

Point-stationarity in d dimensions and Palm theory

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This paper extends to $d > 1$ dimensions the concept of point-stationarity, which formalizes the intuitive idea of a point process for which the behaviour relative to a given point of the process is independent of the point selected as origin. After defining point-stationarity, this concept is characterized in several ways and the characterizations then used to extend to d dimensions a particular approach to Palm theory, producing two dualities between stationary and point-stationary processes with quite different interpretations. The dualities coincide in the ergodic case.

Keywords: coupling; Palm theory; point process; random field; stationarity; stochastic geometry

1. Introduction

The main purpose of this paper is to extend to $d > 1$ dimensions the concept of point-stationarity, which formalizes the intuitive idea of a simple point process for which the behaviour relative to a given *point of the process* is independent of the point selected as origin.

Note that point-stationarity is different from stationarity. Stationarity means that the behaviour of the point process relative to any given *non-random site* is independent of the site selected as origin. For instance, when $d = 1$, the Poisson process with constant intensity is *stationary* since if the origin is moved from 0 to any other fixed site t then we still have a Poisson process with the same intensity. But if a point is added at 0 then we obtain a non-stationary process since if the origin is moved from the point which we placed at 0 to any other fixed site t then there need not be a point there. This new process, however, is *point-stationary* since, seen from the point which we placed at 0, the intervals between points are i.i.d. exponential and this will still be true if we move the origin, for instance, to the first *point* on the right of the point at 0.

The point-stationarity problem is presented in Section 2 and solved in Section 3.

After defining point-stationarity in Section 3 the concept is further characterized in Sections 4, 5 and 6. In Sections 8 and 9 these characterizations are then used to extend to $d > 1$ dimensions a particular approach to the Palm theory of stationary point processes presented in Thorisson (1995) for the case $d = 1$, while Section 7 contrasts this approach to the conventional one.

Section 10 concludes with comments on the two Palm dualities presented in Sections 8 and 9, on references, and on possible extensions of the point-stationarity concept, for

instance, to the zero set of Brownian motion discussed by Mandelbrot (1983) and Kagan and Vere-Jones (1988). These are in fact the only references that this author is aware of where the point-stationarity problem is even mentioned.

The paper is fairly self-contained. In particular, no prior knowledge of Palm theory is assumed. Only in the proof of the final two theorems do we refer to Thorisson (1996) for a coupling result and to Daley and Vere-Jones (1988) for an ergodic theorem.

2. The point-stationarity problem

This section explains the point-stationarity problem in full detail, moving from the obvious one-dimensional case to the not so obvious higher-dimensional case. Necessary notation is introduced along the way.

2.1. The simple point processes N and N°

Intuitively a simple point process in d dimensions, $d \geq 1$, is a countable set of isolated points scattered over the d -dimensional Euclidean space \mathbb{R}^d in some random manner (like planets scattered over space). This random set of points can be represented by a collection of random variables

$$N = (N(B) : B \in \mathcal{B}^d),$$

where \mathcal{B}^d are the Borel subsets of \mathbb{R}^d and $N(B)$ is the number of points in the set B . More precisely, a *simple point process* in d dimensions is a random element N in the measurable space (M, \mathcal{M}) , where M is the set of all simple counting measures (that is, integer-valued measures μ on $(\mathbb{R}^d, \mathcal{B}^d)$ with $\mu(B) < \infty$ for all bounded $B \in \mathcal{B}^d$ and $\mu(\{t\}) = 0$ or 1 for all $t \in \mathbb{R}^d$) and \mathcal{M} is the product σ -algebra on M (the smallest σ -algebra such that, for each $B \in \mathcal{B}^d$, the projection mapping from M to \mathbb{R} taking μ to $\mu(B)$ is \mathcal{M}/\mathcal{B} measurable). The σ -algebra \mathcal{M} is also generated by the usual topology on M , but we shall not need this fact.

We shall write N° to indicate that one of the points is placed at 0 (the origin of \mathbb{R}^d), that is,

$$N^\circ(\{0\}) \equiv 1.$$

We shall regard N° as a random element in $(M^\circ, \mathcal{M}^\circ)$, where M° is the subset of M containing the simple counting measures having mass one at the origin,

$$M^\circ = \{\mu \in M : \mu(\{0\}) = 1\},$$

and \mathcal{M}° is the trace of \mathcal{M} on M° , that is, $\mathcal{M}^\circ = \mathcal{M} \cap M^\circ$.

Let (Ω, \mathcal{F}, P) be a common probability space supporting N and N° , and all other random elements in this paper. (In Sections 8 and 9 we break this convention by letting there be two choices of P).

2.2. Sites and points

Call a (non-random) element t of \mathbb{R}^d a *site* (and not a point) to distinguish it from the points of the point process. Thus a random element T in \mathbb{R}^d is a *random site*. Call a random site Π a *point* or a *random point* of N only if the point process N has a point at Π , that is, only if

$$N(\{\Pi\}) \equiv 1.$$

Similarly, call a random site Π^o a *point* or a *random point* of N^o only if the point process N^o has a point at Π^o , that is, only if

$$N^o(\{\Pi^o\}) \equiv 1.$$

A point process typically does not look the same seen from a non-random observation *site* as seen from an observation *point* (the universe does not look the same seen from space as seen from a planet).

2.3. Site-shifts and stationarity of N

For $\mu \in M$ let $s(\mu)$ denote the set of μ -points (the *point pattern* or the support of μ):

$$s(\mu) = \{p \in \mathbb{R}^d : \mu(\{p\}) = 1\},$$

For $t \in \mathbb{R}^d$, define the *shift* or *site-shift* θ_t taking $\mu \in M$ to $\theta_t\mu \in M$ by

$$\theta_t\mu(B) = \mu(t + B), \quad B \in \mathcal{B}^d,$$

where $t + B = \{t + s : s \in B\}$. The point process N is called *stationary* if

$$\theta_t N \stackrel{D}{=} N, \quad t \in \mathbb{R}^d,$$

where ‘ $\stackrel{D}{=}$ ’ denotes identity in distribution. Note that θ_t shifts (or translates) the point pattern by $-t$. In other words, θ_t shifts the origin (the observation site) to t . Thus one could say that stationarity of the point process N means that it looks the same from all non-random observation *sites*.

2.4. The point-stationarity problem for N^o

Similarly, it would be natural to say that the point process N^o is *point-stationary* if it looks the same from all observation *points*. What this means has not been clear, however – except in one dimension.

When $d = 1$, point-stationarity means that N^o is *interval-stationary*, that is, the intervals between the points form a stationary sequence:

$$(\Pi_{n+k}^o - \Pi_{n+k-1}^o)_{-\infty < k < \infty} \stackrel{D}{=} (\Pi_k^o - \Pi_{k-1}^o)_{-\infty < k < \infty}, \quad n \in \mathbb{Z},$$

where

$$\dots < \Pi_{-2}^o < \Pi_{-1}^o < \Pi_0^o = 0 < \Pi_1^o < \Pi_2^o < \dots$$

are the points of N^o written as an increasing sequence. This definition of point-stationarity when $d = 1$ can be rewritten as

$$\theta_{\Pi_n^o} N^o \stackrel{D}{=} N^o, \quad n \in \mathbb{Z}.$$

In other words, if the observer moves from the point at the origin to the n th point on the right (or left) of the origin then the probability distribution of the point pattern that he sees does not change: the point process N^o looks the same from all observation points.

When $d > 1$, this definition of point-stationarity does not work since then there are no intervals between the points of N^o to form a stationary sequence, and the observer cannot use the simple point-selection rule ‘move from the point at the origin to the n th point on the right (or left)’ since there is typically no n th point on the right (or left). But is there some similar way of moving between points in higher dimensions?

2.5. The point-stationarity problem in the Poisson case

Note that even the Poisson process seems to present a problem. Let N be a stationary Poisson process. When $d = 1$ we obtain a point-stationary N^o by adding a point at the origin to N ,

$$N^o \equiv N + \delta_0, \tag{2.1}$$

where δ_0 is the measure with mass one at 0. It is well known that the intervals between points are then i.i.d. exponential and point-stationarity follows. But when $d > 1$ (in the plane, for instance) how can we shift the origin to another point of N^o without spoiling the distribution of the point pattern that we see?

‘Why not shift the origin to the closest point?’ tends to be the first reaction to this question. In order to indicate why this does not work, let us consider the following example.

Example 2.1. Consider the case when $d = 1$ and let N^o be the Poisson process with a point at the origin defined in (2.1). We know that N^o is point-stationary so if shifting to the closest point is to be the way to shift in higher dimensions it ought to work in one dimension also, that is, it should not change the distribution of N^o .

Shift the origin of N^o to the closest point to obtain $\theta_{\Pi_{\text{closest}}^o} N^o$, where

$$\Pi_{\text{closest}}^o = \begin{cases} \Pi_1^o & \text{if } \Pi_1^o \leq -\Pi_{-1}^o, \\ \Pi_{-1}^o & \text{if } \Pi_1^o > -\Pi_{-1}^o. \end{cases}$$

Then you are sure to see, either to the right or to the left of the new origin, an interval followed by a longer interval. This is definitely not a property of N^o which in both directions from the origin has an exponential interval followed by an independent exponential interval. Thus $\theta_{\Pi_{\text{closest}}^o} N^o$ does not have the same distribution as N^o , that is, shifting the origin to the closest point does not preserve the distribution of this point-stationary N^o .

2.6. Point-maps and point-shifts

Example 2.1 illustrates that the *selection of a point* to shift the origin to is the key issue when defining point-stationarity. We cannot select points in any old way. In order to discuss this problem, we need the following terminology.

Call an $\mathcal{M}^0/\mathcal{B}^d$ measurable mapping π from M^0 to \mathbb{R}^d an N^0 -point-map if it selects a point, that is, if

$$\mu(\{\pi(\mu)\}) = 1, \quad \mu \in M^0.$$

Call the mapping θ_π from M^0 to M^0 defined by

$$\theta_\pi \mu = \theta_{\pi(\mu)} \mu, \quad \mu \in M^0,$$

an N^0 -point-shift.

Note that θ_π shifts the origin from a point to a point. Call θ_π the point-shift associated with π . It is readily checked that θ_π is $\mathcal{M}^0/\mathcal{M}^0$ measurable.

When $d = 1$, examples of N^0 -point-maps are the π_n , $-\infty < n < \infty$, defined for $\mu \in M^0$ by

$$\pi_n(\mu) = \begin{cases} nth \ \mu \text{ point to the right of } 0, & n > 1, \\ 0, & n = 0, \\ -nth \ \mu \text{ point to the left of } 0, & n < -1. \end{cases}$$

The associated N^0 -point-shifts translate the origin to the n th point on the right (or $-n$ th on the left). Note that the random points Π_n^0 introduced in Section 2.4 can be written as $\Pi_n^0 = \pi_n(N^0)$.

When $d \geq 1$, an example of an N^0 -point-map is the shift to the closest point, π_{closest} , defined for $\mu \in M^0$ by

$$\pi_{\text{closest}}(\mu) = \text{the } \mu\text{-point having the lexicographically highest order among the non-zero } \mu\text{-points being at shortest distance from the origin.}$$

The lexicographic rule here is just to make sure that $\pi_{\text{closest}}(\mu)$ is uniquely defined. The random point Π_{closest}^0 in Example 2.1 can be written as $\Pi_{\text{closest}}^0 = \pi_{\text{closest}}(N^0)$.

2.7. What is wrong with shifting the origin to the closest point?

In Example 2.1 shifting the origin to the closest point changed the distribution of an interval-stationary (that is, point-stationary) N^0 . So what is wrong with this point-shift?

