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The domain of attraction of the α-sun operator for type II and type III distributions

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Let (Y_n) be a sequence of independent random variables with common distribution F and define the iteration: $X_0 = x_0$, $X_n := X_{n-1} \vee (\alpha X_{n-1} + Y_n)$, $\alpha \in [0, 1)$. We denote by $\mathscr{D}(\Phi_\gamma)$ the domain of maximal attraction of Φ_γ, the extreme value distribution of the first type. Greenwood and Hooghiemstra showed in 1991 that for $F \in \mathscr{D}(\Phi_\gamma)$ there exist norming constants $a_n > 0$ and $b_n \in \mathbf{R}$ such that $a_n^{-1}\{X_n - b_n/(1-\alpha)\}$ has a non-degenerate (distributional) limit. In this paper we show that the same is true for $F \in \mathscr{D}(\Psi_\gamma) \cup \mathscr{D}(\Lambda)$, the type II and type III domains. The method of proof is entirely different from the method in the aforementioned paper. After a proof of tightness of the involved sequences we apply (modify) a result of Donnelly, concerning weak convergence of Markov chains with an entrance boundary.

Keywords: extremal limits; self-similar Markov processes; weak convergence

1. Introduction

Let $(Y_n)_{n\geqslant 1}$ be a sequence of independent random variables with common distribution function F and define the iteration

$$X_0 = x_0,\ X_n := X_{n-1} \vee (\alpha X_{n-1} + Y_n),\ n \geqslant 1,\ \alpha \in [0, 1). \tag{1}$$

We denote by $\mathscr{D}(G)$ the domain of maximal attraction of the distribution G, where G is one of the extreme value distributions. For $F \in \mathscr{D}(G)$ and $a_n > 0$, $b_n \in \mathbf{R}$ such that $F^n(a_n x + b_n) \to G(x)$, for all x, we define, for $n \geqslant 1$,

$$Y_{n,j} := \frac{Y_j - b_n}{a_n}, \quad j = 1, 2, \ldots .$$

For $\alpha \in [0, 1)$ and $x_0 \in \mathbf{R}$, the random element $X_n(\cdot) \in D[0, \infty)$ (the space of cadlag functions, equipped with the Skorohod topology) is defined by

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$$X_n(t) := \begin{cases} a_n^{-1}\left(x_0 - \dfrac{b_n}{1-\alpha}\right), & 0 \leq t < n^{-1}, \\ X_n\left(\dfrac{j-1}{n}\right) \vee \left\{\alpha X_n\left(\dfrac{j-1}{n}\right) + Y_{n,j}\right\}, & \dfrac{j}{n} \leq t < \dfrac{j+1}{n}, \, j = 1, 2, \ldots. \end{cases} \tag{2}$$

Observe that the relation between the sequence of random variables X_n given by (1) and the sequence of processes $X_n(\cdot)$ is

$$X_n\left(\frac{j}{n}\right) = a_n^{-1}\left(X_j - \frac{b_n}{1-\alpha}\right). \tag{3}$$

The motivation for studying recursive sequences such as (1) comes from a stochastic solar energy model (cf. Haslett 1980). Note that for $\alpha = 0$ the sequence X_n is the sequence of partial maxima:

$$X_n = x_0 \vee Y_1 \vee \cdots \vee Y_n,$$

whereas for $\alpha = 1$ (this value is not included in the definition (1)) we obtain

$$X_n = x_0 + [Y_1]^+ + \cdots + [Y_n]^+ \qquad ([x]^+ = x \vee 0, \, x \in \mathbf{R}).$$

Hence the sequence X_n defined by (1) is between maxima and sums of independent random variables, and from that viewpoint of theoretical interest.

