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On the growth of variances in a central limit theorem for strongly mixing sequences

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In a central limit theorem under certain strong mixing conditions, one does not quite have an asymptotic linear growth of the variance of the partial sums.

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1. Introduction

Suppose that $(\Omega, \mathscr{F}, P)$ is a probability space. For any two σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, define the following measures of dependence:

$$\alpha(\mathscr{A}, \mathscr{B}) := \sup |P(A \cap B) - P(A)P(B)|,\ A \in \mathscr{A},\ B \in \mathscr{B},$$

$$\rho(\mathscr{A}, \mathscr{B}) := \sup |\mathrm{Corr}(V, W)|,\ V \in \mathscr{L}^2(\mathscr{A}),\ W \in \mathscr{L}^2(\mathscr{B}),$$

$$\beta(\mathscr{A}, \mathscr{B}) := \sup \tfrac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|,$$

where this latter sup is taken over all pairs of partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω such that $A_i \in \mathscr{A}$ for all i and $B_j \in \mathscr{B}$ for all j.

The σ-field of events generated by a given family $(X_j, j \in S)$ of random variables on $(\Omega, \mathscr{F}, P)$ will be denoted by $\sigma(X_j, j \in S)$.

Suppose that $X := (X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on $(\Omega, \mathscr{F}, P)$. For each positive integer n, define the mixing coefficients

$$\alpha(n) := \alpha(\sigma(X_k, k \leqslant 0), \sigma(X_k, k \geqslant n)),$$

$$\rho(n) := \rho(\sigma(X_k, k \leqslant 0), \sigma(X_k, k \geqslant n)),$$

$$\beta(n) := \beta(\sigma(X_k, k \leqslant 0), \sigma(X_k, k \geqslant n)),$$

$$\rho^*(n) := \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)),$$

where this latter sup is taken over all pairs of non-empty disjoint sets $S, T \subset \mathbb{Z}$ such that

$$\mathrm{dist}(S, T) := \min_{j \in S, k \in T} |j - k| \geqslant n.$$

Of course, for each $n \geqslant 1$, one has that

$$0 \leqslant 4\alpha(n) \leqslant \rho(n) \leqslant \rho^*(n) \leqslant 1 \tag{1.1}$$

and

$$2\alpha(n) \leqslant \beta(n) \leqslant 1. \tag{1.2}$$

With a class of examples constructed by Bradley (1996), it was shown that, for a given strictly stationary sequence X, there are "almost" no other restrictions on the simultaneous behaviour of the mixing coefficients $\alpha(n)$, $\rho(n)$ and $\rho^*(n)$, $n = 1, 2, 3, \ldots,$ besides (1.1), $\alpha(n) \geqslant \alpha(n+1)$, $\rho(n) \geqslant \rho(n+1)$ and $\rho^*(n) \geqslant \rho^*(n+1)$.

The random sequence X is said to be "strongly mixing" if $\alpha(n) \to 0$ and $n \to \infty$, "ρ-mixing" if $\rho(n) \to 0$ as $n \to \infty$, "absolutely regular" if $\beta(n) \to 0$ as $n \to \infty$, and "ρ^*-mixing" if $\rho^*(n) \to 0$ as $n \to \infty$.

For each positive integer n, define the partial sum $S_n := X_1 + X_2 + \cdots + X_n$. Consider the following two theorems.

Theorem A. *Suppose that* $(X_k, k \in \mathbb{Z})$ *is a strictly stationary* ρ^**-mixing sequence of random variables such that* $\mathrm{E}X_0 = 0$*,* $\mathrm{E}X_0^2 < \infty$*, and* $\mathrm{E}S_n^2 \to \infty$ *as* $n \to \infty$*. Then* $\sigma^2 := \lim_{n\to\infty} n^{-1}\mathrm{E}S_n^2$ *exists in* $(0, \infty)$*, and* $S_n/n^{1/2}\sigma$ *converges in distribution to* $N(0, 1)$ *as* $n \to \infty$.

This can be seen from Bradley (1992, Theorems 1, 3 and 4). (Recall that, if a random sequence X is stationary, centred, with finite second moments, and has a spectral density which is continuous at 0, then by Fejér's theorem, $\lim_{n\to\infty} n^{-1}\mathrm{E}S_n^2$ exists.) The next theorem is due to Peligrad (1996, Corollary 2.3) and is based partly on a moment inequality of Bryc and Smolenski (1993).

