Bernoulli **12**(1), 2006, 101–111

The reversible nearest particle system on a finite interval

DAYUE CHEN[1*], JUXIN LIU[2] and FUXI ZHANG[1**]

[1]*LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China. E-mail: *dayue@math.pku.edu.cn; **zhangfxi@math.pku.edu.cn*
[2]*Department of Statistics, University of British Columbia, Vancouver BC, Canada V6T 1Z2. E-mail: jliu@stat.ubc.ca*

We study a one-parameter family of attractive reversible nearest particle systems on a finite interval. As the length of the interval increases, the time at which the nearest particle system first hits the empty set increases from logarithmic to exponential depending on the intensity of interaction. In the critical case, the first hitting time is polynomial in the interval length.

Keywords: first hitting time; nearest particle system

1. Introduction

A nearest particle system on $S = \{1, 2, \ldots, N\}$ is a continuous-time Markov chain with the state space $\{A : A \subset S\}$. The jump rates are specified as follows:

$$\begin{aligned} &q(A, A\backslash\{x\}) = 1, && \text{if } x \in A; \\ &q(A, A \cup \{x\}) = \beta(l_x(A), r_x(A)), && \text{if } x \in S\backslash A; \\ &q(A, B) = 0 && \text{otherwise.} \end{aligned}$$

Here $l_x(A)$ and $r_x(A)$ are the distances from x to the nearest points in A to the left and right respectively, with the convention that $l_x(A)$ $(r_x(A))$ is ∞ if $y > x$ $(y < x)$ for all $y \in A$. We assume that:

1. $\beta(l, r) = \beta(r, l)$;
2. $\beta(l, r)$ is decreasing in l and in r;
3. $\beta(\infty, \infty) = 0$, $\beta(l, \infty) > 0$;
4. $\sum_l \beta(l, \infty) < \infty$.

There are many choices of $\beta(\cdot, \cdot)$ satisfying the above assumptions.

Example 1 *The one-dimensional contact process*. $\beta(1, 1) = 2\lambda$, $\beta(1, r) = \beta(l, 1) = \lambda$ for $l, r > 1$, and $\beta(l, r) = 0$ otherwise.

Example 2 *The uniform birth rate*. $\beta(l, r) = \lambda/(l + r - 1)$.

Example 3 *The reversible case*.

$$\beta(l, r) = \lambda \frac{\psi(l)\psi(r)}{\psi(l+r)}, \qquad \beta(l, \infty) = \beta(\infty, l) = \lambda\psi(l),$$

where

$$\psi(\cdot) > 0, \qquad \sum_{n=1}^{\infty} \psi(n) = 1, \qquad \frac{\psi(n)}{\psi(n+1)} \searrow 1.$$

Assume further that

$$\sum_n n^2 \psi(n) < \infty. \tag{1}$$

For example, $\psi(n) = cn^{-\alpha}$ for some $\alpha > 3$ satisfies the above requirement.

It is helpful to associate a subset A of S with an element ξ of $\{0, 1\}^S$ and use them interchangeably: $\xi(x) = 1$ if and only if $x \in A$. Each configuration ξ is given an occupancy interpretation. We say there is a particle at x if $\xi(x) = 1$, and we say the site is vacant if $\xi(x) = 0$. Then the above transition mechanism is interpreted as follows: each particle disappears at rate 1 independently, and a particle is born at vacant site x at rate $\beta(l_x(A), r_x(A))$.

The transition mechanisms also make sense if we replace $S = \{1, 2, \ldots, N\}$ with the integer lattice $\mathbb{Z}$. The state space $\{0, 1\}^{\mathbb{Z}}$ consists of four disjoint subspaces:

(i) all finite subsets of $\mathbb{Z}$;
(ii) all subsets of $\mathbb{Z}$ with infinitely many particles both to the left and to the right of the origin;
(iii) all infinite subsets of $\mathbb{Z}$ with finitely many particles to the right of the origin; and
(iv) all infinite subsets of $\mathbb{Z}$ with finitely many particles to the left of the origin.

Because of assumption 4, a nearest particle system will remain for ever in one of the four subspaces. The processes taking values in subspaces (i) or (ii) are called *finite* and *infinite* nearest particle systems, respectively. A comprehensive account can be found in Chapter 7 of Liggett (1985). The processes taking values in subspaces (iii) or (iv) share many properties of *finite* and *infinite* nearest particle systems, and are indispensable on some occasions (see, for example, Lemma 4.1 below).