Greenwood and Hooghiemstra (1991) showed that for $F \in \mathscr{D}(\Phi_\gamma)$, where

$$\Phi_\gamma(x) := \exp(-x^{-\gamma}) 1_{[0,\infty)}(x),$$

the process $X_n(\cdot)$ converges weakly in $D[0, \infty)$ to a self-similar Markov process $Z(\cdot)$. Furthermore the distribution of $Z(1)$ admits a density h_α on $(0, \infty)$, given as the unique density solution of the equation

$$h_\alpha(x) = \frac{\gamma}{x}\int_0^x (x - \alpha u)^{-\gamma} h_\alpha(u)\, \mathrm{d}u, \qquad x > 0.$$

In this case $X_n(0) = a_n^{-1}\{x_0 - b_n/(1-\alpha)\} \to 0$, and the proof proceeds by showing that the functional induced by (2) on the point process $\sum \delta_{(j/n, Y_{n,j})}$ is continuous.

In this paper we prove weak convergence of $X_n(\cdot)$ for $F \in \mathscr{D}(\Psi_\gamma) \cup \mathscr{D}(\Lambda)$, where

$$\Psi_\gamma(x) := \exp[-(-x)^\gamma] 1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x),$$

$$\Lambda(x) := \exp(-\mathrm{e}^{-x}).$$

For $F \in \mathscr{D}(\Psi_\gamma) \cup \mathscr{D}(\Lambda)$ we have $X_n(0) = a_n^{-1}\{x_0 - b_n/(1-\alpha)\} \to -\infty$. In these cases the method of proof is entirely different from that in the work of Greenwood and Hooghiemstra (1991). It is based on the weak convergence of Markov processes to a limiting Markov process with entrance boundary. The proof uses monotonicity of the relevant Markov process and tightness of the sequence $X_n(t)$ for fixed positive t. In Sections 2 and 3 we prove weak convergence, aside from the tightness of $X_n(t)$, which we postpone to Section 4.

2. The convergence result for type II distributions

Let $F \in \mathscr{D}(\Psi_\gamma)$; then $r := \sup\{x: F(x) < 1\} < \infty$, and $1 - F(r - x^{-1}) = x^{-\gamma}L(x)$, with L slowly varying at infinity. Set $b_n \equiv r$ and $a_n := r - \inf\{y: 1 - F(y) \leqslant n^{-1}\}$. The points $(j/n, Y_{n,j})$, $n \geqslant 1$, $j = 1, 2, \ldots$ are contained in $E := (0, \infty) \times (-\infty, 0)$. To prepare for the formulation of the convergence result we first specify what will be the limiting Markov process. Denote by N a Poisson point process on E with intensity measure the product of Lebesgue measure $\mathrm{d}t$ and the measure $\mathrm{d}\mu$, where

$$\mu(y, 0) = |y|^\gamma, \; y < 0.$$

For $x < 0$ we denote by N_x the points of N in the strip $(0, \infty) \times [x, 0)$. We order the points of N_x according to the first coordinate and denote them by $(t_1, j_1), (t_2, j_2), \ldots,$ where $0 < t_1 < t_2 < \cdots$ and $j_k \in [x, 0)$. The continuous-time Markov process $Z_x(\cdot)$ with state space $[x, 0)$ is defined by

$$Z_x(t) := \begin{cases} x, & 0 \leqslant t < t_1, \\ Z_x(t_{k-1}) \vee \{\alpha Z_x(t_{k-1}) + j_k\}, & t_k \leqslant t < t_{k+1}. \end{cases} \tag{4}$$

We shall show that, for $x \to -\infty$, the process $Z_x(\cdot)$ converges almost surely to a process $Z(\cdot)$ with $Z(0) = -\infty$, almost surely, whereas, for any $t > 0$, we have $-\infty < Z(t) < 0$, almost surely, and where the conditional distribution of $(Z(s)|Z(t) = x)$ is given by the distribution of $Z_x(s - t)$, $s > t$. This final statement is clear from the definition of Z_x. The process $Z(\cdot)$ will be the limit of $X_n(\cdot)$ on $D(0, \infty)$. Here is a proof of the statements concerning $Z(\cdot)$.