In order to answer this question let us first consider another one: when $d = 1$, what is so special about shifting the origin of N^0 to the n th point on the right? The essential property of this point-shift is the following. By knowing the point-selection rule (select the n th point on the *right*) and by looking at the point pattern from the new origin you can always tell from what point you came and shift the origin back to the n th point on the *left* of the new origin: we first shift the origin of μ to $\pi_n(\mu)$ to obtain $\theta_{\pi_n} \mu$ and then the origin of $\theta_{\pi_n} \mu$ to

$$\pi_{-n}(\theta_{\pi_n}\mu) = -\pi_n(\mu)$$

to obtain

$$\theta_{\pi_{-n}}\theta_{\pi_n}\mu = \mu, \quad \mu \in M^0.$$

Thus μ is the *only* element of M^0 that θ_{π_n} shifts to $\mu' := \theta_{\pi_n}\mu$. Also *any* element μ' of M^0 can arise from this point-shift since taking $\mu := \theta_{\pi_{-n}}\mu'$ yields

$$\theta_{\pi_n}\mu = \mu'.$$

Thus, shifting the origin to the n th point on the right is a *bijjective* point-shift.

When $d = 1$, it can be shown that an interval-stationary (that is, point-stationary) N^0 is distributionally invariant under the group of all bijjective N^0 -point-shifts (see Remark 3.1).

Shifting the origin to the closest point is wrong because this point-shift is not bijjective: there can be more than one point of a point pattern having a particular point p as the closest point.

2.8. Bijjective point-shifts?

It is natural to apply bijjective point-shifts when defining point-stationarity since under a bijjective point-shift θ_π all points are *equally important*, or get *equal attention*, in the following sense: with $\mu \in M^0$ fixed, the mapping

$$p \mapsto p + \pi(\theta_p\mu)$$

is a bijection from $s(\mu)$ to $s(\mu)$, that is, under this mapping each point is the image of a unique point. All this suggests that we should define point-stationarity in higher dimensions by requiring that N^0 is distributionally invariant under bijjective N^0 -point-shifts.

But for this definition to make sense we must find some bijjective point-shifts, other than the trivial θ_0 . Having searched for such point-shifts for quite some time, the author doubts that they exist and therefore makes the following guess, without further motivation.

Conjecture 2.1. No nontrivial bijjective N^0 -point-shifts exist when $d > 1$.

If this is true, how then can we define point-stationarity in higher dimensions? The answer is that bijjective point-shifts exist if we consider N^0 against an independent stationary background. We explain what this means in the next section.

3. Definition of point-stationarity

In order to highlight the point-stationarity problem, we have up to now avoided saying that the point process N^0 will be regarded in association with a random field X^0 . Considering N^0 jointly with X^0 does not solve our problem but is conceptually a step in the right direction.

3.1. The associated random fields X and X°

We shall henceforth consider a pair (N, X) where N is a simple counting process and

$$X = (X_s)_{s \in \mathbb{R}^d}$$

is a random field with an arbitrary state space (E, \mathcal{E}) and path space (H, \mathcal{H}) , where H is a shift-invariant subset of $E^{\mathbb{R}^d}$ and \mathcal{H} is the σ -algebra on H generated by the projection mappings taking $x = (x_s)_{s \in \mathbb{R}^d}$ in H to x_t in E , $t \in \mathbb{R}^d$.

In order to be able to apply random shifts we need the minimal regularity condition (satisfied in the standard settings, for instance when (E, \mathcal{E}) is Polish and the paths right-continuous) that X is *canonically measurable*, that is, the mapping from $H \times \mathbb{R}^d$ to E taking (x, t) to x_t is $\mathcal{H} \otimes \mathcal{B}^d / \mathcal{E}$ measurable. For $t \in \mathbb{R}^d$, let θ_t denote the *shift* or *site-shift* from H to H defined by

$$\theta_t x = (x_{t+s})_{s \in \mathbb{R}^d}, \quad x \in H.$$

Canonical measurability is equivalent to the mapping from $H \times \mathbb{R}^d$ to H taking (x, t) to $\theta_t x$ being $\mathcal{H} \otimes \mathcal{B}^d / \mathcal{H}$ measurable (*shift-measurability*).

(Considering N in association with a random field is equivalent to considering a marked point process in the following sense. When X is given we could define the mark of a point Π to be $\theta_\Pi X$. Conversely, when the marks are given we could define X by letting X_s be the marked point process with origin shifted to s .)

We shall not assume any functional connection between N and X . At one extreme, N could be determined by X . For instance, when $d = 1$, the points could be the times when a stochastic process enters a given state. At the other extreme, X could be identically constant, which boils down to regarding N alone (not in association with any random field) as we did in Section 2.

When we consider a random field in association with N° we shall denote it X° . Thus the $^\circ$ on X° is not to indicate a property of X° . It only indicates that X° is considered jointly with N° (while as before the $^\circ$ on N° is to indicate that N° has a point at the origin).

3.2. Extended point-maps and point-shifts

Let θ_t denote the *shift* or *site-shift* from $M \times H$ to $M \times H$ defined, for $t \in \mathbb{R}^d$, by

$$\theta_t(\mu, x) = (\theta_t \mu, \theta_t x), \quad (\mu, x) \in M \times H.$$

Call an $\mathcal{M}^\circ \otimes \mathcal{H} / \mathcal{B}^d$ measurable mapping π from $M^\circ \times H$ to \mathbb{R}^d an (N°, X°) -*point-map* if it selects a point, that is, if

$$\mu(\{\pi(\mu, x)\}) = 1, \quad (\mu, x) \in M^\circ \times H.$$

Call the mapping θ_π from $M^\circ \times H$ to $M^\circ \times H$ defined by

$$\theta_\pi(\mu, x) = \theta_{\pi(\mu, x)}(\mu, x), \quad (\mu, x) \in M^\circ \times H,$$

an (N°, X°) -*point-shift*.

Note that θ_π shifts the origin from a point to a point. Call θ_π the point-shift associated

with π . The $\mathcal{M}^\circ \otimes \mathcal{H} / \mathcal{M}^\circ \otimes \mathcal{H}$ measurability of θ_π follows from the canonical measurability of X° .

In order to distinguish (N°, X°) -point-maps and -shifts from N° -point-maps and -shifts we shall sometimes call them *extended point-maps* and *extended point-shifts*.

3.3. The point-stationarity problem for (N°, X°)

The pair (N, X) is *stationary* if it looks the same from all observation sites, that is,

$$\theta_t(N, X) \stackrel{D}{=} (N, X), \quad t \in \mathbb{R}^d.$$

Similarly, (as in the case of N° alone) it would be natural to say that the pair (N°, X°) is *point-stationary* if it looks the same from all observation *points*.

When $d = 1$, point-stationarity means that (N°, X°) is *cycle-stationary*, that is, the points of N° split X° into a stationary sequence of cycles. This definition of point-stationarity, when $d = 1$, can be rewritten as

$$\theta_{\Pi_n^\circ}(N^\circ, X^\circ) \stackrel{D}{=} (N^\circ, X^\circ), \quad n \in \mathbb{Z},$$

and it can be shown that this is equivalent to (N°, X°) being distributionally invariant under the group of all bijective (N°, X°) -point-shifts (again see Remark 3.1).

Under a bijective (N°, X°) -point-shift θ_π all points are *equally important*, or get *equal attention*, in the same sense as in the N° case: with $(\mu, x) \in M^\circ \times H$ fixed, the mapping

$$p \mapsto p + \pi(\theta_p(\mu, x))$$

is a bijection from $s(\mu)$ to $s(\mu)$, that is, under this mapping each point is the image of a unique point.

This again suggests that we should define point-stationarity for $d > 1$ by requiring that (N°, X°) is distributionally invariant under bijective (N°, X°) -point-shifts. Even if Conjecture 2.1 is true, non-trivial bijective (N°, X°) -point-shifts might exist for certain random fields X° . Indeed, we shall give an example in Section 3.6.

Although this will not give us a definition of point-stationarity for arbitrary pairs (N°, X°) it is a step half-way towards the solution.

3.4. Intuitive motivation of the solution

The solution to the point-stationarity problem is to consider (N°, X°) against any independent stationary background, that is, consider (N°, X°) jointly with an arbitrary independent stationary (canonically measurable) random field

$$Y^\circ = (Y_s^\circ)_{s \in \mathbb{R}^d}.$$

Let (L, \mathcal{L}) be the path space of Y° . Note that the $^\circ$ on Y° (like the $^\circ$ on X°) is only to indicate that Y° is considered in association with N° and does not imply that Y° and N° are

functionally connected (while the \circ on the point process N° itself indicates that N° has a point at the origin).

Intuitively, if the triple $(N^\circ, X^\circ, Y^\circ)$ looks the same from all the points of N° then so, in particular, does the pair (N°, X°) . Conversely, if (N°, X°) looks the same from all the points of N° then so will $(N^\circ, X^\circ, Y^\circ)$ because (due to the stationarity of Y°) Y° looks the same from all random sites that are independent of Y° and thus (due to the independence of Y° and (N°, X°) and the fact that (N°, X°) looks the same from all the points of N°) the triple $(N^\circ, X^\circ, Y^\circ)$ should look the same from all points of N° .

In other words, (N°, X°) should be point-stationary if and only if $(N^\circ, X^\circ, Y^\circ)$ is point-stationary.

3.5. Solution of the point-stationarity problem

This suggests that we call the pair (N°, X°) point-stationary if the triple $(N^\circ, X^\circ, Y^\circ)$ is distributionally invariant under all bijective $(N^\circ, X^\circ, Y^\circ)$ -point-shifts for all canonically measurable random fields Y° which are stationary and independent of (N°, X°) . The above discussion motivates this definition intuitively, while the theory established in the upcoming sections motivates it practically. Here is the definition stated in full detail.

Definition 3.1. Call (N°, X°) *point-stationary* if

$$\theta_{\Pi^\circ}(N^\circ, X^\circ, Y^\circ) \stackrel{D}{=} (N^\circ, X^\circ, Y^\circ) \tag{3.1}$$

for all random points Π° of the form

$$\Pi^\circ = \pi(N^\circ, X^\circ, Y^\circ), \tag{3.2}$$

where Y° is any canonically measurable random field which is stationary and independent of (N°, X°) and π is any $(N^\circ, X^\circ, Y^\circ)$ -point-map (that is, π is an $\mathcal{M}^\circ \otimes \mathcal{H} \otimes \mathcal{L} | \mathcal{B}^d$ measurable mapping from $M^\circ \times H \times L$ to \mathbb{R}^d which selects a point:

$$\mu(\{\pi(\mu, x, y)\}) = 1, \quad (\mu, x, y) \in M^\circ \times H \times L$$

such that the associated point-shift (that is, the $\mathcal{M}^\circ \otimes \mathcal{H} \otimes \mathcal{L} | \mathcal{M}^\circ \otimes \mathcal{H} \otimes \mathcal{L}$ measurable mapping θ_π from $M^\circ \times H \times L$ to $M^\circ \times H \times L$ defined by

$$\theta_\pi(\mu, x, y) = \theta_{\pi(\mu, x, y)}(\mu, x, y), \quad (\mu, x, y) \in M^\circ \times H \times L$$

is a bijection.

Remark 3.1. When $d = 1$, this point-stationarity definition is equivalent to the apparently weaker property of cycle-stationarity since both properties are equivalent to (4.3) in Theorem 4.1 below.

Remark 3.2. Definition 3.1 is equivalent (see Lemma 4.1) to the apparently weaker condition

$$\theta_{\Pi^0}(N^0, X^0) \stackrel{D}{=} (N^0, X^0), \tag{3.3}$$

for all random points

$$\Pi^0 = \kappa_n(N^0, Y^0),$$

where Y^0 and κ_n , $n \in \mathbb{Z}$, are from the family of random fields and point-shifts (indexed by $h > 0$) defined in the next subsection.