Theorem B (Peligrad). *Suppose that* $(X_k, k \in \mathbb{Z})$ *is a strictly stationary, strongly mixing sequence of random variables such that* $\mathrm{E}X_0 = 0$*,* $\mathrm{E}X_0^2 < \infty$*,* $\sigma_n^2 := \mathrm{E}S_n^2 \to \infty$ *as* $n \to \infty$*, and* $\rho^*(n) < 1$ *for some* $n \geqslant 1$*. Then*

$$0 < \liminf_{n\to\infty} n^{-1}\sigma_n^2 \leqslant \limsup_{n\to\infty} n^{-1}\sigma_n^2 < \infty, \tag{1.3}$$

and S_n/σ_n *converges to* $N(0, 1)$ *in distribution as* $n \to \infty$.

(For the last inequality in (1.3), see Bradley (1992, Lemma 2).)

Under the hypothesis of Theorem B, together with the extra "covariance" assumption that $\sum_{n=1}^{\infty}|\mathrm{E}X_0X_n| < \infty$, one has that $\sigma^2 := \lim_{n\to\infty} n^{-1}\mathrm{E}S_n^2$ exists in $(0, \infty)$, and that $S_n/n^{1/2}\sigma$ converges to $N(0, 1)$ in distribution as $n \to \infty$. This can be derived either as a corollary of Theorem B itself or as a special case of a similar but more general central limit theorem for random fields that was proved by Perera (1997). It would be nice if such a σ^2 still exists under the hypothesis of Theorem B, without the extra "covariance" assumption. Unfortunately, in general, things do not work out in that way, even under certain extra mixing assumptions. Here is our main result.

Theorem 1. *Suppose that* $\epsilon > 0$. *Then there exists a stationary, absolutely regular Gaussian sequence* $X := (X_k,\ k \in \mathbb{Z})$ *with* $\mathrm{E}X_0 = 0$, *such that* $\rho^*(1) \leqslant \epsilon$ *and*

$$\liminf_{n\to\infty} n^{-1}\mathrm{E}S_n^2 < \limsup_{n\to\infty} n^{-1}\mathrm{E}S_n^2. \tag{1.4}$$

Of course the sequence X in Theorem 1 is strongly mixing (by (1.2)), and hence also ρ mixing by a well-known result of Kolmogorov and Rozanov (1960) for Gaussian random sequences.

Theorem 1 will be proved in Section 2. With an example from Ibragimov and Rozanov (1978, p. 180, Example 2), it was shown that, if a stationary Gaussian sequence is strongly mixing (equivalently, ρ mixing) and its spectral density is bounded between two positive constants, the spectral density can (in a non-trivial way) still fail to be continuous. The construction here in Section 2 is a somewhat embellished, slightly modified version of that example. At a critical point, it will involve an application of the theorem of Ibragimov and Solev (1969) that characterized the stationary, absolutely regular Gaussian sequences in terms of properties of the spectral density. Further pertinent comments will be made in Remarks 1 and 2 at the end of the article.

Throughout the proof, quantities of the form a_b will often be written as $a(b)$ for typographical convenience.

2. Proof of Theorem 1

First, a few preliminary items will be dealt with.

A (real) stationary Gaussian sequence $(X_k,\ k \in \mathbb{Z})$ is said to have a spectral density $f\colon [-\pi, \pi] \to [0, \infty)$ if

$$\mathrm{cov}(X_0, X_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\lambda} f(\lambda)\,\mathrm{d}\lambda \tag{2.1}$$

holds for every integer k. It is understood that f is a real non-negative Borel integrable function which is symmetric about 0, i.e. $f(-\lambda) = f(\lambda)$ for all $\lambda \in [-\pi, \pi]$.

Lemma 1. *Suppose that* b *and* B *are positive numbers such that* $b < B$. *Suppose that* $X := (X_k,\ k \in \mathbb{Z})$ *is a stationary Gaussian random sequence with a spectral density* f *such that* $b \leqslant f(\lambda) \leqslant B$ *for all* $\lambda \in [-\pi, \pi]$. *Then* X *satisfies* $\rho^*(1) \leqslant 1 - b/B$.

In one form or another this seems to be part of the folklore. By a well-known theorem of Kolmogorov and Rozanov (1960) for Gaussian random sequences, it suffices to prove that

$$\left|\mathrm{Corr}\left(\sum_{k\in S} a_k X_k,\ \sum_{k\in T} a_k X_k\right)\right| \leqslant 1 - \frac{b}{B}, \tag{2.2}$$

where S and T are two arbitrary non-empty disjoint finite sets of integers and $(a_k,\ k \in S \cup T)$

are arbitrary real numbers. The proof of (2.2) involves (2.1) and arguments from Kolmogorov and Rozanov (1960) and Rosenblatt (1985); it is essentially the argument given in a closely related context by Bradley (1992, p. 365).

The following technical lemma will be useful.