Since we have, for $x < y$ and each $t \geqslant 0$,

$$Z_x(t) \leqslant Z_y(t) \leqslant 0,$$

the almost sure convergence of $Z_x(t)$ to a value $Z(t)$, possibly $-\infty$, follows. As for each x the process $Z_x(\cdot)$ is non-decreasing we obtain that $Z(\cdot)$ is non-decreasing and we hence conclude that $Z_x(\cdot)$ converges almost surely to a non-decreasing random function $Z(\cdot)$, as $x \to -\infty$. If we show that for arbitrary $t > 0$ the collection $\text{II} := \{Z_x(t), x < 0\}$ is uniformly tight, then $-\infty < Z(t) \leqslant 0$, $t > 0$. The tightness of II is a consequence of the three lemmas below, the first of which goes back to Rényi and is well known.

Lemma 1. *Fix $x < 0$. Let σ_j, $j = 1, 2, \ldots$ be the points of a Poisson process on $\mathbf{R}^+$ with intensity $|x|^\gamma$. Independent of this Poisson process we define an independent, identically distributed sequence $\beta_1, \beta_2, \ldots$ with distribution*

$$P(\beta_1 \leqslant y) = 1 - \left|\frac{y}{x}\right|^\gamma, \qquad x \leqslant y \leqslant 0.$$

Then the point process $N'_x := \sum_j \delta_{(\sigma_j, \beta_j)}$ is equal in distribution to N_x.

Lemma 2. *Let (X_n) be defined by* (1) *with initial value $X_1 = -1$, and with (Y_n) an independent, identically distributed sequence with distribution*

$$F(y) = 1 - |y|^\gamma, \qquad -1 \leqslant y \leqslant 0. \tag{5}$$

Then

$$\sup_{n\geqslant 1} n^{1/\gamma}\mathrm{E}X_n \geqslant A,$$

where $A<0$ *is given by* $|A|^\gamma := \{(1+\gamma)/\gamma\}(1-\alpha)^{-1-\gamma}$.

Remark 1. Note that F given in (5) belongs to $\mathscr{D}(\Psi_\gamma)$ and that for this specific distribution the norming constants are given by $b_n = 0$ and $a_n = n^{-1/\gamma}$. The proof below is equal to the tightness proof of Theorem 3 in Section 4 for F given in (5). Because of the smoothness of F the proof of Lemma 2 is easier than that of Theorem 3.

Proof. The conditional expectation $\mathrm{E}(X_{n+1}|X_n) = X_n + \int_{(1-\alpha)X_n}^0 \{1-F(y)\}\,\mathrm{d}y$; so by taking double expectations and using the Jensen inequality

$$\mathrm{E}X_{n+1} = \mathrm{E}g(X_n) \geqslant g(\mathrm{E}X_n), \tag{6}$$

where $g(u) := u + \{(1-\alpha)|u|\}^{1+\gamma}/(1+\gamma)$, $-1 \leqslant u \leqslant 0$. Put $u_n := \mathrm{E}X_n$ and $v_n := An^{-1/\gamma}$. We shall prove by induction that $u_n \geqslant v_n$ for all $n \geqslant 1$. For $n=1$, $u_1 = -1$ and $v_1 = A < -1$. Assume that $u_n \geqslant v_n$ for some n. By (6) and the monotonicity of g,

$$u_{n+1} \geqslant g(u_n) \geqslant g(v_n).$$

The inequality $g(v_n) \geqslant v_{n+1}$ follows because $n[1-\{n/(n+1)\}^{1/\gamma}] \leqslant 1/\gamma$, for all $n \geqslant 1$ and $\gamma > 0$. □

Lemma 3. *For any* $t>0$,

$$\lim_{M\to\infty}\lim_{x\to-\infty} P(Z_x(t) \geqslant -M) = 1. \tag{7}$$

Proof. By monotonicity it is sufficient to show (7) for a sequence $x_n \to -\infty$. Let

$$\tau_n := \inf\{s>0\colon \#\text{ points of } N \text{ contained in the set } (0,s]\times[-n^{1/\gamma},0) = n\}.$$