3.6. A key example of extended bijective point-shifts

We shall now construct a random field Y^0 and a two-sided sequence of (N^0, Y^0) -point-maps κ_n , $n \in \mathbb{Z}$, such that the associated (N^0, Y^0) -point-shifts are bijections. Both Y^0 and the point-maps will depend on a fixed constant $h > 0$. Thus we are really defining a family of random fields and point-shifts indexed by $h > 0$, although we suppress the parameter h in the notation.

We start by constructing Y^0 which will simply represent the stationary point-pattern $h\mathbb{Z}^d - hU$, where U is uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]^d$. Let $b = (b_s)_{s \in \mathbb{R}^d}$ be the fixed function from \mathbb{R}^d to \mathbb{R}^d defined by

$$b_s = \text{the vector from the site } s \text{ to the closest element of } h\mathbb{Z}^d.$$

If there are more than one such $h\mathbb{Z}^d$ -elements, let b_s be the vector to the element of highest lexicographic order (thus b_s is right-continuous in all the d coordinates of s).

Let Y^0 have state space $(\mathbb{R}^d, \mathcal{B}^d)$ and have paths in

$$L = \{\theta_t b : t \in [-h/2, h/2]^d\} = \{\theta_t b : t \in \mathbb{R}^d\}.$$

Let U be a random site which is uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]^d$ and independent of (N^0, X^0) , and define

$$Y^0 = \theta_{Uh} b.$$

Clearly Y^0 is stationary and independent of (N^0, X^0) , and canonical measurability follows from right-continuity of the paths.

We now turn to constructing the bijective (N^0, Y^0) -point-maps κ_n , $n \in \mathbb{Z}$. Fix $(\mu, y) \in M^0 \times L$. Call a site $t \in \mathbb{R}^d$ such that $y_t = 0$ a y -centre and the associated set $t + [-h/2, h/2]^d$ a y -box. Note that y_0 is the centre of the y -box containing the origin. Let k be the number of μ -points in that box,

$$k = \mu(y_0 + [-h/2, h/2]^d),$$

($k \geq 1$ since $0 \in s(\mu)$ and $y_0 \in (-h/2, h/2]^d$). Let p_0, \dots, p_{k-1} be the μ -points in $s(\mu) \cap (y_0 + [-h/2, h/2]^d)$ ordered lexicographically. Let m denote the index of the μ -point at the origin, that is, $p_m = 0$. For $n \in \mathbb{Z}$, put

$$\kappa_n(\mu, y) = p_{m+n} \pmod k,$$

where

$$(m + n) \bmod k = \inf(m + n - k\mathbb{Z}) \cap [0, \infty).$$

That is, $\kappa_n(\mu, y)$ is the n th point *after* the point at the origin (if $n < 0$, interpret this as the $-n$ th point *before* the point at the origin) in a circular enumeration of the points in the box containing the origin. Thus the point at the origin is the n th point *before* $\kappa_n(\mu, y)$ in that same circular enumeration. Thus

$$\theta_{k-n}\theta_{k_n}(\mu, y) = (\mu, y) \quad (\mu, y) \in M^o \times L.$$

Thus (μ, y) is the *only* element of $M^o \times L$ that θ_{k_n} shifts to $(\mu', y') := \theta_{k_n}(\mu, y)$. Also *any* element (μ', y') of $M^o \times L$ can arise from θ_{k_n} since taking $(\mu, y) := \theta_{k-n}(\mu', y')$ yields $\theta_{k_n}(\mu, y) = (\mu', y')$. Thus θ_{k_n} is bijective.

4. Palm characterization of point-stationarity

In this section we establish a pleasant characterization of point-stationarity in terms of non-random site-shifts. We shall call it a *Palm characterization* because it is the key to the Palm dualities presented in Sections 8 and 9. At the end of the section we state a dual Palm characterization of stationarity.

We start by introducing an important concept, so-called *Voronoi cells*, which play in some respects the same role in higher dimensions as the intervals between points in one dimension.

4.1. Voronoi cells

Consider a point pattern in \mathbb{R}^d represented by a simple counting measure $\mu \in M$. Note that the point pattern $s(\mu)$ need not have a point at the origin. With each point $p \in s(\mu)$ associate the set of sites $t \in \mathbb{R}^d$ which are *strictly closer* to p than to any other point. This set is the *open* Voronoi cell with point p . Define the Voronoi cells themselves by extending the open Voronoi cells in such a way that a site $t \in \mathbb{R}^d$ which is at equal minimal distance to two or more points belongs to the cell with point p having the highest lexicographic order.

Thus with each point $p \in s(\mu)$ there is associated a Voronoi cell. There are finitely or countably many of these Voronoi cells, they are disjoint, and their union is \mathbb{R}^d . In other words, the Voronoi cells form a finite or countable partition of \mathbb{R}^d .

Note that, when $d = 1$, the Voronoi cells are intervals with the points in the interior while one usually considers intervals with the points at the ends. For further information on Voronoi cells, see for instance Okabe *et al.* (1992).

In what follows the Voronoi cell containing the origin is of key importance. For N put

$$C_0 = \text{the Voronoi cell of } N \text{ which contains the origin}$$

and

$$\Pi_0 = \text{the } N\text{-point of } C_0.$$

For N^o put

C_0° = the Voronoi cell of N° which contains the point at the origin.

4.2. Shifting the origin to and from Π_0

From now on the pairs (N, X) and (N°, X°) will be functionally linked as follows. Regard (N, X) as given. Define (N°, X°) and a C_0° -valued random site S by

$$(N^\circ, X^\circ) \equiv \theta_{\Pi_0}(N, X), \quad \text{and } S \equiv -\Pi_0.$$

Thus $((N^\circ, X^\circ), S)$ is $((N, X), 0)$ seen from Π_0 .

Conversely, regard (N°, X°) and a C_0° -valued random site S as given. Define (N, X) by

$$(N, X) \equiv \theta_S(N^\circ, X^\circ)$$

(this implies that $\Pi_0 \equiv -S$). Thus (N, X) is (N°, X°) seen from S .

Under this functional one-to-one correspondence between (N, X) and $((N^\circ, X^\circ), S)$ we have

$$C_0^\circ \equiv C_0 - \Pi_0$$

or, equivalently,

$$C_0 \equiv C_0^\circ - S.$$

Thus, in particular, the Voronoi cells C_0 and C_0° have the same volume:

$$\lambda(C_0^\circ) \equiv \lambda(C_0)$$

where λ is Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}^d)$.

4.3. The Palm characterization of point-stationarity

We shall now show that point-stationarity of the pair (N°, X°) means that if we shift the origin to a site selected uniformly at random in C_0° then the pair will look the same from all observation *sites* provided we *volume-bias* by the volume of C_0° .

Theorem 4.1. *Let (N°, X°) , S and (N, X) be linked as in Section 4.2, and suppose $\lambda(C_0)$ is finite with probability one. Then*

$$(N^\circ, X^\circ) \text{ is point-stationary} \tag{4.1}$$

and

$$\text{the conditional distribution of } S \text{ given } (N^\circ, X^\circ) \text{ is uniform on } C_0^\circ \tag{4.2}$$

if and only if

$$E[f(\theta_t(N, X))\lambda(C_0)] = E[f(N, X)\lambda(C_0)], \quad f \in \mathcal{M} \otimes \mathcal{H}^+, t \in \mathbb{R}^d. \tag{4.3}$$

We prove Theorem 4.1 in the next three subsections.

4.4. First part of proof of Theorem 4.1: (4.1) and (4.2) imply (4.3)

We shall first assume that (4.1) and (4.2) hold and show that this implies (4.3). Take $t \in \mathbb{R}^d$ and $f \in \mathcal{M} \otimes \mathcal{H}^+$ and use the conditional uniformity (4.2), and $\lambda(C_0^o) \equiv \lambda(C_0)$, to obtain the second identity in

$$\begin{aligned} E[\lambda(C_0)f(\theta_t(N, X))] &= E[\lambda(C_0)f(\theta_{t+S}(N^o, X^o))] \\ &= E\left[\int_{t+C_0^o} f(\theta_s(N^o, X^o))ds\right]. \end{aligned} \tag{4.4}$$

Let κ_n and Y^o be as in Section 3.6; furthermore, let C^n denote the Voronoi cell of N^o associated with the point $\kappa_n(N^o, Y^o)$,

$$K = N^o(Y_0^o + [-h/2, h/2]^d)$$

denote the number of N^o -points in the Y^o -box containing the origin, and

$$A = \bigcup_{n=0}^{K-1} C^n$$

denote the union of cells with points in the Y^o -box containing the origin.

Note that the number of N^o -points in the box containing the origin remains (since the point-map κ_n selects a point in that box) the same after the shift by θ_{κ_n} , namely K . Thus $1_{\{K=k\}} \int_{(t+C_0^o) \cap C^n} f(\theta_s(N^o, X^o))ds$ is obtained from (N^o, X^o, Y^o) by the same function in $\mathcal{M}^o \otimes \mathcal{H} \otimes \mathcal{L}^+$ as $1_{\{K=k\}} \int_{(t+C^{-n}) \cap C_0^o} f(\theta_s(N^o, X^o))ds$ from $\theta_{\kappa_n}(N^o, X^o, Y^o)$. Due to (4.1), (N^o, X^o, Y^o) and $\theta_{\kappa_n}(N^o, X^o, Y^o)$ have the same distribution, and thus

$$E\left[1_{\{K=k\}} \int_{(t+C_0^o) \cap C^n} f(\theta_s(N^o, X^o))ds\right] = E\left[1_{\{K=k\}} \int_{(t+C^{-n}) \cap C_0^o} f(\theta_s(N^o, X^o))ds\right].$$

Recall that A is the union of C^n over $0 \leq n < K$ and note that A is also the union of C^{-n} over $0 \leq n < K$. Thus summing first over $0 \leq n < k$ and then over $1 \leq k < \infty$ yields

$$E\left[\int_{(t+C_0^o) \cap A} f(\theta_s(N^o, X^o))ds\right] = E\left[\int_{(t+A) \cap C_0^o} f(\theta_s(N^o, X^o))ds\right].$$

Now A in fact depends on the parameter h and expands to the union of all the Voronoi cells as $h \rightarrow \infty$. Thus both A and $t + A$ expand to \mathbb{R}^d as $h \rightarrow \infty$. Thus by monotone convergence

$$E\left[\int_{t+C_0^o} f(\theta_s(N^o, X^o))ds\right] = E\left[\int_{C_0^o} f(\theta_s(N^o, X^o))ds\right].$$

Apply this to (4.4) to obtain that $E[\lambda(C_0)f(\theta_t(N, X))]$ does not depend on t , that is, (4.3) holds.