Lemma 2. *Suppose that N and M are positive integers such that $N \leqslant M$, and $(a_N, a_{N+1}, a_{N+2}, \ldots, a_M)$ is a (finite) non-increasing sequence of non-negative real numbers. Then, for every $\lambda \in [-\pi, \pi] - \{0\}$, one has that $|\sum_{k=N}^{M} a_k \, \mathrm{e}^{\mathrm{i}k\lambda}| \leqslant \pi a_N/|\lambda|$.*

The case $N \geqslant 2$ reduces trivially to the case $N = 1$ through the representation

$$\sum_{k=N}^{M} a_k \, \mathrm{e}^{\mathrm{i}k\lambda} = \mathrm{e}^{\mathrm{i}(N-1)\lambda} \sum_{k=1}^{M-(N-1)} a_{k+(N-1)} \, \mathrm{e}^{\mathrm{i}k\lambda}.$$

For (say) the case $N = 1$, Lemma 2 is simply an application of Zygmund (1959, p. 3, Theorem 2.2), with $a_k := 0$ for $k \geqslant M + 1$. (First recall that in the work of Zygmund (1959, p. 3, Equation (2.3)), if $u_k := \mathrm{e}^{\mathrm{i}k\lambda}$ and $U_k := \sum_{j=1}^{k} \mathrm{e}^{\mathrm{i}j\lambda}$ where $0 < |\lambda| \leqslant \pi$, then $\sup_k |U_k| \leqslant 2/|\mathrm{e}^{\mathrm{i}\lambda} - 1| \leqslant \pi/|\lambda|$.)

For each positive number δ and each integer $M \geqslant 2$, define the function $g_{\delta,M}$: $[-\pi, \pi] \to \mathbb{R}$ by

$$g_{\delta,M}(\lambda) := \delta \sum_{k=2}^{M} \frac{1}{k \log k} \cos(k\lambda). \tag{2.3}$$

(Throughout this paper, log denotes the natural logarithm.) Define the (finite positive) number Q by

$$Q := \sum_{k=2}^{\infty} \frac{1}{(k \log k)^2}. \tag{2.4}$$

Now let us use the fact that the functions $\cos(k\lambda)$, $k = 1, 2, 3, \ldots$ are orthogonal to each other on the interval $[-\pi, \pi]$, and that $(2\pi)^{-1} \int_{-\pi}^{\pi} \cos^2(k\lambda)\, \mathrm{d}\lambda = \frac{1}{2}$ for $k = 1, 2, 3, \ldots$. By a simple calculation, for each $\delta > 0$ and each integer $M \geqslant 2$, one has that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g_{\delta,M}^2(\lambda)\, \mathrm{d}\lambda = \frac{\delta^2}{2} \sum_{k=2}^{M} \frac{1}{(k \log k)^2},$$

and hence, by (2.4) and Hölder's inequality,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g_{\delta,M}(\lambda)|\, \mathrm{d}\lambda \leqslant \delta Q^{1/2}. \tag{2.5}$$

This completes the preliminary work. Now we are ready to begin the main part of the proof of Theorem 1.

Referring to the hypothesis of Theorem 1, let A be a positive number such that

$$0 < A \leqslant \tfrac{1}{3} \tag{2.6}$$

and

$$e^{-3A} \geqslant 1 - \epsilon. \tag{2.7}$$

This number A will play a key role throughout the proof.

For each $L = 1, 2, 3, \ldots,$ we need to choose a positive number θ_L, an integer $M_L \geqslant 2$, the positive number δ_L such that $g_{\delta(L),M(L)}(0) = A$, and a positive integer N_L. The definition of these numbers will be recursive. In this definition, we shall use the fact that $\sum_{k=2}^{\infty} 1/(k \log k) = \infty$, and also the fact that, for each $\delta > 0$ and each integer $M \geqslant 2$, the function $g_{\delta,M}$ is continuous and

$$g_{\delta,M}(0) = \delta \sum_{k=2}^{M} \frac{1}{k \log k}. \tag{2.8}$$

We start with $L = 1$. Define the positive number θ_1 by

$$\theta_1 := \tfrac{1}{3}. \tag{2.9}$$

Let $M_1 \geqslant 2$ be an integer such that

$$2^{-3}\theta_1 \sum_{k=2}^{M(1)} \frac{1}{k \log k} \geqslant 1. \tag{2.10}$$

Referring to (2.8), let δ_1 be the positive number such that

$$g_{\delta(1),M(1)}(0) = \delta_1 \sum_{k=2}^{M(1)} \frac{1}{k \log k} = A. \tag{2.11}$$

Define (just as a formality) the positive integer

$$N_1 := 1. \tag{2.12}$$

Now suppose that $L \geqslant 2$ and that, for each $\ell = 1, \ldots, L-1$, the following have been defined: the positive number θ_ℓ, the integer $M_\ell \geqslant 2$, the positive number δ_ℓ such that $g_{\delta(\ell),M(\ell)}(0) = A$, and the positive integer N_ℓ.