Observe from Lemma 1 that, for $x_n = -n^{1/\gamma}$, there holds $Z_{x_n}(\tau_n) \stackrel{d}{=} n^{1/\gamma}X_n$, if $X_1 := -1$ and F given in (5). Because N is a Poisson process with intensity $\mathrm{d}t\times\mathrm{d}\mu$ the random variable τ_n is the sum of n independent and exponentially distributed random variables each with parameter n. It is straightforward that $\tau_n \to 1$, a.s. Hence it follows from Lemma 2 and the monotonicity of $Z_x(\cdot)$ that for each $t>1$ the statement (7) holds. The result for $0<t\leqslant 1$ is easily obtained by noting that for any subsequence n_k we have, with $m_k = [n_k t]$,

$$\lim_{k\to\infty} n_k^{1/\gamma} X_{[n_k t]} = t^{-1/\gamma}\lim_{k\to\infty} m_k^{1/\gamma} X_{m_k}.$$ □

We now formulate and prove our main result for $F \in \mathscr{D}(\Psi_\gamma)$.

Theorem 1. *Let* $F\in\mathscr{D}(\Psi_\gamma)$ *and* $x_0 < r/(1-\alpha)$. *On* $D(0,\infty)$ *we have*

$$X_n(\cdot) \stackrel{d}{\to} Z(\cdot), \tag{8}$$

where $Z(\cdot)$ *is the Markov process with entrance boundary introduced above.*

Proof. The coordinate projection $X_n(t)$ at time $t > 0$ is uniformly tight as a consequence of Theorem 3 in Section 4, because

$$\lim_{n\to\infty} \frac{a_{[nt]}}{a_n} = t^{-1/\gamma},$$

and

$$X_n(t) = a_n^{-1}\left(X_{[nt]} - \frac{r}{1-\alpha}\right) = \frac{a_{[nt]}}{a_n} a_{[nt]}^{-1}\left(X_{[nt]} - \frac{r}{1-\alpha}\right).$$

Next we check that the sequence $X_n(\cdot)$ is tight in $D[a, b]$, the space of cadlag functions with $t \in [a, b]$ for each pair a, b with $0 < a < b < \infty$. Given that $X_n(a) = x \in [-M, 0]$, the process $X_n(t)$, $t \geqslant a$, is non-decreasing and converges weakly to $Z_x(t-a)$, $t \geqslant a$, because of convergence of the underlying point processes and continuity of the map $(x, y) \to x \vee (\alpha x + y)$. Hence, if n_k is a subsequence for which $X_{n_k}(a)$ converges weakly on $\mathbf{R}$, then $X_{n_k}(\cdot)$ converges weakly on $D[a, b]$. Consequently the sequence X_n is relatively compact on $D[a, b]$ (and hence tight by Prohorov's theorem).

Take a particular weakly convergent subsequence of $X_n(\cdot)$ and denote its limit by $\hat{Z}(\cdot) \in D(0, \infty)$ (for convenience we shall also index the subsequence by n). For $t > 0$ we denote by $\mathscr{C}_t$ the set of continuity points of the distribution of $\hat{Z}(t)$. We shall show that the process $\hat{Z}(\cdot)$ satisfies the following.

(i) For each $M > 0$, $\lim_{h\downarrow 0} P(\hat{Z}(h) \leqslant -M) = 1$.
(ii) For $0 < s < t$, $x \in \mathscr{C}_s$ and $y \in \mathscr{C}_t$,

$$P(\hat{Z}(s) \leqslant x,\ \hat{Z}(t) \leqslant y) = \int_{-\infty}^{z} P(\hat{Z}(s) \in \mathrm{d}u) P(Z_u(t-s) \leqslant y).$$

(iii) The finite-dimensional distributions of $\hat{Z}(\cdot)$ coincide with those of $Z(\cdot)$.

From (iii) the theorem follows, because the finite-dimensional distributions form a determining class.

If $-M \in \mathscr{C}_h$, then

$$\begin{aligned} P(\hat{Z}(h) \leqslant -M) &= \lim_{n\to\infty} P(X_n(h) \leqslant -M) \\ &\geqslant \lim_{n\to\infty} P(\sup_{1\leqslant j\leqslant [nh]} Y_{n,j} \leqslant -M(1-\alpha)) \\ &= \exp\{-hM^\gamma (1-\alpha)^\gamma\} \to 1,\ h \downarrow 0. \end{aligned}$$

This proves (i).