4.5. Middle part of proof of Theorem 4.1: (4.3) implies (4.2)

We shall now assume that (4.3) holds and show that this implies (4.2). For $t \in \mathbb{R}^d$, write C_t for the Voronoi cell of N containing the site t , and Π_t for the N -point of C_t . From (4.3) and $P(\lambda(C_0) < \infty) = 1$ we have

$$E[1_{\{\lambda(C_t)=\infty\}}\lambda(C_0)] = E[1_{\{\lambda(C_0)=\infty\}}\lambda(C_0)] = 0,$$

which, together with $\lambda(C_0) > 0$, yields

$$P(\lambda(C_t) < \infty) = 1, \quad t \in \mathbb{R}^d.$$

Let g be a non-negative measurable real-valued function defined on \mathbb{R}^d and let $f \in \mathcal{M} \otimes \mathcal{H}^+$. Note that $\lambda(C_0)^{-1}g(S)f(N^o, X^o)$ is obtained from (N, X) by the same function in $\mathcal{M} \otimes \mathcal{H}^+$ as $\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))$ from $\theta_t(N, X)$. Due to (4.3), this implies that

$$E[g(S)f(N^o, X^o)] = E[\lambda(C_0)\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))].$$

Integrating over $0 \leq t < 1$ and interchanging integration and expectation yields

$$E[g(S)f(N^o, X^o)] = E\left[\lambda(C_0)\int_{0 \leq t < 1} \lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt\right]. \tag{4.5}$$

Take $i \in \mathbb{Z}^d$ and note that

$$\int_{0 \leq t < 1} 1_{\{i \leq \Pi_t < i+1\}}\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt$$

is obtained from (N, X) by the same function in $\mathcal{M} \otimes \mathcal{H}^+$ as

$$\int_{-i \leq t < -i+1} 1_{\{0 \leq \Pi_t < 1\}}\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt$$

from $\theta_{-i}(N, X)$. Applying (4.3), with t replaced by $-i$, yields

$$\begin{aligned} E\left[\lambda(C_0)\int_{0 \leq t < 1} 1_{\{i \leq \Pi_t < i+1\}}\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt\right] \\ = E\left[\lambda(C_0)\int_{-i \leq t < -i+1} 1_{\{0 \leq \Pi_t < 1\}}\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt\right]. \end{aligned}$$

Sum over $i \in \mathbb{Z}^d$ to obtain, due to (4.5) and since \mathbb{R}^d is the union of the disjoint sets, $[-i, -i + 1)^d$,

$$E[g(S)f(N^o, X^o)] = E\left[\lambda(C_0)\int_{t \in \mathbb{R}^d} 1_{\{0 \leq \Pi_t < 1\}}\lambda(C_t)^{-1}g(t - \Pi_t)f(\theta_{\Pi_t}(N, X))dt\right].$$

Since $\{t \in \mathbb{R}^d : 0 \leq \Pi_t < 1\}$ is the disjoint union of C_p over the points p in $[0, 1)^d$ and since

$$\begin{aligned} \lambda(C_t) = \lambda(C_p) \quad \text{and} \quad f(\theta_{\Pi_t}(N, X)) = f(\theta_p(N, X)), \quad t \in C_p, \\ g(t - \Pi_t) = g(t - p), \quad t \in C_p, \end{aligned}$$

we obtain

$$E[g(S)f(N^\circ, X^\circ)] = E\left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} f(\theta_p(N, X))\lambda(C_p)^{-1} \int_{t \in C_p} g(t-p)dt\right]. \quad (4.6)$$

Taking $g \equiv 1$ yields

$$E[f(N^\circ, X^\circ)] = E\left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} f(\theta_p(N, X))\right], \quad f \in \mathcal{M} \otimes \mathcal{H}^+. \quad (4.7)$$

Replacing $f(N^\circ, X^\circ)$ by $f(N^\circ, X^\circ)\lambda(C_0)^{-1} \int_{t \in C_0^\circ} g(t)dt$ in (4.7) yields

$$\begin{aligned} E\left[f(N^\circ, X^\circ)\lambda(C_0)^{-1} \int_{t \in C_0^\circ} g(t)dt\right] \\ = E\left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} f(\theta_p(N, X))\lambda(C_p)^{-1} \int_{t \in C_p} g(t-p)dt\right]. \end{aligned}$$

Compare this and (4.6) to obtain

$$E[g(S)f(N^\circ, X^\circ)] = E\left[f(N^\circ, X^\circ)\lambda(C_0)^{-1} \int_{t \in C_0^\circ} g(t)dt\right], \quad g \in \mathcal{B}^{d^+}, f \in \mathcal{M} \otimes \mathcal{H}^+,$$

which is equivalent to (4.2).

4.6. Final part of proof of Theorem 4.1: (4.3) implies (4.1)

Finally, we assume that (4.3) holds and show that this implies (4.1). Let Y° and π be as in Definition 3.1. Assume also that Y° is independent of (N, X) , that is, not only independent of (N°, X°) but also of $((N^\circ, X^\circ), S)$. (This is no restriction since if this were not the case we could replace Y° by a random field which is independent of (N, X) and has the same distribution as Y° : (3.1) holds for Y° if and only if it holds with Y° replaced by this copy of Y° .) Put

$$Y = \theta_S Y^\circ$$

and note that since Y° is stationary and independent of $((N^\circ, X^\circ), S)$, so is Y .

Since Y is stationary and independent of (N, X) we obtain from (4.3) that

$$E[f(\theta_t(N, X, Y))\lambda(C_0)] = E[f(N, X, Y)\lambda(C_0)], \quad f \in \mathcal{M} \otimes \mathcal{H} \otimes \mathcal{L}^+, t \in \mathbb{R}^d. \quad (4.8)$$

In the same way as (4.7) followed from (4.3), we obtain that (4.8) implies

$$E[f(N^\circ, X^\circ, Y^\circ)] = E\left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} f(\theta_p(N, X, Y))\right], \quad f \in \mathcal{M} \otimes \mathcal{H} \otimes \mathcal{L}^+. \quad (4.9)$$

Replace f by $f(\theta_\pi(\cdot))$ in (4.9) to obtain

$$E[f(\theta_\pi(N^\circ, X^\circ, Y^\circ))] = E \left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} f(\theta_\pi \theta_p(N, X, Y)) \right]. \tag{4.10}$$

Take $i \in \mathbb{Z}^d$ and note that

$$\sum_{p \in s(N) \cap [0,1]^d} 1_{\{i \leq p + \pi(\theta_p(N, X, Y)) < i+1\}} f(\theta_\pi \theta_p(N, X, Y))$$

is obtained from (N, X, Y) by the same function in $\mathcal{M} \otimes \mathcal{H} \otimes \mathcal{L}^+$ as

$$\sum_{p \in s(N) \cap [-i, -i+1]^d} 1_{\{0 \leq p + \pi(\theta_p(N, X, Y)) < 1\}} f(\theta_\pi \theta_p(N, X, Y))$$

from $\theta_{-i}(N, X, Y)$. Applying (4.8), with t replaced by $-i$, yields

$$\begin{aligned} & E \left[\lambda(C_0) \sum_{p \in s(N) \cap [0,1]^d} 1_{\{i \leq p + \pi(\theta_p(N, X, Y)) < i+1\}} f(\theta_\pi \theta_p(N, X, Y)) \right] \\ &= E \left[\lambda(C_0) \sum_{p \in s(N) \cap [-i, -i+1]^d} 1_{\{0 \leq p + \pi(\theta_p(N, X, Y)) < 1\}} f(\theta_\pi \theta_p(N, X, Y)) \right]. \end{aligned}$$

Sum over $i \in \mathbb{Z}^d$ and compare with (4.10) to obtain

$$E[f(\theta_\pi(N^\circ, X^\circ, Y^\circ))] = E \left[\lambda(C_0) \sum_{p \in s(N)} 1_{\{0 \leq p + \pi(\theta_p(N, X, Y)) < 1\}} f(\theta_\pi \theta_p(N, X, Y)) \right].$$

Since θ_π is bijective we have that for each point $q \in s(N) \cap [0, 1]^d$ there is a unique point $p \in s(N)$ such that $q = p + \pi(\theta_p(N, X, Y))$. Applying this on the right-hand side yields (note that $\theta_\pi \theta_p(N, X, Y) = \theta_{p + \pi(\theta_p(N, X, Y))}(N, X, Y)$)

$$E[f(\theta_\pi(N^\circ, X^\circ, Y^\circ))] = E \left[\lambda(C_0) \sum_{q \in s(N) \cap [0,1]^d} f(\theta_q(N, X, Y)) \right].$$

Compare this and (4.9) to obtain that

$$E[f(\theta_\pi(N^\circ, X^\circ, Y^\circ))] = E[f(N^\circ, X^\circ, Y^\circ)], \quad f \in \mathcal{M} \otimes \mathcal{H} \otimes \mathcal{L}^+,$$

holds for Y° and π as in Definition 3.1. Thus (4.1) holds and the proof of Theorem 4.1 is complete.

4.7. The backgrounds and point-maps in Section 3.6 suffice

The family of background fields Y° and point-maps κ_n , $n \in \mathbb{Z}$, defined in Section 3.6, depend on a parameter $0 < h < \infty$ suppressed in the notation. This family played a crucial role in the above proof and does in fact suffice to characterize point-stationarity. Below (in the proof of Theorem 5.1) we shall need the following slightly more general result.

Lemma 4.1. *Let V be a $[0, 1]$ -valued random variable which is independent of (N°, X°) and the family of random fields Y° in Section 3.6. Let V° be the stationary random field defined by $V_t^\circ \equiv V$, $t \in \mathbb{R}^d$. The pair (N°, X°) is point-stationary if and only if*

$$\theta_{\Pi_n^\circ}(N^\circ, X^\circ) \stackrel{D}{=} (N^\circ, X^\circ) \tag{4.11}$$

for all random points Π_n° of the form

$$\Pi_n^\circ = \alpha_n(N^\circ, X^\circ, Y^\circ, V^\circ)$$

where α_n , $n \in \mathbb{Z}$, are the $(N^\circ, X^\circ, Y^\circ, V^\circ)$ -point-maps defined by

$$\alpha_n(\mu, x, y, v) = \kappa_{[v_0 k] + n}(\mu, y),$$

with $k = \mu(y_0 + [-h/2, h/2]^d)$ and $[\cdot]$ denoting the integer part.

Proof. The ‘only if’ part holds because (Y°, V°) , regarded as a bivariate random field, $(Y_t^\circ, V_t^\circ)_{t \in \mathbb{R}^d}$, is stationary and independent of (N°, X°) and θ_{α_n} is bijective for the same reason as θ_{κ_n} is: see the end of Section 3.6 and replace the origin in that argument by $\alpha_0(\mu, x, y, v)$.

In order to establish the ‘if’ part, let S be a random site such that the conditional distribution of S given (N°, X°) is uniform on C_0° . Replace Y° by (Y°, V°) and κ_n by α_n in the argument in Section 4.4 to obtain that (4.3) holds. By Theorem 4.1 this implies that (N°, X°) is point-stationary. \square

4.8. Palm characterization of stationarity

Modifying the proof of Theorem 4.1 in an obvious way yields the following dual result.

Theorem 4.2. *Let (N, X) , (N°, X°) and S be linked as in Section 4.2 and suppose $\lambda(C_0) < \infty$ with probability one. Then (N, X) is stationary if and only if the conditional distribution of S given (N°, X°) is uniform on C_0° and*

$$E[f(\theta_{\Pi^\circ}(N^\circ, X^\circ))/\lambda(C_0)] = E[f(N^\circ, X^\circ)/\lambda(C_0)], \quad f \in \mathcal{M}^\circ \otimes \mathcal{H}^+,$$

for all random points Π° as in Definition 3.1.

Proof. Simply replace $E[\cdot]$ by $E[\cdot/\lambda(C_0)]$ throughout the proof of Theorem 4.1. \square

Thus stationarity of the pair (N, X) means that if we view it from the point of the Voronoi cell where the origin lies then the origin is located uniformly at random in the cell and, moreover, the pair looks the same from all observation *points* provided we *volume-debias* by the volume of the cell.

We shall need the following result in Section 6.