For interacting particle systems one is most concerned with the existence of phase transition and the critical value. For the infinite nearest particle system with uniform birth rate (Example 2), the critical value is 1 (see Mountford 1992). For the reversible nearest particle system (Example 3), the critical value is also 1. For the contact process (Example 1), the critical value is unknown but is between 1.5 and 2, and is denoted as λ_c throughout this paper.

Can the critical value of an infinite model be detected by its counterpart on a finite interval? This interplay was first explored for the contact process in a series of papers by Durrett and co-workers (Durrett and Liu 1988; Durrett and Schonmann 1988; Durrett *et al.* 1989). The main results are summarized as follows. Let $\{\zeta_t^N : t \geqslant 0\}$ be the contact process on $\{1, 2, \ldots, N\}$ with the parameter λ starting from all sites occupied, and τ_N be the first time it hits the empty set.

Theorem 1.1. (i) *If* $\lambda < \lambda_c$, *then there is a constant* $\gamma_1(\lambda) > 0$ *such that as* $N \to \infty$, $\tau_N/\log N \to 1/\gamma_1(\lambda)$ *in probability (Durrett and Liu 1988, Theorem 1).*

(ii) *If* $\lambda > \lambda_c$, *then there is a constant* $\gamma_2(\lambda) > 0$ *such that as* $N \to \infty$, $(\log \tau_N)/N \to \gamma_2(\lambda)$ *in probability (Durrett and Schonmann 1988, Theorem 2).*

(iii) *If* $\lambda = \lambda_c$ *and* a, $b \in (0, \infty)$, *then* $P(aN \leqslant \tau_N \leqslant bN^4) \to 1$ *as* $N \to \infty$ *(Durrett et al. 1989, Theorem 1.6).*

We believe that these statements hold for a large class of interacting particle systems. In this paper we study the asymptotic behaviour of the hitting time σ_N of the reversible nearest particle systems (Example 3) on a finite interval, as the length of the interval increases. The results read as follows.

Theorem 1.2. *Suppose the initial state is* $\{1, 2, \ldots, N\}$.

(i) *If*

$$\lambda < \min\left\{1, \min_n \frac{\psi(n)}{\sum_{l+r=n} \psi(l)\psi(r)}\right\},$$

then $\mathrm{E}\sigma_N \leqslant C \log N$ *for some constant C which is independent of N, and*

$$\lim_{N\to\infty} P(\sigma_N \leqslant C_N \log N) = 1,$$

where $\{C_N : N \geqslant 1\}$ *is any sequence of increasing numbers such that* $\lim_{N\to\infty} C_N = \infty$.

(ii) *If* $\lambda > 1$, *then there is a constant* $\gamma > 0$ *such that* $\lim_{N\to\infty} P(\sigma_N \geqslant \mathrm{e}^{\gamma N}) = 1$.

Remark. It is not difficult to establish estimates in the opposite direction (see Theorems 2.3 and 2.4). We have thus shown that σ_N increases logarithmically if λ is small enough and exponentially if $\lambda > 1$.

For any non-empty set $A = \{x_1, x_2, \ldots, x_k\}$, we assume without loss of generality that $x_1 < x_2 < \ldots < x_k$ and define

$$\nu_\psi(A) = \begin{cases} \psi(x_2 - x_1)\psi(x_3 - x_2) \cdots \psi(x_k - x_{k-1}), & \text{if } k > 1, \\ 1 & \text{if } k = 1. \end{cases}$$

Let $\mathcal{S}_N = \{0, 1\}^{\{1,\ldots,N\}}$, $K_N = \sum_{A \in \mathcal{S}_N \setminus \{\varnothing\}} \nu_\psi(A)$, and $\pi(A) = \nu_\psi(A)/K_N$. Then π is a probability measure on $\mathcal{S}_N$, with $\pi(\varnothing) = 0$.

Theorem 1.3. *Suppose that* $\lambda = 1$ *and the initial distribution is* π. *Let* $\{C_N : N \geqslant 1\}$ *be any sequence of increasing numbers such that* $\lim_{N\to\infty} C_N = \infty$. *Then*

$$\lim_{N\to+\infty} P\left(\frac{N}{C_N} \leqslant \sigma_N \leqslant C_N N^2\right) = 1.$$

We now proceed to prove Theorems 1.2 and 1.3 by three different approaches.