For $0 < s < t$, $x \in \mathscr{C}_s$ and $y \in \mathscr{C}_t$,

$$\begin{aligned} P(\hat{Z}(s) \leqslant x,\ \hat{Z}(t) \leqslant y) &= \lim_{n\to\infty} P(X_n(s) \leqslant x,\ X_n(t) \leqslant y) \\ &= \lim_{n\to\infty} \int_{-\infty}^{x} P(X_n(s) \in \mathrm{d}u) P(X_n(t) \leqslant y | X_n(s) = u). \end{aligned}$$

Since for each u we have $P(X_n(t) \leqslant y | X_n(s) = u) \to P(Z_u(t-s) \leqslant y)$ and, since the map $u \to P(Z_u(t) \leqslant x)$ is bounded and continuous, we obtain (ii) from the definition of weak convergence.

In order to prove (iii) for the one-dimensional distributions write, for $0 < h < t$ and $x \in \mathscr{C}_t$,

$$\begin{aligned} P(\hat{Z}(t) \leqslant x) &= \int_{-\infty}^{0} P(\hat{Z}(h) \in \mathrm{d}u) P(Z_u(t-h) \leqslant x) \\ &\geqslant \int_{-\infty}^{-M} P(\hat{Z}(h) \in \mathrm{d}u) P(Z_u(t-h) \leqslant x) \\ &\geqslant P(Z_{-M}(t-h) \leqslant x) P(\hat{Z}(h) \leqslant -M) \to P(Z(t) \leqslant x), \end{aligned}$$

by letting first $h \downarrow 0$ and then $M \to \infty$. On the other hand

$$\begin{aligned} P(\hat{Z}(t) \leqslant x) &= \int_{-\infty}^{0} P(\hat{Z}(h) \in \mathrm{d}u) P(Z_u(t-h) \leqslant x) \\ &\leqslant P(Z(t-h) \leqslant x) \to P(Z(t) \leqslant x). \end{aligned}$$

Hence the distribution of $\hat{Z}(t)$ coincides with that of $Z(t)$. Statement (iii) for *two*-dimensional distributions and also for arbitrary finite-dimensional distributions is now an easy consequence of (ii) and the equality of the one-dimensional distributions at each positive time t. □

Remark 2. The above proof is an adaption of the proof of Theorem 1 of Donnelly (1991). One of the differences is that in the present paper the state space of the Markov process is a subset of **R**, whereas Donnelly treats countable state spaces; also the way we prove tightness on $D(0, \infty)$ differs from Donnolly's approach.

Corollary 1. *For* $F \in \mathscr{D}(\Psi_\gamma)$ *and* (X_n), *with* $x_0 < r/(1-\alpha)$, *the sequence defined in* (1), *we have*

$$a_n^{-1}\left(X_n - \frac{b_n}{1-\alpha}\right) \xrightarrow{d} X,$$

where the limit X *has density* h_α *on* $(-\infty, 0)$, *given by the unique density solution of the functional equation*

$$h_\alpha(x) = \frac{\gamma}{|x|} \int_{x/\alpha}^{x} |x - \alpha u|^\gamma h_\alpha(u)\, \mathrm{d}u, \qquad x < 0. \tag{9}$$

Proof. For $x < 0$, an elementary argument using the definition of $Z(\cdot)$ gives, for $h \to 0$,

$$P(Z(t+h) > x) - P(Z(t) > x) = h \int_{x/\alpha}^{x} |x - \alpha u|^\gamma P(Z(t) \in \mathrm{d}u) + \mathrm{o}(h).$$

This equation can be rewritten, using the self-similarity of $Z(\cdot)$,

$$P(Z(1) > x(t+h)^{1/\gamma}) - P(Z(1) > xt^{1/\gamma}) = h\int_{x/\alpha}^{x} |x - \alpha u|^{\gamma} P(Z(1) \in t^{1/\gamma}\,\mathrm{d}u) + \mathrm{o}(h).$$