Lemma 4.2. *If (N, X) is stationary then*

$$E[f(N^\circ, X^\circ)/\lambda(C_0)] = E\left[\sum_{p \in s(N) \cap [0,1]^d} f(\theta_p(N, X))\right], \quad f \in \mathcal{M} \otimes \mathcal{H}^+. \tag{4.12}$$

In particular,

$$E[1/\lambda(C_0)] = E[N([0, 1]^d)]. \tag{4.13}$$

Proof. In the proof of (4.7) replace $E[\cdot]$ by $E[\cdot/\lambda(C_0)]$ to obtain (4.12). Then take $f \equiv 1$ to obtain (4.13). □

The expected number of points in a unit box, $E[N([0, 1]^d)]$, is called the *intensity* of the stationary point process N . According to (4.13) it can be calculated by taking the expectation of one over the volume of the Voronoi cell containing the origin, $E[1/\lambda(C_0)]$.

5. Point-stationarity characterized by randomized point-shifts

In this section we shall show that point-stationarity means distributional invariance under doubly randomized point-shifts: a shift to a uniformly selected *site* followed by a shift to a uniformly selected *point*.

5.1. The characterization result

It turns out that a point-stationary (N°, X°) is characterized by the following property: if (N°, X°) is first shifted by an independent site U selected uniformly at random in any bounded Borel set B of positive volume and the origin is then shifted to a $\theta_{-U}N^\circ$ -point Π picked uniformly at random among the points $s(\theta_{-U}N^\circ) \cap B$ that ended up in B , then the distribution of (N°, X°) does not change. Note that there is at least one $\theta_{-U}N^\circ$ -point in B , the one at U which initially was at the origin:

$$\theta_{-U}N^\circ(\{U\}) = N^\circ(\{0\}) = 1.$$

The *first* randomized shift would not change the distribution of a *stationary* pair: if (N, X) is stationary then, since U is independent of (N, X) ,

$$\theta_{-U}(N, X) \stackrel{D}{=} (N, X).$$

The first randomized shift does, however, change the distribution of a *point-stationary* pair since $\theta_{-U}(N^\circ, X^\circ)$ has no point at the origin, unlike (N°, X°) . But when the first randomized shift is followed by the *second* then the distribution is restored: if (N°, X°) is point-stationary then

$$\theta_\Pi \theta_{-U}(N^\circ, X^\circ) \stackrel{D}{=} (N^\circ, X^\circ).$$

Note that $\Pi^\circ = \Pi - U$ is uniform on $s(N^\circ) \cap (B - U)$ so we can describe this characteri-

zation of a point-stationary (N°, X°) alternatively as follows: if we place any bounded Borel set of positive volume uniformly at random around the N° -point at the origin and shift the origin to a point Π° selected uniformly at random among the N° -points in that set then the distribution of (N°, X°) does not change.

Here is a more formal statement.

Theorem 5.1. *Let $B \in \mathcal{B}^d$ be bounded and such that $\lambda(B) > 0$. Let U be a random site that is uniform on B and independent of (N°, X°) . Let Π° be a random point of N° such that the conditional distribution of Π° given $((N^\circ, X^\circ), U)$ is uniform on the finite set of points of N° lying in $B - U$, that is, uniform on*

$$s(N^\circ) \cap (B - U).$$

Then (N°, X°) is point-stationary if and only if for each such B

$$(\theta_{\Pi^\circ}(N^\circ, X^\circ), \Pi^\circ + U) \stackrel{D}{=} ((N^\circ, X^\circ), U). \tag{5.1}$$

In fact, (N°, X°) is point-stationary if (5.1) holds for all B of the form

$$B = [-h/2, h/2)^d, \quad h > 0.$$

Note that it follows from (5.1) that if we place a bounded Borel set of positive volume uniformly at random around the point at the origin then the point at the origin is located uniformly at random among the points in the set.

5.2. Proof of Theorem 5.1

Let Y° be as in Section 3.6 and choose h large enough for B to be contained in $[-h/2, h/2)^d$. Put

$$B' = [-h/2, h/2)^d \setminus B.$$

Let β_n , $n \in \mathbb{Z}$, be the following modification of the point-shifts κ_n in Section 3.6: if $0 \in y_0 + B$ let $\beta_n(\mu, y)$ be the n th point after the point at the origin in the circular lexicographic enumeration of the μ -points in $y_0 + B$; if $0 \in y_0 + B'$ let $\beta_n(\mu, y)$ be the n th point after the point at the origin in the circular lexicographic enumeration of the μ -points in $y_0 + B'$. Then θ_{β_n} is bijective for the same reason as θ_{κ_n} is bijective (see Section 3.6).

Note that conditionally on $-Y_0^\circ \in B$, $Y_0^\circ + B$ is B placed uniformly at random around the origin. In order to obtain a point placed uniformly at random among the points in $Y_0^\circ + B$, let V be uniform of $[0, 1)$ and independent of $(N^\circ, X^\circ, Y^\circ)$, and define a stationary random field V° by

$$V_t^\circ \equiv V, \quad t \in \mathbb{R}^d.$$

For each $n \in \mathbb{Z}$, define $(N^\circ, X^\circ, Y^\circ, V^\circ)$ -point-maps α_n as follows: if $0 \in y_0 + B$ let $\alpha_n(\mu, x, y, v) = \beta_{[v_0 k] + n}(\mu, y)$, where $k = \mu(y_0 + B)$, and if $0 \in y_0 + B'$ let $\alpha_n(\mu, x, y, v) =$

$\beta_n(\mu, \nu)$. Then θ_{α_n} is bijective and (Y°, V°) , regarded as a bivariate random field, $(Y_t^\circ, V_t^\circ)_{s \in \mathbb{R}^d}$, is stationary and independent of (N°, X°) .

With

$$\Pi_n^\circ = \alpha_n(N^\circ, X^\circ, Y^\circ, V^\circ)$$

we have

$$P(((N^\circ, X^\circ), \Pi_n^\circ - Y_0^\circ, -Y_0^\circ) \in \cdot \mid -Y_0^\circ \in B) = P(((N^\circ, X^\circ), \Pi^\circ + U, U) \in \cdot). \tag{5.2}$$

Suppose (N°, X°) is point-stationary. Then

$$\theta_{\Pi_n^\circ}(N^\circ, X^\circ, Y^\circ, V^\circ) \stackrel{D}{=} (N^\circ, X^\circ, Y^\circ, V^\circ),$$

which implies

$$(\theta_{\Pi_n^\circ}(N^\circ, X^\circ), -Y_{\Pi_n^\circ}^\circ) \stackrel{D}{=} ((N^\circ, X^\circ), -Y_0^\circ). \tag{5.3}$$

Now

$$\{-Y_0^\circ \in B\} = \{-Y_{\Pi_n^\circ}^\circ \in B\}, \quad -Y_{\Pi_n^\circ}^\circ = \Pi_n^\circ - Y_0^\circ, \tag{5.4}$$

which yields the second identity in

$$\begin{aligned} P((\theta_{\Pi^\circ}(N^\circ, X^\circ), \Pi^\circ + U) \in \cdot) &= P((\theta_{\Pi_n^\circ}(N^\circ, X^\circ), \Pi_n^\circ - Y_0^\circ) \in \cdot \mid -Y_0^\circ \in B) && \text{(due to (5.2))} \\ &= P((\theta_{\Pi_n^\circ}(N^\circ, X^\circ), -Y_{\Pi_n^\circ}^\circ) \in \cdot \mid -Y_{\Pi_n^\circ}^\circ \in B) && \text{(due to (5.4))} \\ &= P(((N^\circ, X^\circ), -Y_0^\circ) \in \cdot \mid -Y_0^\circ \in B) && \text{(due to (5.3))} \\ &= P(((N^\circ, X^\circ), U) \in \cdot) && \text{(due to (5.2)).} \end{aligned}$$

Thus point-stationarity implies (5.1) for all bounded B of positive Lebesgue measure.

Conversely, suppose (5.1) holds for $B = [-h/2, h/2]^d$. With this B (5.2) becomes

$$P(((N^\circ, X^\circ), \Pi_n^\circ - Y_0^\circ, -Y_0^\circ) \in \cdot) = P(((N^\circ, X^\circ), \Pi^\circ + U, U) \in \cdot),$$

which implies $\theta_{\Pi_n^\circ}(N^\circ, X^\circ) \stackrel{D}{=} \theta_{\Pi^\circ}(N^\circ, X^\circ)$. Thus, due to (5.1),

$$\theta_{\Pi_n^\circ}(N^\circ, X^\circ) \stackrel{D}{=} (N^\circ, X^\circ),$$

that is, (4.11) in Lemma 4.1 holds. Also, when $B = [-h/2, h/2]^d$ the α_n satisfy the condition in Lemma 4.1. Thus (N°, X°) is point-stationary.

6. Point-stationarity and the invariant σ -algebras \mathcal{T} and \mathcal{J}

In this section we first define the invariant σ -algebras \mathcal{T} and \mathcal{J} , then extend Theorem 5.1

slightly, and finally show that several properties are preserved under conditioning on \mathcal{I} , the invariant σ -algebra of both (N, X) and $(N^\circ, X^\circ) \equiv \theta_{\Pi_0}(N, X)$.

6.1. The invariant σ -algebras \mathcal{I} and \mathcal{J}

Let \mathcal{I} be the invariant σ -algebra on $\mathcal{M} \otimes \mathcal{H}$, namely the following sub- σ -algebra of $\mathcal{M} \otimes \mathcal{H}$:

$$\mathcal{I} = \{B \in \mathcal{M} \otimes \mathcal{H} : \theta_t B = B, t \in \mathbb{R}^d\}.$$

Let \mathcal{J} be the invariant σ -algebra of (N, X) , namely the following sub- σ -algebra of \mathcal{F} :

$$\mathcal{J} = (N, X)^{-1}\mathcal{I}, \tag{6.1}$$

(short for $\mathcal{J} = \{(N, X) \in B\} : B \in \mathcal{I}$). Note that, for any random site T and any $B \in \mathcal{I}$,

$$\begin{aligned} \{\theta_t(N, X) \in B, T = t\} &= \{(N, X) \in \theta_{-t}B, T = t\} \\ &= \{(N, X) \in B, T = t\}, \quad t \in \mathbb{R}^d, \end{aligned}$$

and thus taking union over $t \in \mathbb{R}^d$ yields

$$\{\theta_T(N, X) \in B\} = \{(N, X) \in B\}, \quad B \in \mathcal{I}. \tag{6.2}$$

In particular, with $(N^\circ, X^\circ) \equiv \theta_{\Pi_0}(N, X)$ this yields

$$\mathcal{J} = (N^\circ, X^\circ)^{-1}\mathcal{I}, \tag{6.3}$$

that is, when (N, X) and (N°, X°) are linked as in Section 4.2 then \mathcal{J} is also the invariant σ -algebra of (N°, X°) .

Note that although we have chosen to regard N° as a random element in $(M^\circ, \mathcal{M}^\circ)$, and not in (M, \mathcal{M}) , the invariant σ -algebra of $(N^\circ, X^\circ) \equiv \theta_{\Pi_0}(N, X)$ is still \mathcal{J} since

$$\mathcal{I}^\circ = \text{the invariant } \sigma\text{-algebra on } (M^\circ, \mathcal{M}^\circ) = \text{the trace of } M^\circ \times H \text{ on } \mathcal{I}$$

and thus $\mathcal{J} = (N^\circ, X^\circ)^{-1}\mathcal{I} = (N^\circ, X^\circ)^{-1}\mathcal{I}^\circ$.

6.2. Extension of Theorem 5.1

We shall now extend Theorem 5.1 by allowing the set B to be expanded by an invariant random variable. This result will be used in the proof of Theorem 9.3.