2. Comparison by coupling

We will prove the first part of Theorem 1.2 by establishing a more general conclusion (Theorem 2.2). Let $\{X_t : t \geqslant 0\}$ be a birth and death process on $\{0, 1, \ldots, N\}$ with death rate

$$a_i = i, \qquad \text{for } i = 1, \ldots, N,$$

and birth rate

$$b_i = (i+1)\alpha, \qquad \text{for } i = 0, \ldots, N-1.$$

Let $\tau = \inf\{t > 0 : X_t = 0\}$ be the first time that $\{X_t : t \geqslant 0\}$ hits 0. Let E^N be the conditional expectation on $X_0 = N$.

Lemma 2.1. *Suppose that $X_0 = N$. For large N,*

$$\mathrm{E}^N\tau \leqslant \begin{cases} (2\log N)/(1-\alpha), & \text{if } \alpha < 1, \\ 2N\log N, & \text{if } \alpha = 1, \\ \alpha^N\alpha/(\alpha-1)^2, & \text{if } \alpha > 1. \end{cases}$$

Furthermore,

$$\mathrm{E}^N\tau^2 \leqslant 2(\mathrm{E}^N\tau)^2. \tag{2}$$

Proof. Let P^i be the conditional probability distribution on the initial state i, E^i be the expectation with respect to P^i, and $m_i = \mathrm{E}^i\tau$ for $i = 0, \ldots, N$. It is shown in Wang (1980) that

$$\mathrm{E}^N\tau = \sum_{i=1}^N e_i, \qquad \mathrm{E}^N\tau^2 = \sum_{i=1}^N \varepsilon_i,$$

where

$$e_i = \frac{1}{a_i} + \sum_{k=0}^{N-1-i} \frac{b_ib_{i+1}\cdots b_{i+k}}{a_ia_{i+1}\cdots a_{i+k}a_{i+k+1}} = \frac{1}{i}(1+\alpha+\alpha^2+\cdots\alpha^{N-i}), \tag{3}$$

$$\varepsilon_i = \frac{2m_i}{a_i} + \sum_{k=0}^{N-1-i} \frac{2b_ib_{i+1}\cdots b_{i+k}m_{i+k+1}}{a_ia_{i+1}\cdots a_{i+k}a_{i+k+1}}.$$

Notice that $m_i \leqslant m_N$ for any $i \leqslant N$. It follows that $\varepsilon_i \leqslant 2m_Ne_i$. Therefore,

$$\mathrm{E}^N\tau^2 = \sum_{i=1}^{N} \varepsilon_i \leqslant 2m_N \sum_{i=1}^{N} e_i \leqslant 2m_N \mathrm{E}^N\tau = 2\big(\mathrm{E}^N\tau\big)^2.$$

If $\alpha < 1$, then

$$\mathrm{E}^N\tau = \sum_{i=1}^{N} e_i = \sum_{i=1}^{N} \frac{1}{i}\,\frac{1-\alpha^{M+1-i}}{1-\alpha} \leqslant \sum_{i=1}^{N} i^{-1}/(1-\alpha) \leqslant \frac{2\log N}{1-\alpha}.$$

If $\alpha = 1$, by (3), $e_i = (N - i + 1)/i$, and for large N,

$$\mathrm{E}^N\tau = \sum_{i=1}^{N} e_i \leqslant N \sum_{i=1}^{N} i^{-1} \leqslant 2N\log N.$$

If $\alpha > 1$, then $\mathrm{E}^N\tau = \sum_{i=1}^{N} e_i \leqslant (\alpha-1)^{-1}\sum_{i=1}^{N}\alpha^{N-i+1} \leqslant \alpha^{N+1}/(\alpha-1)^2$. □

Consider a nearest particle system $\{\xi_t^N : t \geqslant 0\}$ on $\{1, 2, \ldots, N\}$ starting from $\{1, 2, \ldots, N\}$ (not necessarily reversible). Let σ_N be the first time ξ_t^N hits the empty set, and

$$M = \max\left\{\max_n \sum_{l+r=n} \beta(l, r), \sum_{l} \beta(l, \infty)\right\}.$$

Theorem 2.2. *Suppose the initial state is* $\{1, 2, \ldots, N\}$*. If* $M < 1$*, then* $\mathrm{E}\sigma_N \leqslant (2\log N)/(1-M)$*; and for any sequence* $\{C_N : N \geqslant 1\}$ *of increasing numbers such that* $\lim_{N\to\infty} C_N = \infty$*,* $\lim_{N\to\infty} P(\sigma_N \leqslant C_N \log N) = 1$.

