and hence $y_j = x_j$ for $j = 0, 1, \ldots, k$ which implies $d(T(\vec{u}), T(\vec{v})) < \varepsilon$. The continuity of T^{-1} follows from the fact that B^∞ is compact and metric. $\square$

In general, a function T corresponding to (2) need not be one-to-one or onto, but it will be continuous.

The primary problem of interest in this paper is the inverse problem, i.e., which functions $T : B^\infty \to B^\infty$ can be represented as nonlinear feedback systems (1) or (2). In the next section we show that not all maps defined on the sequence space can be realized as nonlinear feedback systems. Then in Section 3 we characterize all systems which can be realized as nonlinear feedback systems.

2. Nonlinear feedback systems. Let $T : B^\infty \to B^\infty$ be the mapping associated with (2). Since $\vec{x}_0$ is given, and all functions are polynomial, there exists a polynomial t_0 such that

$$\begin{aligned} y_0 &= h(\vec{x}_0) + u_0 p(\vec{x}_0) \\ &= t_0(u_0) \end{aligned}$$

and there exists a polynomial t_1 such that

$$\begin{aligned} y_1 &= h(\vec{x}_1) + u_1 p(\vec{x}_1) \\ &= h(f(\vec{x}_0) + u_0 g(\vec{x}_0)) + u_1 p(f(\vec{x}_0) + u_0 g(\vec{x}_0)) \\ &= t_1(u_0, u_1) \end{aligned}$$

and so on. In general,

$$y_k = t_k(u_0, u_1, \ldots, u_k) \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{4}$$

where each t_k is a polynomial in $\{u_0, u_1, \ldots, u_k\}$. For such a T we write $T = \{t_k\}$.

THEOREM 2.1. *There exist $T = \{t_k\}$ which cannot be realized as nonlinear feedback systems of the form (1) or (2).*

Proof. We give three distinct proofs of this result. The simplest proof is to observe that the number of nonlinear feedback systems in (1) and (2) is countable, while the number of T is uncountable.

A more instructive proof is obtained if we let $\delta \in B^\infty$ be a random sequence and take

$$y_i = u_i + \delta_i. \tag{5}$$

This system maps $\vec{0}$ to δ. Letting $\vec{u} = \vec{0}$ in (1) or (2) yields

$$\begin{aligned} \vec{x}_{k+1} &= f(\vec{x}_k) \\ y_k &= h(\vec{x}_k) = \delta_k. \end{aligned} \tag{6}$$

But then (6) is a recursive machine capable of generating a random sequence, which is impossible, and hence (5) cannot be realized as a nonlinear feedback system.

Finally, a somewhat different example can be given. Suppose T is defined by

$$y_k = t_k(u_0, u_1, \ldots, u_k) = u_k + \delta_k \tag{7}$$

where, for example,

$$\delta = (0\ 1\ 00\ 11\ 000\ 111\ 0000\ 1111 \ldots).$$

In this case T is a homeomorphism of B^∞ onto B^∞ with $T(\vec{0}) = \delta$. Again, letting $\vec{u} = \vec{0}$ in (1) or (2) gives (6). But since the state space is finite, the output sequence $\{y_k\}$ is ultimately periodic, i.e., there exists an m such that $y_m, y_{m+1}, y_{m+2}, \ldots$ is periodic. Since δ is obviously not eventually periodic, (7) cannot be realized as a nonlinear feedback system. □

COROLLARY 2.1. *Not all T which are homeomorphisms of B^∞ onto B^∞ can be realized as nonlinear feedback systems.*

In the next section we characterize those T which can be realized as nonlinear feedback systems.

3. A characterization of nonlinear feedback systems. Let $T = \{t_k\}$ be of the form (4).

DEFINITION 3.1. *We say that $T = \{t_k\}$ is proper if*

1. *Each $t_k(u_0, u_1, \ldots, u_k)$ is a polynomial in the preceeding $m+1$ variables $u_{k-m}, \ldots, u_k$, so that t_k is determined by a list of $n = 2^{m+1}$ coefficients*

$$t_k \leftrightarrow \begin{pmatrix} a_{1,k} \\ \vdots \\ a_{n,k} \end{pmatrix} = A_k.$$

2. *The t_k's are given recursively by*

$$A_{k+1} = F(A_k) + u_k G(A_k). \tag{8}$$

THEOREM 3.1. *If $T = \{t_k\}$ is proper, T can be realized as a nonlinear feedback system of the form (2).*

Proof. Let

$$\vec{x}_k = \begin{pmatrix} u_{k-1} \\ \vdots \\ u_{k-m} \\ \hline a_{1,k} \\ \vdots \\ a_{n,k} \end{pmatrix} = \begin{pmatrix} U_k \\ \hline A_k \end{pmatrix}.$$

By letting S be the shift, i.e., $S(U_k) = (0, u_{k-1}, \ldots, u_{k-m+1})^T$, we have

$$\begin{aligned} \vec{x}_{k+1} = \begin{pmatrix} U_{k+1} \\ \hline A_{k+1} \end{pmatrix} = \begin{pmatrix} u_k \\ \vdots \\ u_{k-m+1} \\ \hline A_{k+1} \end{pmatrix} &= \begin{pmatrix} S(U_k) \\ \hline F(A_k) \end{pmatrix} + u_k \begin{pmatrix} e_1 \\ \hline G(A_k) \end{pmatrix} \\ &= R(\vec{x}_k) + u_k H(\vec{x}_k), \end{aligned}$$

for some polynomials R and H.