Theorem 6.1. *Let $g \in \mathcal{I}$ be a strictly positive and finite function and put*

$$G \equiv g(N^\circ, X^\circ).$$

Let $B \in \mathcal{B}^d$ be bounded and such that $\lambda(B) > 0$. Let U be a random site that is uniform on B and independent of (N°, X°) . Let Π° be a random point of N° such that the conditional distribution of Π° given $((N^\circ, X^\circ), U)$ is uniform on the finite set of points of N° lying in $G(B - U)$; that is, let Π° be uniform on

$$s(N^\circ) \cap G(B - U).$$

Then (N°, X°) is point-stationary if and only if for each such B

$$(\theta_{\Pi^\circ}(N^\circ, X^\circ), \Pi^\circ + GU) \stackrel{D}{=} ((N^\circ, X^\circ), GU). \tag{6.4}$$

Proof. First assume that g is bounded and repeat the proof of Theorem 5.1 with B replaced by $g(\mu, x)B$. Now $g \in \mathcal{F}$ means that $g(\theta_t(\mu, x))B = g(\mu, x)B$ for all $t \in \mathbb{R}^d$. This fact – that the set $g(\mu, x)B$ does not change by shifting (μ, x) – is needed to deduce that θ_{β_n} is still bijective. Thus (6.4) holds for bounded g . If g is not bounded, take a finite constant $a > 0$ and apply (6.4), with g replaced by $g \wedge a$, to obtain

$$\begin{aligned} & \mathbb{E}[f(\theta_{\Pi^\circ}(N^\circ, X^\circ), \Pi^\circ + GU) 1\{g(\theta_{\Pi^\circ}(N^\circ, X^\circ)) < a\}] \\ &= \mathbb{E}[f((N^\circ, X^\circ), GU) 1\{g(N^\circ, X^\circ) < a\}], \quad f \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{B}^{d+}. \end{aligned}$$

Send a to infinity to obtain (6.4). □

6.3. Conditioning on the invariant σ -algebra \mathcal{F}

We shall now show that all characterizations still hold when we condition on the invariant σ -algebra \mathcal{F} of (N, X) and of $(N^\circ, X^\circ) \equiv \theta_{\Pi^\circ}(N, X)$. This result will be used in Section 9.

Theorem 6.2. *Let (N°, X°) , S and (N, X) be linked as in Section 4.2.*

- (a) *The pair (N°, X°) is point-stationary under P if and only if it is so under $P(\cdot|\mathcal{F})$.*
- (b) *The pair (N, X) is stationary under P if and only if it is so under $P(\cdot|\mathcal{F})$.*
- (c) *Formula (4.3) holds if and only if it holds with \mathbb{E} replaced by $\mathbb{E}[\cdot|\mathcal{F}]$.*
- (d) *Formulae (5.1) and (6.4) hold if and only if they hold when P is replaced by $P(\cdot|\mathcal{F})$.*
- (e) *If (N, X) is stationary under P then*

$$\mathbb{E}[N([0, 1)^d)|\mathcal{F}] = \mathbb{E}[1/\lambda(C_0)|\mathcal{F}] \text{ a.s. } P.$$

Proof. Formula (3.1) in the definition of point-stationarity is equivalent to

$$\begin{aligned} & \mathbb{E}[f(\theta_{\Pi^\circ}(N^\circ, X^\circ, Y^\circ)) 1_{\{\theta_{\Pi^\circ}(N^\circ, X^\circ) \in B\}}] \\ & \mathbb{E}[f(N^\circ, X^\circ, Y^\circ) 1_{\{(N^\circ, X^\circ) \in B\}}], \quad B \in \mathcal{F}, f \in \mathcal{M} \otimes \mathcal{H}^+. \end{aligned}$$

Due to (6.2), we have that

$$1_{\{\theta_{\Pi^\circ}(N^\circ, X^\circ) \in B\}} = 1_{\{(N^\circ, X^\circ) \in B\}}, \quad B \in \mathcal{F},$$

and thus (3.1) is equivalent to

$$\mathbb{E}[f(\theta_{\Pi^\circ}(N^\circ, X^\circ, Y^\circ))|\mathcal{F}] = \mathbb{E}[f(N^\circ, X^\circ, Y^\circ)|\mathcal{F}], \quad f \in \mathcal{H} \otimes \mathcal{H}^+.$$

This yields (a). We obtain (b), (c) and (d) in a similar way.

In order to obtain (e), take $f \in \mathcal{T}$ in Lemma 4.2. Then $f(\theta_p(N, X)) = f(N^\circ, X^\circ)$ and we obtain from (4.12) that

$$E[f(N^\circ, X^\circ)/\lambda(C_0)] = E[f(N^\circ, X^\circ)N([0, 1)^d)], \quad f \in \mathcal{T},$$

that is, (e) holds. □

7. On Palm theory of stationary point processes

We are now ready to extend to $d > 1$ dimensions the approach to Palm theory presented in Thorisson (1995) for $d = 1$. Since this approach differs from the conventional one, let us first have a brief look at the conventional theory.

7.1. On the conventional Palm theory of stationary point processes

The so-called *Palm version* (N°, X°) of a stationary pair (N, X) having finite intensity:

$$E[1/\lambda(C_0)] = E[N([0, 1)^d)] < \infty$$

is usually defined by specifying the expectations of $f(N^\circ, X^\circ)$ to coincide with

$$E \left[\sum_{p \in s(N) \cap [0, 1)^d} f(\theta_p(N, X)) \right] / E[N([0, 1)^d)], \quad f \in \mathcal{M} \otimes \mathcal{H}^+ \tag{7.1}$$

(compare with (4.12) and Theorem 4.2). This (N°, X°) has the following interpretation: (N°, X°) behaves as (N, X) when N happens to have a point at the origin. (That is, the distribution of (N°, X°) is the conditional distribution of (N, X) when it is given that N has a point at the origin. Since the probability of N having a point at the origin is zero this interpretation is informal and must be motivated, for instance, by a limit theorem.)

The distribution of the Palm version (N°, X°) can be *reversed* to get back the distribution of the stationary pair (N, X) . Thus there is a duality (one-to-one correspondence) between the distributions of stationary pairs (N, X) with finite intensity and their Palm versions (N°, X°) . This is not a very satisfying duality since the Palm version is *derived* from the stationary (N, X) . In other words, the Palm version does not have a defining property without reference to its stationary dual and thus the duality is rather half-baked.

When $d = 1$, however, it is known that the Palm version (N°, X°) is cycle-stationary (that is, *point-stationary*) with cycle lengths having finite mean. Moreover, it is known that the reversed Palm of any cycle-stationary (N°, X°) with cycle lengths having finite mean yields a stationary pair (N, X) with finite intensity. Thus, when $d = 1$, there is a *Palm duality* between *stationary* (N, X) with finite intensity and *point-stationary* (N°, X°) with cycle lengths having finite mean.

When $d > 1$, there should be a Palm duality between two classes of processes of equal status: stationary and point-stationary.

7.2. The approach of this paper

In this paper we have given meaning to *point-stationarity* when $d > 1$. The task now is to extend the above-mentioned Palm duality to $d > 1$. We shall do so along the lines of Thorisson (1995). This approach differs from the conventional one in two ways.

On the one hand, rather than defining the distribution of the Palm version by the single formula (7.1) (and then getting the stationary process back through a single reversion formula) we define the Palm duality in two separate steps. The first step is *measure-free*, involving only the paths (shifting to or from a point; Section 4.2). The other is a *biasing* step involving the probability measure and the volume of the Voronoi cell containing the origin. (Other expressions for *biasing* are weighting, Radon–Nikodym, change of measure.)

On the other hand, we present two Palm dualities rather than only one. These dualities have quite different interpretations:

- (1) The first duality has the conventional informal interpretation that the point-stationary pair behaves like its stationary dual *conditioned on having a point at the origin*. In particular, this means that the point-stationary pair in this duality is the conventional Palm version of the stationary pair.
- (2) The second duality has the informal interpretation that the point-stationary pair behaves like its stationary dual *observed from a point selected uniformly at random among all the points*. Conversely, the stationary pair behaves like its point-stationary dual observed from a site selected uniformly at random among all sites in \mathbb{R}^d .

In other words:

- (1) If we watch a stationary pair (N, X) from the origin *and there happens to be a point there* then (N, X) behaves as its first point-stationary dual.
- (2) If we watch a stationary pair (N, X) from a *uniformly selected point* then (N, X) behaves as its second point-stationary dual.

The two dualities coincide in the ergodic case (when the invariant σ -algebra is trivial). The exact condition for coincidence is more general; see Section 10.1.

We shall present these two Palm dualities in the next two sections. The two dualities are indicated by subscripts on the probability measures.

8. The first Palm duality

We start by extending from $d = 1$ to $d > 1$ the conventional (point-at-zero) Palm duality between stationarity and point-stationarity.

8.1. The first Palm duality theorem

Let (N, X) and $((N^\circ, X^\circ), S)$ be defined on the same measurable space (Ω, \mathcal{F}) and linked as in Section 4.2, that is to say,

$$(N^\circ, X^\circ) \equiv \theta_{\Pi_0}(N, X) \quad \text{and} \quad S \equiv -\Pi_0$$

or equivalently

$$(N, X) \equiv \theta_S(N^\circ, X^\circ).$$

Recall that C_0 and C_0° , respectively, are the Voronoi cells of (N, X) and (N°, X°) containing the origin and that they have the same Lebesgue measure $\lambda(C_0^\circ) \equiv \lambda(C_0)$.

Let P_1 and P_1° be two probability measures on (Ω, \mathcal{F}) satisfying

$$E_1[1/\lambda(C_0)] < \infty$$

and

$$dP_1^\circ = \frac{1}{\lambda(C_0)E_1[1/\lambda(C_0)]} dP_1 \quad (\text{volume-debiasing } P_1);$$

or, equivalently,

$$E_1^\circ[\lambda(C_0)] < \infty$$

and

$$dP_1 = \frac{\lambda(C_0)}{E_1^\circ[\lambda(C_0)]} dP_1^\circ \quad (\text{volume-biasing } P_1^\circ).$$

Note that

$$E_1[1/\lambda(C_0)] = \frac{1}{E_1^\circ[\lambda(C_0)]}$$

and that the conditional distribution of S given (N°, X°) is the same under P_1 as under P_1° [since $\lambda(C_0)$ is determined by (N°, X°)].

Theorem 8.1. *We have that*

$$(N, X) \text{ is stationary under } P_1$$

if and only if

$$(N^\circ, X^\circ) \text{ is point-stationary under } P_1^\circ$$

and the conditional distribution of S given (N°, X°) is uniform on C_0° .

Proof. This is an immediate consequence of Theorem 4.1. □

8.2. Interpretation and properties

Theorem 8.1 yields a duality between the *distributions* of stationary (N, X) with finite intensity and point-stationary (N°, X°) with finite-mean Voronoi cell volume, as follows. If the distribution of the stationary (N, X) is given we obtain the distribution of a point-stationary (N°, X°) immediately from the theorem. If, on the other hand, the distribution of

the point-stationary (N°, X°) is given then we can introduce a random site S with a uniform conditional distribution on C_0° , given (N°, X°) , to obtain a distribution of a stationary (N, X) .

The following theorem yields an unusually strong limit motivation of the conventional point-at-zero interpretation of this duality. Put $\Pi_0 = t = 0$ in (c) to obtain the informal expression:

$$P_1((N, X) \in \cdot | \Pi_0 = 0) = P_1^\circ((N^\circ, X^\circ) \in \cdot).$$

(This expression is informal because $P_1(\Pi_0 = 0) = 0$.)