Let θ_L be a positive number satisfying the following three conditions:

$$0 < \theta_L < \theta_{L-1}, \tag{2.13}$$

$$\theta_L \leqslant 2^{-L} N_{L-1}^{-1} A \tag{2.14}$$

and

$$\forall \lambda \in [-\theta_L, \theta_L], \; |g_{\delta(L-1),M(L-1)}(\lambda) - A| \leqslant 2^{-L} A. \tag{2.15}$$

Let $M_L \geqslant 2$ be an integer such that

$$2^{-L-2}\theta_L N_{L-1}^{-1} \sum_{k=2}^{M(L)} \frac{1}{k \log k} \geqslant 1. \tag{2.16}$$

Referring to (2.8), let δ_L be the positive number such that

$$g_{\delta(L),M(L)}(0) = \delta_L \sum_{k=2}^{M(L)} \frac{1}{k \log k} = A. \tag{2.17}$$

Let N_L be a positive integer such that

$$N_L > N_{L-1} \tag{2.18}$$

and

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{N(L)} \frac{\sin^2\{N(L)\lambda/2\}}{\sin^2(\lambda/2)} \exp\left(\sum_{\ell=1}^{L} (-1)^{\ell} g_{\delta(\ell),M(\ell)}(\lambda) \right) \mathrm{d}\lambda - \exp\left[\sum_{\ell=1}^{L} (-1)^{\ell} g_{\delta(\ell),M(\ell)}(0) \right] \right|$$
$$\leqslant 2^{-L}. \tag{2.19}$$

To obtain (2.19), we are using Fejér's theorem.

By (2.11) and (2.17), the requirement $g_{\delta(L),M(L)}(0) = A$ is met for each $L \geqslant 1$. The recursive definition of θ_L, M_L, δ_L and N_L is complete.

A couple of comments on this recursive definition are in order. Referring to (2.19), one has that

$$\exp\left(\sum_{\ell=1}^{L} (-1)^{\ell} g_{\delta(\ell),M(\ell)}(0) \right) = \begin{cases} 1 \text{ if } L \text{ is even} \\ \exp(-A) \text{ if } L \text{ is odd} \end{cases} \tag{2.20}$$

by (2.11) and (2.17). Also, for each $L = 1, 2, 3, \ldots$, $\theta_L \leqslant \frac{1}{3}$ by (2.9) and (2.13), and hence

$$\begin{gathered} \delta_1 \leqslant 2^{-3}\theta_1 A \leqslant 2^{-3} A, \\ \forall L \geqslant 2,\ \delta_L \leqslant 2^{-L-2}\theta_L N_{L-1}^{-1} A \leqslant 2^{-L-2} A \end{gathered} \tag{2.21}$$

by (2.10) and (2.11) for $L = 1$ and by (2.16) and (2.17) for $L \geqslant 2$.

For each $L = 1, 2, 3, \ldots$, define the function h_L: $[-\pi, \pi] \to \mathbb{R}$ by

$$h_L(\lambda) := \sum_{\ell=1}^{L} (-1)^{\ell} g_{\delta(\ell),M(\ell)}(\lambda). \tag{2.22}$$

Lemma 3.

(a) For each even positive integer L, $h_L(0) = 0$ and, for each odd positive integer L, $h_L(0) = -A$.

(b) For every $\lambda \in [-\pi, \pi]$, one has that

$$\forall L = 1, 2, 3, \ldots,\ -2A \leqslant h_L(\lambda) \leqslant A. \tag{2.23}$$

(c) For every $\lambda \in [-\pi, \pi] - \{0\}$, $H(\lambda) := \lim_{L\to\infty} h_L(\lambda)$ exists in $[-2A, A]$.

Proof of Lemma 3. Statement (a) follows immediately from (2.22), (2.11) and (2.17). Hence also (2.23) holds for $\lambda = 0$. Now let $\lambda \in [-\pi, \pi] - \{0\}$ be arbitrary but fixed. To complete the proof of Lemma 3, it suffices to prove (2.23) and the conclusion of statement (c) for this λ.

Refer again to (2.3) and (2.22). For each $\delta > 0$ and each integer $M \geqslant 2$, the function $g_{\delta,M}$ is symmetric about 0. Hence, for each $L \geqslant 1$, the function h_L is symmetric about 0. Hence, without loss of generality, we assume that $0 < \lambda \leqslant \pi$.

Consider first the case where $\theta_1 \leqslant \lambda \leqslant \pi$. In this case, for each $L = 1, 2, 3, \ldots,$ by (2.13), (2.21) and Lemma 2,

$$\begin{aligned}|g_{\delta(L),M(L)}(\lambda)| &\leqslant \frac{1}{\lambda}\pi\delta_L\frac{1}{2\log 2}\\ &\leqslant \frac{1}{\theta_1}4(2^{-L-2}\theta_L A)1\\ &\leqslant 2^{-L}A.\end{aligned}$$

Hence $\sum_{L=1}^{\infty}|g_{\delta(L),M(L)}(\lambda)| \leqslant A$ and, by (2.22), one has (2.23) and the conclusion of statement (c) (in lemma 3) for our given λ.