Proof. Let $|\xi|$ be the cardinality of the set $\{x : \xi(x) = 1, 1 \leqslant x \leqslant N\}$. For any configuration ξ such that $|\xi| = i$, there are at most $i + 1$ intervals of consecutive vacant sites, separated by occupied sites; and the rate at which a new particle in each interval is born is no more than M. Hence the rate at which $|\xi_t^N|$ increases by 1 is no more than $(i + 1)M$. On the other hand, when $|\xi_t| = i$, the rate at which $|\xi_t|$ decreases by 1 is equal to i, the total number of particles. Compare $|\xi_t|$ with a birth and death process X_t with parameter $\alpha = M$. Since initially $X_0 = |\xi^N|$, there is a coupling of $\{X_t : t \geqslant 0\}$ and $\{\xi_t^N : t \geqslant 0\}$ such that

$$P^{N,\xi^N}(X_t \geqslant |\xi_t^N|, \forall\ t \geqslant 0) = 1, \tag{4}$$

where P^{N,ξ^N} is the coupling measure with initial state (N, ξ^N). By (4), σ_N is stochastically dominated by τ, that is, for any $t \geqslant 0$,

$$P(\sigma_N > t) \leqslant P^N(\tau \geqslant t).$$

Therefore,

$$\mathrm{E}\sigma_N \leqslant \mathrm{E}^N\tau \leqslant \frac{2\log N}{1-M}$$

by Lemma 2.1. By the Chebyshev inequality and (2), for any $c_N > 0$,

$$P(\sigma_N > c_N \mathrm{E}^N\tau) \leqslant P^N(\tau \geqslant c_N \mathrm{E}^N\tau) \leqslant \frac{\mathrm{E}^N\tau^2}{(c_N\mathrm{E}^N\tau)^2} \leqslant \frac{2}{c_N^2}. \tag{5}$$

For any sequence $C_N \to \infty$ as $N \to \infty$, choose $c_N = C_N(1-M)/2$. Then an upper estimate of σ_N may be taken as $c_N\mathrm{E}^N\tau$, and the claims in Theorem 2.2 hold by (5) and Lemma 2.1. □

By the same argument it is not difficult to establish the following estimates, though a renormalization argument is used in the proof of the second part of Theorem 2.4. We skip the proof since it is not needed in proving Theorems 1.2 and 1.3.

Theorem 2.3. *Suppose the initial state is* $\{1, 2, \dots, N\}$.

(i) *If* $M = 1$, *then* $\mathrm{E}\sigma_N \leqslant 2N\log N$; *and* $\lim_{N\to\infty} P(\sigma_N \leqslant C_N N\log N) = 1$.
(ii) *If* $M > 1$, *then* $\mathrm{E}\sigma_N \leqslant M^{N+1}/(M-1)^2$; *and there is a constant* $\gamma_1 > 0$ *such that*

$$\lim_{N\to\infty} P(\sigma_N \leqslant \mathrm{e}^{\gamma_1 N}) = 1.$$

Theorem 2.4. *Suppose the initial state is* $\{1, 2, \dots, N\}$.

(i) *For any* $\varepsilon > 0$, $\lim_{N\to\infty} P(\sigma_N > (1-\varepsilon)\log N) = 1$.
(ii) *If* $\max_n \min\{\frac{1}{2}\sum_{l=n}^{2n}\beta(l, 3n-l), \sum_{l=n}^{2n}\beta(l,\infty)\}$ *is larger than the critical value of the contact process on* $\mathbb{Z}$, *then there is a constant* $\gamma > 0$ *such that*

$$\lim_{N\to\infty} P(\sigma_N \geqslant \mathrm{e}^{\gamma N}) = 1.$$

3. A lower estimate of σ_N

We first extend the notation introduced before Theorem 1.3. For any non-empty set $A = \{x_1, x_2, \dots, x_k\}$, $x_1 < x_2 < \dots < x_k$, define

$$\nu_{\psi,\lambda}(A) = \begin{cases} \lambda^{k-1}\psi(x_2 - x_1)\psi(x_3 - x_2)\cdots\psi(x_k - x_{k-1}), & \text{if } k > 1, \\ 1 & \text{if } k = 1. \end{cases}$$

Let $\mathcal{S}_N = \{0,1\}^{\{1,\dots,N\}}$, $K_N(\lambda) = \sum_{A\in\mathcal{S}_N\setminus\{\varnothing\}}\nu_{\psi,\lambda}(A)$, and $\pi(A) = \nu_{\psi,\lambda}(A)/K_N(\lambda)$. Then π is a probability measure on $\mathcal{S}_N$, with $\pi(\varnothing) = 0$.