Theorem 8.2. *Suppose that the conditional distribution of S given (N°, X°) is uniform on C_0° under P_1 (or, equivalently, under P_1°). Then the following claims hold:*

(a) *Under P_1 the random site S is continuous with density*

$$\frac{P_1(S \in ds)}{ds} = \frac{P_1^\circ(s \in C_0^\circ)}{E_1^\circ[\lambda(C_0)]}, \quad s \in \mathbb{R}^d.$$

(b) *For each $A \in \mathcal{M} \otimes \mathcal{H}$,*

$$P_1((N^\circ, X^\circ) \in A | S = s) = P_1^\circ((N^\circ, X^\circ) \in A | s \in C_0^\circ), \quad s \in \mathbb{R}^d.$$

(c) *For each $A \in \mathcal{M} \otimes \mathcal{H}$, there is a version of $P_1(\theta_{\Pi_0}(N, X) \in A | \Pi_0 = \cdot)$ such that*

$$P_1(\theta_{\Pi_0}(N, X) \in A | \Pi_0 = t) \rightarrow P_1^\circ((N^\circ, X^\circ) \in A), \quad |t| \downarrow 0.$$

Proof. (a) For $B \in \mathcal{B}^d$,

$$\begin{aligned} P_1(S \in B) &= E_1[E_1[1_{\{S \in B\}} | (N^\circ, X^\circ)]] \\ &= E_1[\lambda(B \cap C_0^\circ) / \lambda(C_0)] \\ &= E_1^\circ[\lambda(B \cap C_0^\circ)] / E_1^\circ[\lambda(C_0)] \\ &= \int_B P_1^\circ(s \in C_0^\circ) ds / E_1^\circ[\lambda(C_0)]. \end{aligned}$$

(b) With $f \in \mathcal{M} \otimes \mathcal{H}^+$ put

$$g_f(s) = E_1^\circ[f(N^\circ, X^\circ) | s \in C_0^\circ], \quad s \in \mathbb{R}^d.$$

Let $h \in \mathcal{B}^d$ be a non-negative function and apply (a) for the first step in

$$\begin{aligned} E_1[h(S)g_f(S)] &= \int_{\mathbb{R}^d} h(S)E_1^{\circ}[f(N^{\circ}, X^{\circ})|s \in C_0^{\circ}]P_1^{\circ}(s \in C_0^{\circ})/E_1^{\circ}[\lambda(C_0)]ds \\ &= E_1^{\circ} \left[f(N^{\circ}, X^{\circ}) \int_{\mathbb{R}^d} h(s)1_{\{s \in C_0^{\circ}\}} ds \right] / E_1^{\circ}[\lambda(C_0)] \\ &= E_1 \left[f(N^{\circ}, X^{\circ}) \int_{\mathbb{R}^d} h(s)1_{\{s \in C_0^{\circ}\}} ds / \lambda(C_0) \right]. \end{aligned}$$

Apply the conditional uniformity of S to obtain the desired result on the following form:

$$E_1[h(S)g_f(S)] = E_1[h(S)f(N^{\circ}, X^{\circ})], \quad f \in \mathcal{M} \otimes \mathcal{H}^+.$$

(c) This follows from (b) since $\Pi_0 = -S$, and $1_{\{-t \in C_0^{\circ}\}} \rightarrow 1$ pointwise as $|t| \downarrow 0$. \square

9. The second Palm duality

We now extend the less known duality from $d = 1$ to $d > 1$. It is obtained by conditioning on the invariant σ -algebra \mathcal{F} before biasing. Thus, in particular, the two dualities coincide in the ergodic case (when \mathcal{F} is trivial).

9.1. The second Palm duality theorem

Again let (N, X) and $((N^{\circ}, X^{\circ}), S)$ be defined on the same measurable space (Ω, \mathcal{F}) and linked as in Section 4.2, that is,

$$(N^{\circ}, X^{\circ}) \equiv \theta_{\Pi_0}(N, X) \quad \text{and} \quad S \equiv -\Pi_0$$

or equivalently

$$(N, X) \equiv \theta_S(N^{\circ}, X^{\circ}).$$

Recall that C_0 and C_0° , respectively, are the Voronoi cells of (N, X) and (N°, X°) containing the origin and that they have the same Lebesgue measure $\lambda(C_0^{\circ}) \equiv \lambda(C_0)$. Recall from Section 6.1 that (N, X) and (N°, X°) have the same invariant σ -algebra, namely

$$\mathcal{F} = (N, X)^{-1}\mathcal{F} = (N^{\circ}, X^{\circ})^{-1}\mathcal{F},$$

where

$$\mathcal{F} = \{B \in \mathcal{M} \otimes \mathcal{H}^+ : \theta_t B = B, t \in \mathbb{R}^d\}.$$

Let P_2 and P_2° be probability measures on (Ω, \mathcal{F}) satisfying

$$E_2[1/\lambda(C_0)|\mathcal{F}] < \infty \text{ a.s. } P_2$$

and

$$dP_2^0 = \frac{1}{\lambda(C_0)E_2[1/\lambda(C_0)|\mathcal{F}]} dP_2 \quad (\text{volume-debiasing } P_2 \text{ given } \mathcal{F})$$

or, equivalently,

$$E_2^\circ[\lambda(C_0)|\mathcal{F}] < \infty \text{ a.s. } P_2^\circ$$

and

$$dP_2 = \frac{\lambda(C_0)}{E_2^\circ[\lambda(C_0)|\mathcal{F}]} dP_2^\circ \quad (\text{volume-biasing } P_2^\circ \text{ given } \mathcal{F}).$$

Note that

$$P_2 = P_2^\circ \text{ on } \mathcal{F}, \quad P_2((N, X) \in \cdot) = P_2^\circ((N^\circ, X^\circ) \in \cdot) \text{ on } \mathcal{F}, \tag{9.1}$$

$$E_2^\circ[V\lambda(C_0)|\mathcal{F}] = \frac{E_2[V|\mathcal{F}]}{E_2[1/\lambda(C_0)|\mathcal{F}]}, \quad V \in \mathcal{F}^+. \tag{9.2}$$

and that the conditional distribution of S given (N°, X°) is the same under P_2 as under P_2° (since $\lambda(C_0)$ is determined by (N°, X°) and since \mathcal{F} is contained in $\sigma\{(N^\circ, X^\circ)\}$).

Theorem 9.1. *We have that*

(N, X) is stationary under P_2

if and only if

(N°, X°) is point-stationary under P_2°

and the conditional distribution of S given (N°, X°) is uniform on C_0° .

Proof. By Theorem 6.2(b), (N, X) is stationary under P_2 if and only if

$$E_2[f(\theta_t(N, X))|\mathcal{F}] = E_2[f(N, X)|\mathcal{F}], \quad f \in \mathcal{M} \otimes \mathcal{H}^+, t \in \mathbb{R}^d.$$

By (9.2) this is equivalent to

$$E_2^\circ[f(\theta_t(N, X))\lambda(C_0)|\mathcal{F}] = E_2^\circ[f(N, X)\lambda(C_0)|\mathcal{F}], \quad f \in \mathcal{M} \otimes \mathcal{H}^+, t \in \mathbb{R}^d.$$

Combining this, Theorem 6.2(c) and Theorem 4.1 completes the proof. □

9.2. Shift-coupling the stationary and point-stationary duals

Theorem 9.1 yields (in the same way as described at the beginning of Section 8.2) a duality between the *distributions* of stationary (N, X) with finite intensity given \mathcal{F} and point-stationary (N°, X°) with finite-mean Voronoi cell volume given \mathcal{F} .

According to the following theorem the stationary and point-stationary duals in this second duality are really the same, only seen from different sites.

Theorem 9.2. *The probability space $(\Omega, \mathcal{F}, P_2)$ can be extended to support a random point Π of N such that*

$$\theta_{\Pi}(N, X) \text{ under } P_2 \stackrel{D}{=} (N^{\circ}, X^{\circ}) \text{ under } P_2^{\circ},$$

and the probability space $(\Omega, \mathcal{F}, P_2^{\circ})$ can be extended to support a random site T° such that

$$\theta_{T^{\circ}}(N^{\circ}, X^{\circ}) \text{ under } P_2^{\circ} \stackrel{D}{=} (N, X) \text{ under } P_2.$$

Proof. Due to (9.1), this is a consequence of Theorem 2 in Thorisson (1996). □

9.3. Interpretation

The following theorem yields a limit motivation of the randomized-origin interpretations of the second duality. Put $h = \infty$ and $G \equiv 1$ in (a) to obtain the informal expression:

$$P_2^{\circ}(\theta_{\text{uniform site in } \mathbb{R}^d}(N^{\circ}, X^{\circ}) \in \cdot) = P_2((N, X) \in \cdot)$$

and put $h = \infty$ and $G \equiv 1$ in (b), or $n = \infty$ in (c), to obtain the informal expression:

$$P_2(\theta_{\text{uniform point of } N}(N, X) \in \cdot) = P_2^{\circ}((N^{\circ}, X^{\circ}) \in \cdot).$$

(These expressions are informal because there is neither a uniform distribution on \mathbb{R}^d nor a uniform distribution on a countable set of points.)

Theorem 9.3. *Suppose the equivalent conditions in Theorem 9.1 hold. Let $B_h \in \mathcal{B}^d$, $0 < h < \infty$, be a family of sets satisfying $0 < \lambda(B_h) < \infty$ and, for all $t \in \mathbb{R}^d$,*

$$\lambda(B_h \cap (t + B_h)) / \lambda(B_h) \rightarrow 1, \quad h \rightarrow \infty,$$

(which holds, for example, when $B_h = hB_1$). Let $g \in \mathcal{F}$ be strictly positive and finite and put

$$G \equiv g(N^{\circ}, X^{\circ}).$$

Then the following limit results hold:

- (a) *Let U_h be uniformly distributed on B_h and independent of (N°, X°) under P_2° . Then*

$$\theta_{GU_h}(N^{\circ}, X^{\circ}) \text{ under } P_2^{\circ} \rightarrow (N, X) \text{ under } P_2$$

in total variation as $h \rightarrow \infty$.

- (b) *Let each B_h be bounded. Let T_h be a random site such that under P_2 the conditional distribution of T_h given (N, X) is uniform on $s(N) \cap GB_h$ when $s(N) \cap GB_h \neq \emptyset$. Then*

$$\theta_{T_h}(N, X) \text{ under } P_2 \rightarrow (N^{\circ}, X^{\circ}) \text{ under } P_2^{\circ}$$

in total variation as $h \rightarrow \infty$.

- (c) *Let the B_h be convex and compact and increase continuously from $\{0\}$ to \mathbb{R}^d as h increases from 0 to ∞ . Let $\Pi_n, n \geq 0$, be the points of N enumerated in the order*

they are hit by B_h as h increases and lexicographically if two or more are hit simultaneously. Let U be uniform on $[0, 1)$ and independent of (N, X) under P_2 . Then

$$\theta_{\Pi_{[U_n]}}(N, X) \text{ under } P_2 \rightarrow (N^\circ, X^\circ) \text{ under } P_2^\circ$$

in total variation as $n \rightarrow \infty$.

Proof. (a) Due to (9.1), this is a consequence of Theorem 2 in Thorisson (1996).