Now we only need to consider the remaining case where $0 < \lambda < \theta_1$. By (2.14), $\theta_L \to 0$ as $L \to \infty$. Referring to (2.13), let J be the (unique) positive integer such that

$$\theta_{J+1} \leqslant \lambda < \theta_J. \tag{2.24}$$

For any $L \geqslant J + 1$, one has that $\lambda \in [\theta_L, \pi]$ by (2.13), and hence

$$\begin{aligned}|g_{\delta(L),M(L)}(\lambda)| &\leqslant \frac{1}{\lambda}\pi\delta_L\frac{1}{2\log 2}\\ &\leqslant \frac{1}{\theta_L}4(2^{-L-2}\theta_L A)1\\ &\leqslant 2^{-L}A\end{aligned} \tag{2.25}$$

by Lemma 2 and (2.21). Hence $\sum_{\ell=1}^{\infty}|g_{\delta(L),M(L)}(\lambda)| < \infty$. Hence by (2.22), if (2.23) is proved for our given λ, then the conclusion of statement (c) (of Lemma 3) will also follow, and the proof of Lemma 3 will be complete. Thus all that remains is to prove (2.23) for our given λ.

Referring to the integer J from (2.24) again, we first need to show that

$$-2^{-J}A \leqslant g_{\delta(J),M(J)}(\lambda) \leqslant A. \tag{2.26}$$

The second inequality in (2.26) holds by (2.3) together with (2.11) (if $J = 1$) or (2.17) (if $J \geqslant 2$). The proof of the first inequality in (2.26) will take a little more work.

From (2.9), (2.13) and (2.24), one has that $2 < 1/\lambda$. Let I denote the greatest element of $\{2, 3, 4, \ldots, M_J\}$ such that $I \leqslant 1/\lambda$. For each $k = 2, 3, \ldots, I$, one has that $k\lambda \leqslant 1$ and hence $\cos(k\lambda) > 0$. If $I = M_J$, then, by (2.3), the first inequality in (2.26) holds with $g_{\delta(J),M(J)}(\lambda) \geqslant 0$. Therefore, let us suppose instead that $I < M_J$. In order to prove the first inequality in (2.26), it now suffices to show that

$$\delta_J \sum_{k=I+1}^{M(J)} \frac{1}{k\log k}\cos(k\lambda) \geqslant -2^{-J}A. \tag{2.27}$$

Now $\lambda < \theta_J \leqslant \frac{1}{3}$ by (2.9), (2.13) and (2.24), and $I + 1 > 1/\lambda > 3$. Hence, by (2.21) and Lemma 2,

$$\begin{aligned}\left|\delta_J \sum_{k=I+1}^{M(J)} \frac{1}{k \log k} \cos(k\lambda)\right| &= \left|\operatorname{Re} \sum_{k=I+1}^{M(J)} \frac{\delta(J)}{k \log k} \mathrm{e}^{\mathrm{i}k\lambda}\right| \\ &\leqslant \frac{1}{\lambda} \pi \frac{\delta(J)}{(I+1)\log(I+1)} \\ &\leqslant \frac{\pi\delta(J)}{\log(I+1)} \\ &\leqslant \frac{4\delta_J}{1} \\ &\leqslant 2^{-J} A.\end{aligned}$$

Thus (2.27) holds. This completes the proof of the first inequality in (2.26).

Now we return to the task of proving (2.23) for our given λ.

Referring to (2.24), consider first the case where $J = 1$. By (2.26), (2.25) (for $L \geqslant J + 1 = 2$) and (2.22), one has that $2^{-1}A \geqslant h_1(\lambda) \geqslant -A$ and, for each $L \geqslant 2$,

$$\sum_{\ell=1}^{L} 2^{-\ell} A \geqslant h_L(\lambda) \geqslant -A - \sum_{\ell=2}^{L} 2^{-\ell} A.$$

Thus (2.23) holds if $J = 1$.

Now suppose instead that $J \geqslant 2$.

For any positive integer $L \leqslant J - 1$, one has that $\lambda \in (0, \theta_{L+1}]$ by (2.24) and (2.13), and hence $|g_{\delta(L),M(L)}(\lambda) - A| \leqslant 2^{-(L+1)}A$ by (2.15). Hence, by (2.22),

$$\forall L = 1, \ldots, J-1, \left|h_L(\lambda) - \sum_{\ell=1}^{L} (-1)^\ell A\right| \leqslant \sum_{\ell=1}^{L} 2^{-\ell} A. \tag{2.28}$$

Recall that $\sum_{\ell=1}^{L} (-1)^\ell A = 0$ or $-A$, according to whether L is even or odd. As a consequence, it follows from (2.28) that (2.23) holds for the case $J \geqslant 2$, $L \leqslant J - 1$. Now all that remains is to prove (2.23) for the case $J \geqslant 2$, $L \geqslant J$.