Lemma 3.1. $K_N(\lambda) \geqslant CN^2\mathrm{e}^{\gamma(\lambda)N}$ *for* $\lambda \geqslant 1$, *where* $\gamma(1) = 0$ *and* $\gamma(\lambda) > 0$ *if* $\lambda > 1$.

Proof. For $\xi \in \mathcal{S}_N$, recall that $|\xi|$ is the cardinality of $\{x : \xi(x) = 1, 1 \leqslant x \leqslant N\}$. We have

$$K_N(\lambda) = \sum_{\xi\in\mathcal{S}_N\setminus\{\varnothing\}} \nu_{\psi,\lambda}(\xi) \geqslant \sum_{x=0}^{[N/3]}\sum_{y=[2N/3]}^{N}\sum_{\xi\in\mathcal{S}_N(x,y)} \lambda^{|\xi|-1}\nu_\psi(\xi), \tag{6}$$

where

$$S_N(x, y) = \{\xi \in \mathcal{S}_N : \xi(x) = \xi(y) = 1, \xi(z) = 0, \forall\ 1 \leqslant z < x, \text{ or } y < z \leqslant N\}.$$

In light of (1), by the renewal theorem, $\nu_\psi(S_N(x, y)) \geqslant 1/(2\sum_n n\psi(n))$ whenever $y - x$ is large enough. If $\lambda = 1$, then $K_N \geqslant CN^2$ when N is large, and we are done. If $\lambda > 1$, we can choose a constant $\delta > 0$ such that

$$\nu_\psi(\{\xi \in S_N(x, y); |\xi| \geqslant \delta|y - x|\}) \geqslant \frac{\nu_\psi(S_N(x, y))}{2}.$$

This, together with (6), implies the desired conclusion. In particular, we may choose $\gamma(\lambda) = (\delta/3)\log\lambda$. $\square$

We now borrow an idea used in proving Theorem 7.1.20 of Liggett (1985) to prove

$$\lim_{N\to\infty} P^\pi\left(\sigma_N \geqslant \frac{K_N(\lambda)}{C_N N}\right) = 1. \tag{7}$$

The first half of Theorem 1.3 readily follows from (7) and Lemma 3.1. Notice that the hitting time of the nearest particle system starting from $\{1, 2, \ldots, N\}$ is stochastically larger than that starting from the initial distribution π. Therefore the second part of Theorem 1.2 also follows, with a small change to γ.

Proof of (7)**.** The reversible nearest particle system $\{\xi_t^N : t \geqslant 0\}$ is a Markov process taking values in $\mathcal{S}_N$ with jump rate

$$q(A, B) = \begin{cases} 1, & \text{if } x \in A,\ B = A\backslash\{x\}, \\ \lambda\dfrac{\psi(l_x(A))\psi(r_x(A))}{\psi(l_x(A) + r_x(A))}, & \text{if } x \notin A,\ B = A \cup \{x\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is reversible with respect to π in the sense that $\pi(A)q(A, B) = \pi(B)q(B, A)$ for $A, B \in \mathcal{S}_N\backslash\{\varnothing\}$.

Let $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ be a Markov process on $\mathcal{S}_N$, which is a modification of $\{\xi_t^N : t \geqslant 0\}$ so that particles can be born from the empty set. More specifically, the transition rate of $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ is defined as follows:

$$\tilde{q}(A, B) = \begin{cases} q(A, B), & \text{if } A \neq \varnothing, \\ q, & \text{if } A = \varnothing \text{ and } |B| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $q > 0$ is a constant to be determined later. Let K_N stand for $K_N(\lambda)$ for simplicity,

$$\nu_\psi(\{\varnothing\}) = q^{-1} \quad \text{and} \quad \tilde{\pi} = \frac{\nu_\psi}{K_N + q^{-1}}.$$

Then $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ is reversible with respect to $\tilde{\pi}$ in the sense that $\tilde{\pi}(A)q(A, B) = \tilde{\pi}(B)q(B, A)$ for any $A, B \in \mathcal{S}_N$.