The functional equation (9) now follows by standard arguments and by using the equality $X \stackrel{d}{=} Z(1)$. That (9) has a unique density solution can be seen by calculating the moments

$$\mu_k := \int_{-\infty}^{0} |x|^{k\gamma} h_\alpha(x)\,\mathrm{d}x, \qquad k = 0, 1, \ldots .$$

It follows from (9) that

$$\mu_k = \mu_{k+1}\int_{\alpha}^{1} \gamma y^{k\gamma-1}(y-\alpha)^{\gamma}\,\mathrm{d}y,$$

and hence by a theorem of Carleman (cf. Feller 1971, p. 227), the moments $\mu_0 = 1$, $\mu_1, \ldots$ uniquely determine the density h_α. □

3. The convergence result for type III distributions

In this section we treat the case where $F \in \mathscr{D}(\Lambda)$. In order to define the limit process of $X_n(\cdot)$ for this case let N be the Poisson process on $(0, \infty) \times \mathbf{R}$ with intensity measure $\mathrm{d}t \times \mathrm{d}\mu$, where $\mu(x, \infty) = \mathrm{e}^{-x}$, $x \in \mathbf{R}$. The point process N_x is the restriction of N to $(0, \infty) \times (x, \infty)$. On the points (t_1, j_1), $t_2, j_2), \ldots,$ of N_x, we define $Z_x(\cdot)$ by (4). Further we denote by $Z(\cdot)$ the almost sure limit of $Z_x(\cdot)$, as $x \to -\infty$. Along the lines of Section 2 we have the following.

Theorem 2. *Let $F \in \mathscr{D}(\Lambda)$ and $x_0 < r/(1-\alpha)$. On $D(0, \infty)$ we have*

$$X_n(\cdot) \xrightarrow{d} Z(\cdot).$$

Corollary 2. *For $F \in \mathscr{D}(\Lambda)$ and (X_n), with $x_0 < r/(1-\alpha)$, the sequence defined in* (1), *we have*

$$a_n^{-1}\left(X_n - \frac{b_n}{1-\alpha}\right) \xrightarrow{d} X,$$

where the limit X has density h_α on $\mathbf{R}$ given by

$$h_\alpha(x) := (1-\alpha)\{\Gamma((1-\alpha)^{-1})\}^{-1}\exp\{-x - \mathrm{e}^{-x(1-\alpha)}\},\ x \in \mathbf{R}, \qquad (10)$$

and where $\Gamma(t) := \int_0^\infty x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x,\ t > 0$.

Proof. For $x \in \mathbf{R}$ and $h \to 0$,

$$P(Z(t+h) > x) - P(Z(t) > x) = h \int_{-\infty}^{x} \exp\{-(x - \alpha u)\} P(Z(t) \in \mathrm{d}u) + \mathrm{o}(h). \qquad (11)$$

From (11) the density of $X \stackrel{d}{=} Z(1)$ can be obtained, using the self-similarity of $\exp\{-Z(t)\}$. □

Remark 3. Note that the density in (10) has the form

$$h_\alpha(x)\,\mathrm{d}x = c \exp(-\alpha x)\,\mathrm{d}\Lambda\{x(1-\alpha)\}, \qquad \alpha \in [0, 1).$$

However, for $\alpha \neq 0$ this density is not of the Gumbel type, i.e., there are no constants a and b such that

$$h_\alpha(x)\,\mathrm{d}x = \mathrm{d}\Lambda(\alpha x + b).$$

4. Tightness of sequences

In this section we prove tightness for the sequence

$$a_n^{-1}\left(X_n - \frac{b_n}{1-\alpha}\right),$$

with (X_n) the sequence given by (1).