Our next task is to show that

$$-A - \sum_{\ell=1}^{J} 2^{-\ell} A \leqslant h_J(\lambda) \leqslant \sum_{\ell=1}^{J} 2^{-\ell} A. \tag{2.29}$$

First note that

$$h_J(\lambda) = h_{J-1}(\lambda) + (-1)^J g_{\delta(J),M(J)}(\lambda). \tag{2.30}$$

If J is even, then by (2.28) (with $L = J - 1$), one has that

$$-A - \sum_{\ell=1}^{J-1} 2^{-\ell} A \leqslant h_{J-1}(\lambda) \leqslant -A + \sum_{\ell=1}^{J-1} 2^{-\ell} A,$$

and hence (2.29) holds by (2.30) and (2.26). If instead J is odd, then, by (2.28) (with $L = J - 1$), $|h_{J-1}(\lambda)| \leqslant \sum_{\ell=1}^{J-1} 2^{-\ell} A$, and hence (2.29) holds by (2.30) and (2.26). This completes the proof of (2.29).

Now for any $L \geqslant J + 1$, one has that

$$h_L(\lambda) = h_J(\lambda) + \sum_{\ell=J+1}^{L} (-1)^\ell g_{\delta(\ell), M(\ell)}(\lambda).$$

Hence, by (2.25) and (2.29),

$$\forall L \geqslant J + 1, \ -A - \sum_{\ell=1}^{L} 2^{-\ell} A \leqslant h_L(\lambda) \leqslant \sum_{\ell=1}^{L} 2^{-\ell} A. \tag{2.31}$$

By (2.31), (2.29) and (2.28) (together with the fact that $\sum_{\ell=1}^{L} (-1)^\ell A = 0$ or $-A$), one has that (2.23) holds for our given λ. This completes the proof of Lemma 3. □

2.1. Continuation of the main argument in the proof of Theorem 1

Let $H(\lambda)$, $\lambda \in [-\pi, \pi] - \{0\}$, denote the (bounded) function defined in statement (c) in Lemma 3. As a consequence of (2.3) and (2.22), the function H is symmetric about 0.

Define the positive bounded symmetric Borel function f on $[-\pi, \pi] - \{0\}$ by

$$f(\lambda) := \mathrm{e}^{H(\lambda)}. \tag{2.32}$$

Let $X := (X_k, k \in \mathbb{Z})$ be a stationary Gaussian sequence with $\mathrm{E}X_0 = 0$ and with spectral density f. We shall prove for this Gaussian sequence X the properties asserted in the statement of Theorem 1.

By Lemma 3, $\mathrm{e}^{-2A} \leqslant f(\lambda) \leqslant \mathrm{e}^{A}$ for every $\lambda \in [-\pi, \pi] - \{0\}$. By (2.7) and Lemma 1, the sequence X satisfies $\rho^*(1) \leqslant \epsilon$. To complete the proof of Theorem 1, we now need to prove absolute regularity and (1.4).

Proof of absolute regularity. In order to prove that the (stationary Gaussian) sequence X satisfies absolute regularity, it suffices to prove that the function H has the Fourier representation

$$H(\lambda) = \sum_{k=0}^{\infty} a_k \cos(k\lambda), \tag{2.33}$$

where the (real) coefficients a_k satisfy

$$\sum_{k=0}^{\infty} k a_k^2 < \infty \tag{2.34}$$

(and, say, the sum in (2.33) converges in $\mathscr{L}^2$ to $H(\lambda)$). This is simply an application of the theorem of Ibragimov and Solev (1969) that characterized the stationary, absolutely regular Gaussian random sequences (see, for example, Ibragimov and Rozanov (1978, p. 129, Theorem 8)).

Define the non-negative numbers $c_{k,\ell}$, $k \in \{0, 1, 2, \ldots\}$, $\ell \in \{1, 2, 3, \ldots\}$, by

$$c_{k,\ell} := \begin{cases} \delta_\ell \cdot \dfrac{1}{k \log k} & \text{if } 2 \leqslant k \leqslant M_\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

For each $\lambda \in [-\pi, \pi]$, one has the following: for each $\ell = 1, 2, 3, \ldots,$

$$g_{\delta(\ell),M(\ell)}(\lambda) = \sum_{k=0}^{\infty} c_{k,\ell} \cos(k\lambda) \tag{2.36}$$

by (2.3), and hence, for each $L = 1, 2, 3, \ldots,$

$$h_L(\lambda) = \sum_{k=0}^{\infty} \left(\sum_{\ell=1}^{L} (-1)^\ell c_{k,\ell} \right) \cos(k\lambda) \tag{2.37}$$

by (2.22). In the sum in (2.36) and in the double sum in (2.37), only finitely many of the numbers $c_{k,\ell}$ are non-zero, since for each $\ell \geqslant 1$ one has that $c_{k,\ell} \neq 0$ for only finitely many k.