Let $\tilde{P}$ be the distribution of $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ with initial distribution $\tilde{\pi}$, and $\tilde{\mathrm{E}}$ be the expectation with respect to $\tilde{P}$. Notice that $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ is stationary under $\tilde{P}$. For any $t > 0$,

$$2t\tilde{\pi}(\{\varnothing\}) = \tilde{\mathrm{E}}\int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s.$$

Introduce the stopping time $\tau = \inf\{t \geqslant 0 : \widetilde{\xi_t^N} = \varnothing\}$. By the strong Markovian property,

$$\begin{aligned}\tilde{\mathrm{E}}\int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s &= \tilde{\mathrm{E}}\tilde{\mathrm{E}}\left(\int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s|\mathcal{F}_\tau\right) \geqslant \tilde{\mathrm{E}}\tilde{\mathrm{E}}\left(1_{\{\tau<t\}}\int_0^{2t} 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s|\mathcal{F}_\tau\right)\\ &\geqslant \tilde{\mathrm{E}}\tilde{\mathrm{E}}\left(1_{\{\tau<t\}}\int_\tau^{\tau+t} 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s|\mathcal{F}_\tau\right) = \tilde{P}(\tau<t)\tilde{\mathrm{E}}\left(\int_0^t 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s|\widetilde{\xi_0^N}=\varnothing\right)\end{aligned}$$

Denote by σ the first time $\{\widetilde{\xi_t^N} : t \geqslant 0\}$ jumps. Then

$$\tilde{\mathrm{E}}\left(\int_0^t 1_{\{\widetilde{\xi_s^N}=\varnothing\}}\,\mathrm{d}s|\widetilde{\xi_0^N}=\varnothing\right) \geqslant \tilde{\mathrm{E}}\left(\sigma 1_{\{\sigma\leqslant t\}}|\widetilde{\xi_0^N}=\varnothing\right) = \int_0^t s\tilde{q}_\varnothing \mathrm{e}^{-\tilde{q}_\varnothing s}\,\mathrm{d}s,$$

where $\tilde{q}_\varnothing = \sum_\xi \tilde{q}(\varnothing, \xi) = Nq$. Hence

$$\tilde{P}(\tau<t) \leqslant \frac{2t\tilde{\pi}(\{\varnothing\})}{\displaystyle\int_0^t s\tilde{q}_\varnothing \mathrm{e}^{-\tilde{q}_\varnothing s}\,\mathrm{d}s} = \frac{2tq^{-1}}{K_N+q^{-1}}\cdot\frac{Nq}{1-\mathrm{e}^{-Nqt}-Nqt\mathrm{e}^{-Nqt}}. \tag{8}$$

On the other hand,

$$\begin{aligned}\tilde{P}(\tau<t) &\geqslant \tilde{P}\left(\tau<t, \widetilde{\xi_0^N}\neq\varnothing\right) = \tilde{P}\left(\widetilde{\xi_0^N}\neq\varnothing\right)\tilde{P}\left(\tau<t|\widetilde{\xi_0^N}\neq\varnothing\right)\\ &= \frac{K_N}{K_N+q^{-1}}P(\sigma_N<t).\end{aligned}$$

This, together with (8), yields that

$$P(\sigma_N<t) \leqslant \frac{2tN}{K_N(1-\mathrm{e}^{-Nqt}-Nqt\mathrm{e}^{-Nqt})}.$$

Letting $q \to \infty$, then

$$P(\sigma_N<t) \leqslant \frac{2tN}{K_N}.$$

This implies (7), by choosing $t = K_N/(C_N N) = K_N(\lambda)/(C_N N)$. □

4. The critical case

In this section we will prove the second half of Theorem 1.3, that is, when $\lambda = 1$,

$$\lim_{N\to\infty} P^{\pi}\big(\sigma_N \leqslant C_N N^2\big) = 1. \tag{9}$$

Let $\{\eta_t : t \geqslant 0\}$ be an infinite reversible nearest particle system on $\mathbb{Z}$ with finitely many particles to the right of the origin (subspace (iii) of Section 1); and r_t the rightmost particle in $\{\eta_t : t \geqslant 0\}$, that is, $r_t := \sup\{x : \eta_t(x) = 1\}$. The properties of r_t of the critical nearest particle system are studied in Schinazi (1992). For a recent survey, see Mountford (2003).