Theorem 3. *For $F \in \mathscr{D}(\Psi_\gamma)$ and $x_0 < r/(1-\alpha)$, there exist norming constants $a_n > 0$ and $b_n \in \mathbf{R}$ such that the sequence $\{X_n - b_n/(1-\alpha)\}/a_n$ is tight on $(-\infty, 0)$. A possible choice of (a_n) and (b_n) is*

$$b_n \equiv r, \; a_n := r - \inf\{x\colon 1 - F(x) \leqslant n^{-1}\}.$$

Proof. Note by induction that $X_n \leqslant x_0 \vee M_n/(1-\alpha)$, where $M_n = Y_1 \vee Y_2 \cdots \vee Y_n$, however, it is not possible to obtain a lower bound for X_n in terms of M_n. From the well known extreme value limit for $(M_n - b_n)/a_n$ we obtain 0 as a distributional upper bound for $\{X_n - b_n/(1-\alpha)\}/a_n$.

Choose a sequence θ_n of positive real numbers with $a_n/\theta_n \to 1$, and satisfying

$$\lim_{n\to\infty} n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right) = \gamma^{-1}. \qquad (12)$$

This is possible since $a_n = a(n)$, where

$$a(y) := r - \inf\{x\colon 1 - F(x) \leqslant y^{-1}\}, \qquad y \geqslant 1,$$

and a is regularly varying; for details see Galambos and Seneta (1973) and de Bruijn (1959). Our goal is to prove that there exists a constant $A_0 > 0$ and an integer n_0 such that

$$\mathrm{E}\,\frac{X_n - r/(1-\alpha)}{\theta_n} \geqslant -A_0, \qquad n \geqslant n_0. \qquad (13)$$

This inequality, together with the upper bound $X_n \leqslant x_0 \vee M_n/(1-\alpha)$, implies tightness of $\{X_n - b_n/(1-\alpha)\}/\theta_n$ and hence of $\{X_n - b_n/(1-\alpha)\}/a_n$, since $a_n/\theta_n \to 1$. So all we need to prove is inequality (13).

Choose $A_1 > 0$ with $A_1^\gamma = \frac{3}{2}[\{(\gamma+1)/\gamma\}(1-\alpha)^{-1-\gamma}]$, and put $\eta = (4\gamma)^{-1}$. Since $n\{1 - F(r - \theta_n z)\}$ converges uniformly to z^γ on compacta, we can find n_1 such that, for $n \geqslant n_1$,

$$\begin{aligned}\frac{1}{A_1}\int_0^{A_1(1-\alpha)} n\{1 - F(r-\theta_n z)\}\,\mathrm{d}z &\geqslant \frac{1}{A_1}\int_0^{A_1(1-\alpha)} (z^\gamma - \eta)\,\mathrm{d}z \\ &= \frac{1}{\gamma+1}A_1^\gamma(1-\alpha)^{\gamma+1} - (1-\alpha)\eta \\ &\geqslant \frac{3}{2\gamma} - \eta \\ &= \frac{5}{4\gamma}.\end{aligned}$$

According to (12) we can find n_2 such that, for $n \geqslant n_2$,

$$n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right) \leqslant \gamma^{-1} + \eta = \frac{5}{4\gamma}.$$

Hence for $n \geqslant n_0 = n_1 \vee n_2$,

$$\frac{1}{A_1}\int_0^{A_1(1-\alpha)} n\{1 - F(r-\theta_n z)\}\,\mathrm{d}z \geqslant \frac{5}{4\gamma} \geqslant n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right). \tag{14}$$

We are now ready to show (13). Note that

$$\mathrm{E}(X_{n+1}|X_n) = X_n + \int_{(1-\alpha)X_n}^r \{1 - F(y)\}\,\mathrm{d}y;$$

so by taking double expectations and using the Jensen inequality

$$\mathrm{E}X_{n+1} = \mathrm{E}g(X_n) \geqslant g(\mathrm{E}X_n), \tag{15}$$

where $g(u) := u + \int_{(1-\alpha)u}^r \{1 - F(y)\}\,\mathrm{d}y$, $u < r/(1-\alpha)$. Put $u_n := \mathrm{E}X_n$ and $v_n := r/(1-\alpha) - A_0\theta_n$, where $A_0 > A_1$ is taken large enough to satisfy

$$u_{n_0} \geqslant v_{n_0}.$$

We shall prove by induction that

$$u_n \geqslant v_n \tag{16}$$

for all $n \geqslant n_0$. Assuming that (16) holds for some $n \geqslant n_0$ it follows from the monotonicity of g on $(-\infty, r/(1-\alpha))$ and (15) that