It remains to show that we can write y_k as

$$y_k = P(\vec{x}_k) + u_k Q(\vec{x}_k).$$

By assumption, $y_k = t_k(u_{k-m}, \ldots, u_k)$, where A_k consists of the coefficients of t_k, so that for any k we have

$$\begin{aligned} y_k = {} & a_{1,k}u_{k-m} + a_{2,k}u_{k-m+1} + \cdots + a_{m+1,k}u_k + a_{m+2,k}u_{k-m}u_{k-m+1} \\ & + \cdots\cdots + a_{n,k}u_{k-m}u_{k-m+1}\cdots u_k. \end{aligned}$$

Factoring out u_k we have the desired result, $y_k = P(\vec{x}_k) + u_k Q(\vec{x}_k)$ where P and Q are polynomials. □

Corollary 3.1 follows form the observation that if (2) is onto (as a map from B^∞ to B^∞) then $p \equiv 1$.

COROLLARY 3.1. *If T is proper and onto, T can be realized as a nonlinear feedback system of the form (1).*

Next, we show that all nonlinear feedback systems are proper. This result, together with Theorem 3.1, provides a nice characterization of nonlinear feedback systems.

THEOREM 3.2. *A nonlinear feedback system as in (1) or (2) can be represented by a proper T.*

Proof. We have the system

$$\begin{aligned} \vec{x}_{k+1} &= f(\vec{x}_k) + u_k g(\vec{x}_k) \\ y_k &= h(\vec{x}_k) + u_k p(\vec{x}_k)\ , \end{aligned} \tag{9}$$

which induces $T = \{t_k\}$ with $y_k = t_k(u_0, \ldots, u_k)$. We must demonstrate a recursion of the form (8)

We write $\vec{x}_k = (x_1^k, x_2^k, \ldots, x_n^k)$ and define A_k to be the 2^n-tuple

$$A_k = \left(x_1^k, \ldots, x_n^k, x_1^k x_2^k, \ldots, x_{n-1}^k x_n^k, \ldots\ldots, x_1^k x_2^k \cdots x_n^k\right)^T.$$

Now the polynomial h in (9) is determined by a list of 2^n coefficients $\vec{h}$, which we list in the order corresponding to A_k. We also denote p by its coefficients $\vec{p}$, so that

$$t_k(u_0, u_1, \ldots, u_k) = \left\langle \vec{h}, A_k \right\rangle + u_k \left\langle \vec{p}, A_k \right\rangle,$$

where $\langle x, y \rangle$ is the inner product $x^T y$. Finally, since $\vec{x}_{k+1} = f(\vec{x}_k) + u_k g(\vec{x}_k)$, it is clear that $A_{k+1} = F(A_k) + u_k G(A_k)$ for an appropriate choice of F and G, and the theorem is proved. □

4. Conclusion. In this paper we considered a class of discrete nonlinear feedback systems and it was shown that these systems can be specialized to several well-known types of systems. We then gave a complete characterization of all systems which can be realized as such nonlinear feedback systems.

The results in this paper are all given in terms of the binary field. However, the results will easily generalize to any setting where all functions are polynomials.

REFERENCES

[1] G. B. Di Masi and A. Gombani, *On the observability of chaotic systems: an example*, Robust Control of Linear Systems and Nonlinear Control: Proceedings of the International Symposium MTNS-89, Vol. II (Progress in Systems and Control Theory, Vol. 4), eds. M. A. Kaashoek, J. H. van Schuppen, and A. C. M. Ran, Birkhäuser, Boston, 1990, pp. 489–496.

[2] G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Mathematical Systems Theory, 3:4(1969), pp. 320–375.

[3] C. F. Martin and M. Stamp, *Constructing polynomials over finite fields*, in: Computation and Control: Proceedings of the Bozeman Conference (Progress in Systems and Control Theory, Vol. 1), eds. K. Bowers and J. Lund, Birkhäuser, Boston, 1989, pp. 233–252.

[4] C. F. Martin and M. Stamp, *Classification and realization of pseudo-random number generators*, System and Control Letters, 14(1990), pp. 169-175.

[5] C. F. Martin and M. Stamp, *Observability of the left-shift by polynomial observers*, submitted.