By (2.37) and a simple calculation, for each $L \geqslant 1$ and each $k \geqslant 0$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} h_L(\lambda)\, d\lambda &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\lambda} h_L(\lambda)\, d\lambda \\ &= \tfrac{1}{2} \sum_{\ell=1}^{L} (-1)^\ell c_{k,\ell}. \end{aligned} \tag{2.38}$$

(For $k = 0$, the factor of $\frac{1}{2}$ in the last term does not belong, but it is harmless, since $c_{0,\ell} = 0$ for all $\ell \geqslant 1$. This should be kept in mind below.)

It will be helpful to observe that, for $k = 0$ or 1,

$$\sum_{\ell=1}^{\infty} |c_{k,\ell}| = \sum_{\ell=1}^{\infty} 0 = 0 \tag{2.39}$$

and, for each $k \geqslant 2$, one has that

$$\sum_{\ell=1}^{\infty} |c_{k,\ell}| \leqslant \sum_{\ell=1}^{\infty} \delta_\ell \frac{1}{k \log k}$$
$$\leqslant \sum_{\ell=1}^{\infty} 2^{-\ell-2} A \frac{1}{k \log k} \tag{2.40}$$
$$\leqslant \frac{1}{k \log k}$$

by (2.35), (2.21) and (2.6).

Now, by (2.38), Lemma 3 and dominated convergence, one has that, for each $k \geqslant 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda} H(\lambda)\, d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\lambda} H(\lambda)\, d\lambda$$
$$= \tfrac{1}{2}\sum_{\ell=1}^{\infty} (-1)^\ell c_{k,\ell}. \tag{2.41}$$

For each $k = 0, 1, 2, \ldots,$ define the number

$$a_k := \sum_{\ell=1}^{\infty} (-1)^\ell c_{k,\ell}. \tag{2.42}$$

Then, by (2.39) and (2.40), $a_0 = a_1 = 0$ and $|a_k| \leqslant 1/(k \log k)$ for each $k \geqslant 2$. For these numbers a_k, (2.34) holds and, by (2.41) and (2.42), the function H has the Fourier representation in (2.33) (with the sum there converging in $\mathscr{L}^2$). This completes the proof that the stationary Gaussian sequence X is absolutely regular. □

To complete the proof of Theorem 1, all that remains is to prove (1.4).

Proof of (1.4). By (2.6) and Lemma 3, for each $L \geqslant 1$ and each $\lambda \in [-\pi, \pi] - \{0\}$,

$$|H(\lambda) - h_L(\lambda)| \leqslant 3A \leqslant 1. \tag{2.43}$$

Also, by a simple calculation, for each real number $x \leqslant 1$, one has that $|e^x - 1| \leqslant 2|x|$. Hence by (2.32), for each $L \geqslant 1$ and each $\lambda \in [-\pi, \pi] - \{0\}$,

$$|f(\lambda) - \exp[h_L(\lambda)]| = \{\exp[h_L(\lambda)]\}|\{\exp[H(\lambda) - h_L(\lambda)]\} - 1|$$
$$\leqslant \{\exp[h_L(\lambda)]\} 2 |H(\lambda) - h_L(\lambda)| \tag{2.44}$$
$$\leqslant 6|H(\lambda) - h_L(\lambda)|,$$

where the last inequality holds by (2.43) and statement (b) of Lemma 3.

We need to derive some $\mathscr{L}^1$ bounds for the terms in (2.44). For each $L \geqslant 1$ and each $\ell \geqslant L + 1$, one has that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g_{\delta(\ell),M(\ell)}(\lambda)|\,\mathrm{d}\lambda &\leqslant \delta_\ell Q^{1/2}\\ &\leqslant 2^{-\ell-2}\theta_\ell N_{\ell-1}^{-1}AQ^{1/2}\\ &\leqslant 2^{-\ell}N_L^{-1}Q^{1/2}\end{aligned}$$

by (2.5) (see (2.4)), (2.21), (2.9), (2.13), (2.18) and (2.43). Hence for each $L \geqslant 1$ and each $I \geqslant L+1$, by (2.22),