(b) Let U_h be as in (a). Let Π_h° be a random point of N° such that the conditional distribution of Π_h° given $((N^\circ, X^\circ), U_h)$ is uniform on $s(N^\circ) \cap G(B_h - U_h)$. Apply Theorem 6.1 to obtain that, under P_2° ,

$$(\theta_{\Pi_h} \theta_{-GU_h}(N^\circ, X^\circ), \Pi_h) \stackrel{D}{=} ((N^\circ, X^\circ), GU_h), \tag{9.3}$$

where

$$\Pi_h = \Pi_h^\circ + GU_h.$$

The conditional distribution of Π_h given $((N^\circ, X^\circ), U_h)$ is uniform on $s(\theta_{-GU_h} N^\circ) \cap GB_h$. This set is determined by $\theta_{-GU_h}(N^\circ, X^\circ) -$ since $g \in \mathcal{S}$ implies $G = g(\theta_{-GU_h}(N^\circ, X^\circ)) -$ and thus the conditional distribution of Π_h given $\theta_{-GU_h}(N^\circ, X^\circ)$ is uniform on $s(\theta_{-GU_h} N^\circ) \cap GB_h$. It follows that, a.s. $P_2^\circ(\theta_{-GU_h}(N^\circ, X^\circ) \in \cdot)$, we have

$$P_2(T_h \in \cdot | (N, X) = \cdot) = P_2^\circ(\Pi_h \in \cdot | \theta_{-GU_h}(N^\circ, X^\circ) = \cdot). \tag{9.4}$$

Thus, with $\|\cdot\|$ denoting the total variation norm,

$$\begin{aligned} & \|P_2(\theta_{T_h}(N, X) \in \cdot) - P_2^\circ((N^\circ, X^\circ) \in \cdot)\| \\ &= \|P_2(\theta_{T_h}(N, X) \in \cdot) - P_2^\circ(\theta_{\Pi_h} \theta_{-GU_h}(N^\circ, X^\circ) \in \cdot)\| \quad (\text{due to (9.3)}) \\ &\leq \|P_2((N, X), T_h) \in \cdot) - P_2^\circ((\theta_{-GU_h}(N^\circ, X^\circ), \Pi_h) \in \cdot)\| \\ &= \|P_2((N, X) \in \cdot) - P_2^\circ(\theta_{-GU_h}(N^\circ, X^\circ) \in \cdot)\| \quad (\text{due to (9.4)}). \end{aligned}$$

Applying (a) yields the desired result.

(c) Due to Theorem 6.2(e), $E_2[N([0, 1)^d] | \mathcal{F}] = E_2[1/\lambda(C_0) | \mathcal{F}]$ and by assumption $E_2[1/\lambda(C_0) | \mathcal{F}] < \infty$. Put $G \equiv E_2[N([0, 1)^d] | \mathcal{F}]^{-1/d}$. Then $N(G \cdot)$ is stationary and

$$E_2[N(G[0, 1)^d] | \mathcal{F}] = G^d E_2[N([0, 1)^d] | \mathcal{F}] = 1 \text{ a.s. } P_2.$$

Let $h(n)$ be such that $\lambda(B_{h(n)}) = n$ and apply Theorem 10.2.IV in Daley and Vere-Jones (1988) to $N(G \cdot)$ to obtain

$$N(GB_{h(n)})/n \rightarrow 1 \text{ a.s. } P_2, \quad n \rightarrow \infty. \tag{9.5}$$

Let T_h be as in (b) with $T_h = 0$ when $s(N) \cap GB_h = \emptyset$. Interpret

$$0^{-1} \sum_1^0 1_{\{\theta_{n_{k-1}}(N, X) \in \cdot\}} = 1_{\{(N, X) \in \cdot\}}$$

and note that, for $n, m \geq 1$ and $0 \leq a_1 \leq 1, 0 \leq a_2 \leq 1, \dots,$

$$n^{-1} \sum_1^n a_k - m^{-1} \sum_1^m a_k \leq \begin{cases} n(n^{-1} - m^{-1}) = 1 - n/m, & n \leq m, \\ (n - m)n^{-1} = 1 - m/n, & m \leq n, \end{cases}$$

to obtain (with \wedge denoting minimum and \vee maximum)

$$\begin{aligned} &P_2(\theta_{\Pi_{\lfloor Un \rfloor}}(N, X) \in \cdot) - P_2(\theta_{T_{h(n)}}(N, X) \in \cdot) \\ &= E_2 \left[n^{-1} \sum_1^n 1_{\{\theta_{\Pi_{k-1}}(N, X) \in \cdot\}} - N(GB_{h(n)})^{-1} \sum_1^{N(GB_{h(n)})} 1_{\{\theta_{\Pi_{k-1}}(N, X) \in \cdot\}} \right] \\ &\leq 1 - E_2 \left[\frac{n \wedge N(GB_{h(n)})}{n \vee N(GB_{h(n)})} \right]. \end{aligned}$$

Thus, with $\|\cdot\|$ denoting the total variation norm,

$$\|P_2(\theta_{\Pi_{\lfloor Un \rfloor}}(N, X) \in \cdot) - P_2(\theta_{T_{h(n)}}(N, X) \in \cdot)\| \leq 2 - 2E_2 \left[\frac{n \wedge N(GB_{h(n)})}{n \vee N(GB_{h(n)})} \right].$$

By (9.5) and bounded convergence the expectation goes to 1 as $n \rightarrow \infty$ and thus

$$\|P_2(\theta_{\Pi_{\lfloor Un \rfloor}}(N, X) \in \cdot) - P_2(\theta_{T_{h(n)}}(N, X) \in \cdot)\| \rightarrow 0, \quad n \rightarrow \infty.$$

This, together with (b), yields the final step in

$$\begin{aligned} &\|P_2(\theta_{\Pi_{\lfloor Un \rfloor}}(N, X) \in \cdot) - P_2^0((N^0, X^0) \in \cdot)\| \\ &\leq \|P_2(\theta_{\Pi_{\lfloor Un \rfloor}}(N, X) \in \cdot) - P_2(\theta_{T_{h(n)}}(N, X) \in \cdot)\| \\ &+ \|P_2(\theta_{T_{h(n)}}(N, X) \in \cdot) - P_2^0((N^0, X^0) \in \cdot)\| \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad \square$$

10. Comments and discussion

We conclude with comments on the two Palm dualities, on references, and on a possible extension of the point-stationarity concept to more general random phenomena.

10.1. The exact condition for coincidence of the two Palm dualities

When is standing at the origin of a stationary point pattern and happening to find a point there equivalent to standing at a point selected uniformly at random from the point pattern?

The two dualities coincide when

$$P_1 = P_2 \Leftrightarrow P_1^0 = P_2^0.$$

This holds if and only if

$$E_2[1/\lambda(C_0)|\mathcal{F}] = E_2[1/\lambda(C_0)] \text{ a.s. } P_2 \tag{10.1}$$

and if and only if

$$E_2^\circ[\lambda(C_0)|\mathcal{F}] = E_2^\circ[\lambda(C_0)] \text{ a.s. } P_2^\circ.$$

In particular, this is true in the ergodic case (that is, when \mathcal{F} is trivial, under either measure) which is a common assumption in the literature.

The first Palm version is well known and widely used, but the second is not so well known. From the point of view of applications the second version, however, is often (for instance when averaging over the points) more natural than the first, which suggests a partial explanation of the common ergodicity assumption.

10.2. A random site change hides the gap between the two dualities

We can make the two Palm dualities coincide by a simple random change of site scale (provided X is measurable under change of site scale, as when it has a Polish state space and right-continuous paths with left-hand limits). Let (N, X) be stationary under a probability measure P with $E[1/\lambda(C_0)] < \infty$ and change the site scale by

$$G \equiv E[1/\lambda(C_0)|\mathcal{F}]^{-1/d}$$

to obtain

$$(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d}).$$

This new pair is stationary and the Voronoi cell containing the origin is C_0/G . Noting that $\lambda(C_0/G) = \lambda(C_0)/G^d$ and $G \in \mathcal{F}^+$ yields

$$E[1/\lambda(C_0/G)|\mathcal{F}] = E[1/\lambda(C_0)|\mathcal{F}]G^d = 1.$$

The invariant σ -algebra of $(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d})$ is contained in \mathcal{F} and thus

$$E[1/\lambda(C_0/G)|(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d})^{-1}\mathcal{F}] = 1.$$

Since also $E[1/\lambda(C_0/G)] = 1$, the coincidence condition holds (see (10.1)). Thus the two point-stationary Palm duals of $(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d})$ coincide, that is, $(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d})$ has only one point-stationary Palm dual.

Note that $\lambda(C_0/G) = \lambda(C_0)E[1/\lambda(C_0)|\mathcal{F}]$ and thus the change of measure used to obtain this common point-stationary dual of $(N(G\cdot), (X_{G_s})_{s \in \mathbb{R}^d})$ is the same as the change of measure used to obtain the point-stationary randomized-origin dual of (N, X) . Thus this procedure *preserves* the randomized-origin duality and *not* the point-at-zero duality.

In fact, we *lose* the point-at-zero duality by this procedure: the point-at-zero duality merges with the randomized-origin duality through the change of site scale and does not reappear when we return to the original time scale after change of measure (as the randomized-origin duality does). Thus the change of site scale is not a way to bridge the gap between the two dualities, it only *hides* it. To bridge the gap we cannot avoid a change of measure.

10.3. On references

For $d = 1$, see Thorisson (1995) for references on the first Palm duality (the conventional point-at-zero duality). The author is aware of the following references on the second (the randomized-origin duality): Nawrotzki (1978) and the recent papers by Glynn and Sigman (1992), Nieuwenhuis (1994; 1998), Thorisson (1995), and Miyazawa *et al.* (1999). The second duality is highlighted in Sigman (1994).

For $d > 1$, see Matthes *et al.* (1978), Daley and Vere-Jones (1988), Stoyan *et al.* (1987) and Cowan (1978) for references on the conventional (point-at-zero) Palm version of stationary point-processes. The author is not aware of references on the other (the randomized-origin) Palm version.

The only references (beyond the final comment in his own 1995 paper) that the author has seen where the point-stationarity problem is mentioned are Mandelbrot (1983) and Kagan and Vere-Jones (1988), which we discuss below.

10.4. Extending point-stationarity to more general random sets?

When working on the revision of this paper the author had the good fortune to meet David Vere-Jones, who informed him about Mandelbrot (1983) and Kagan and Vere-Jones (1988). The following is taken from Kagan and Vere-Jones (1988):

Kagan refers at several points to the suggestion of Mandelbrot (1983) that point process models for self-similar behaviour should be sought not within the class of homogeneous processes [called *stationary* in this paper] but within the class of processes for which the behaviour relative to a given point of the process is independent of the point selected as origin.

The authors then speculate:

One interpretation of this requirement is that the Palm distributions of the process should be the same for all points of the process.

They are on the right track because if we turn this on its head to read ‘One interpretation of this requirement is that the *reversed* Palm distribution of the process should be stationary’, then it becomes similar to defining point-stationarity by its Palm characterization (Theorem 4.1 above). Observe that (due to Theorem 4.1) the reversed Palm distribution can be generalized beyond a probability distribution and thus finite-mean Voronoi cell volume is not needed for this alternative definition.

Further, Kagan and Vere-Jones say:

What hampers us in pursuing this discussion, is that we are not aware of a well established theory for such Palm-stationary point processes. . . . Any references which deal with these and related questions would be appreciated.

Further examples of Palm-stationary processes suggested by Mandelbrot, such as the Lévy dust model and zeros of Brownian motion, have a very complex point set structure, including finite accumulation points, and cannot be modelled within the standard point process framework.

The solution in the present paper of the point-stationarity problem in the case of point

processes in $d > 1$ dimensions (Definition 3.1) suggests that one might attempt to define point-stationarity in these models as follows.

Proposed definition. A random set is *point-stationary* if it is distributionally invariant under bijective point-shifts against any independent stationary background.

Thus our next task is to find such backgrounds and point-shifts!

NOTE ADDED IN PROOF

Conjecture 2.1 has been proved wrong by Olle Haggstrom (personal communication). It is, however, not clear at this point whether the class of bijective \mathbb{N}^0 -point-shifts is rich enough to define point-stationarity.

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