$$u_{n+1} \geqslant g(u_n) \geqslant g(v_n).$$

Hence we shall obtain $u_{n+1} \geqslant v_{n+1}$ if we show that

$$g(v_n) \geqslant v_{n+1}, \qquad \forall n \geqslant n_0. \tag{17}$$

The inequality (17) is equivalent to

$$v_n + \int_{(1-\alpha)v_n}^{r} \{1 - F(y)\}\,\mathrm{d}y \geqslant v_{n+1}, \qquad \forall n \geqslant n_0,$$

or, after setting $y = r - A_0\theta_n z$,

$$\int_0^{1-\alpha} n\{1 - F(r - A_0\theta_n z)\}\,\mathrm{d}z \geqslant n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right), \qquad \forall n \geqslant n_0. \tag{18}$$

Inequality (18), and hence (17), follows from

$$\begin{aligned}\int_0^{1-\alpha} n\{1 - F(r - A_0\theta_n z)\}\,\mathrm{d}z &\geqslant \int_0^{1-\alpha} n\{1 - F(r - A_1\theta_n z)\}\,\mathrm{d}z\\ &= \frac{1}{A_1}\int_0^{(1-\alpha)A_1} n\{1 - F(r - \theta_n z)\}\,\mathrm{d}z\\ &\geqslant n\left(1 - \frac{\theta_{n+1}}{\theta_n}\right),\end{aligned}$$

for all $n \geqslant n_0$, according to (14). □

The proof of tightness of the sequence $a_n^{-1}\{X_n - b_n/(1-\alpha)\}$, in case $F \in D(\Lambda)$, can be given in a similar way; therefore we omit this proof.

Theorem 4. *For* $F \in \mathscr{D}(\Lambda)$ *and* $x_0 < r/(1-\alpha)$, (a_n) *and* (b_n) *such that* $F^n(a_n x + b_n) \to \Lambda(x)$ *we have that* $\{X_n - b_n/(1-\alpha)\}/a_n$ *is tight on* $\mathbf{R}$.

5. Concluding remarks

(i) Together with the paper of Greenwood and Hooghiemstra (1991) this paper gives sufficient conditions on F to ensure that $\{X_n - b_n/(1-\alpha)\}/a_n$ has a distributional limit. It is known that for $\alpha = 0$ these conditions are also necessary. Whether this is also the case for $0 < \alpha < 1$ we do not know.

(ii) The recursion (1) can be written as

$$X_n = X_{n-1} + [Y_n - (1-\alpha)X_n]^+.$$

A description of what kind of results can be expected if we let α depend on n such that $\alpha_n \to 1$ is given in the work of den Hollander *et al.* (1991).

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References

de Bruijn, N.G. (1959) Pairs of slowly oscillating functions occurring in asymptotic problems concerning Laplace transforms. *Nieuw Arch. Wisk.*, **7**, 20–26.

Donnelly, P. (1991) Weak convergence to a Markov chain with an entrance boundary; ancestral processes in population genetics. *Ann. Probab.*, **19**, 1102–1117.

Feller, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. II. New York: Wiley.

Galambos, J. and Seneta, E. (1973) Regularly varying sequences. *Proc. Amer. Math. Soc.*, **41**, 110–116.

Greenwood, P.E. and Hooghiemstra, G. (1991) On the domain of an operator between supremum and sum. *Probab. Theory Related Fields*, **89**, 201–210.

Haslett, J. (1980) Problems in the stochastic storage of solar thermal energy. In O.L.R. Jacobs, M.H.A. Davies, M.A.H. Dempster, C.J. Harris and P.C. Parks (eds) *Analysis and Optimization of Stochastic Systems*, pp. 553–562. London: Academic Press.

den Hollander, F., Hooghiemstra, G., Keane, M. and Resing, J. (1991) Strong law and central limit theorem for a process between maxima and sums. *Probab. Theory Related Fields*, **90**, 37–55.

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