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|h_I(\lambda)-h_L(\lambda)|\,\mathrm{d}\lambda &\leqslant \sum_{\ell=L+1}^{I}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g_{\delta(\ell),M(\ell)}(\lambda)|\,\mathrm{d}\lambda\\ &\leqslant 2^{-L}N_L^{-1}Q^{1/2}.\end{aligned}$$

Hence by Lemma 3 and dominated convergence, for each $L \geqslant 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\lambda)-h_L(\lambda)|\,\mathrm{d}\lambda \leqslant 2^{-L}N_L^{-1}Q^{1/2}.$$

Hence by (2.44), for each $L \geqslant 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\lambda)-\exp[h_L(\lambda)]|\,\mathrm{d}\lambda \leqslant 6\times 2^{-L}N_L^{-1}Q^{1/2}. \tag{2.45}$$

We shall come back to this equation shortly.

In working with the Fejer kernels, it will be handy to use the notation

$$F_n(\lambda) := \frac{1}{n}\frac{\sin^2(n\lambda/2)}{\sin^2(\lambda/2)}$$

for $n \geqslant 1$ and $\lambda \in [-\pi, \pi]$, with $F_n(0)$ defined by continuity to be n. The upper bound $F_n(\lambda) \leqslant n$ is well known, for $n \geqslant 1$ and $\lambda \in [-\pi, \pi]$.

For each positive integer n, one has for our stationary (Gaussian) sequence X that

$$n^{-1}\mathrm{E}S_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}F_n(\lambda)f(\lambda)\,\mathrm{d}\lambda,$$

by a well-known calculation (Ibragimov and Linnik 1971, p. 322, Theorem 18.2.1). (Keep in mind the stipulation $\mathrm{E}X_0 = 0$. Also note that, in the definition of spectral density in (2.1), there is an extra factor of $1/2\pi$.) Consequently, for each $L \geqslant 1$, one has that

$$\begin{aligned}N_L^{-1}\mathrm{E}S_{N(L)}^2 = {}&\frac{1}{2\pi}\int_{-\pi}^{\pi}F_{N(L)}\{\exp[h_L(\lambda)]\}\,\mathrm{d}\lambda\\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi}F_{N(L)}\{f(\lambda)-\exp[h_L(\lambda)]\}\,\mathrm{d}\lambda.\end{aligned} \tag{2.46}$$

Now for each $L \geqslant 1$, by (2.45),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_{N(L)}|f(\lambda) - \exp[h_L(\lambda)]|\,\mathrm{d}\lambda \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} N_L|f(\lambda) - \exp[h_L(\lambda)]|\,\mathrm{d}\lambda$$
$$\leqslant 6 \times 2^{-L}Q^{1/2}. \tag{2.47}$$

For each odd $L \geqslant 3$, one has that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F_{N(L)}(\lambda)\{\exp[h_L(\lambda)]\}\,\mathrm{d}\lambda - \mathrm{e}^{-A}\right| \leqslant 2^{-L}$$

by (2.19), (2.22) and (2.20). Hence, by (2.46) and (2.47),

$$N_L^{-1}\mathrm{E}S_{N(L)}^2 \to \mathrm{e}^{-A} \text{ as } L \to \infty,\ L \text{ odd.} \tag{2.48}$$

For each even $L \geqslant 2$, one has that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F_{N(L)}(\lambda)\{\exp[h_L(\lambda)]\}\,\mathrm{d}\lambda - 1\right| \leqslant 2^{-L}$$

by (2.19), (2.22) and (2.20). Hence, by (2.46) and (2.47),

$$N_L^{-1}\mathrm{E}S_{N(L)}^2 \to 1 \text{ as } L \to \infty,\ L \text{ even.} \tag{2.49}$$

By (2.48) and (2.49), (1.4) holds. This completes the proof of (1.4) and of Theorem 1. □

Remark 1. For stationary Gaussian sequences, the "information regularity" condition (which will not be defined here) is equivalent to absolute regularity (Ibragimov and Rozanov 1978, Chapter 4, Theorems 4, 6 and 8). Hence the sequence X in Theorem 1 satisfies information regularity.

Remark 2. For the sequence X in Theorem 1, one can in addition make $\beta(1)$ (and even $I(1)$, the first dependence coefficient associated with the information regularity condition) arbitrarily small. This can be done by having $\sum_{k=0}^{\infty} ka_k^2$ sufficiently small, where the numbers a_k are as in (2.33) and (2.34). This is based on a technical fact which was pointed out by Bradley (1983, p. 84, Lemma 1.2) and which is just a slight embellishment of some of the arguments of Ibragimov and Solev (1969) and Ibragimov and Rozanov (1978, Chapter 4). Referring to (2.40) and (2.42), one way to have $\sum_{k=0}^{\infty} ka_k^2$ very small is to retain the constant A in the last term in (2.40) and to choose A (see (2.6) and (2.7)) very small to begin with.

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