Lemma 4.1. (Schinazi 1992, Theorem 1). *Let $\{\eta_t : t \geqslant 0\}$ be the critical reversible nearest particle system on $\mathbb{Z}$. Suppose the initial configurations have a particle at the origin and no particle to the right of the origin, and the distribution of particles to the left of the origin follows the renewal measure Ren(ψ) with density $\psi(\cdot)$. Then, as $a \to \infty$, $r_{a^2 t}/a$ converges in distribution to a Brownian motion with diffusion constant $D > 0$ in the Skorohod space.*

Proof of (9)**.** Partition the configurations of $\mathcal{S}_N$ by the position of the rightmost particle. Namely, let

$$A_x = \{\xi \in \mathcal{S}_N : \xi(x) = 1, \text{ and } \xi(y) = 0 \text{ for any } y > x\}$$

be the set of configurations whose rightmost particle is at x. Denote by P the distribution of $\{\xi_t^N : t \geqslant 0\}$ with initial distribution π (introduced before Theorem 1.3), and by $P_{N,x}$ the conditional distribution of the nearest particle system on $\{1, 2, \ldots, N\}$ whose initial configurations are in A_x. Then

$$P = \sum_{x=0}^{N} P(A_x) P_{N,x}. \tag{10}$$

Denote by $\mathbf{P}$ the distribution of the nearest particle system on $\mathbb{Z}$ with the initial distribution in Lemma 4.1, and $\mathbf{P}_x$ the translation of $\mathbf{P}$ by x. Thanks to the attractive property, there is a coupling of $\mathbf{P}_x$ and $P_{N,x}$ such that for all $t > 0$ and all $i \in \mathbb{Z}$,

$$\xi_t^N(i) \leqslant \eta_t(i).$$

Then under this coupling, $\xi_t^N \equiv \varnothing$ once $r_t < 1$, hence $\sigma_N \leqslant \inf\{t : r_t < 1\}$.

Suppose that $\lim_{N\to\infty} C_N = \infty$. For any $C > 0$ and large N,

$$\begin{aligned}
P_{N,x}(\sigma_N \leqslant C_N N^2) &\geqslant P_{N,x}\big(\sigma_N \leqslant C(x-1)^2\big) \\
&\geqslant \mathbf{P}_x\big(\exists\ t \leqslant C(x-1)^2 \text{ such that } r_t < 1\big) \\
&= \mathbf{P}\big(\exists\ t \leqslant C(x-1)^2 \text{ such that } r_t < -(x-1)\big) \\
&= \mathbf{P}\big(\exists\ t \leqslant C \text{ such that } r_{(x-1)^2 t}/(x-1) < -1\big).
\end{aligned}$$

Here the first equality holds because $\mathbf{P}_x$ is the translation of $\mathbf{P}$ by x. This, together with Lemma 4.1, implies that

$$\liminf_{N,x\to+\infty} P_{N,x}\big(\sigma_N \leqslant C_N N^2\big) \geqslant \mathbf{P}(\exists\ t \leqslant C \text{ such that } B_t < -1), \qquad \forall\ C > 0,$$

where $\{B_t : t \geqslant 0\}$ is a Brownian motion with diffusion constant $D > 0$. Letting $C \to +\infty$, the right-hand side of the above equation converges to 1. Hence

$$\lim_{N,x \to +\infty} P_{N,x}\big(\sigma_N \leqslant C_N N^2\big) = 1.$$

Consequently, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that for any $N \geqslant x \geqslant N_0$,

$$P_{N,x}\big(\sigma_N \leqslant C_N N^2\big) > 1 - \varepsilon.$$

This, together with (10), implies that

$$P\big(\sigma_N \leqslant C_N N^2\big) = \sum_{x=1}^{N} P(A_x) P_{N,x}\big(\sigma_N \leqslant C_N N^2\big) \geqslant (1-\varepsilon) \sum_{x=N_0}^{N} P(A_x). \tag{11}$$

On the other hand,

$$\sum_{x=1}^{N_0-1} \nu_\psi(A_x) \leqslant \sum_{x=1}^{N_0-1} \sum_{y=1}^{x} \nu_\psi(S_N(y, x)) \leqslant N_0^2.$$

Therefore, as $N \to \infty$,

$$\sum_{x=N_0}^{N} P(A_x) \geqslant 1 - N_0^2/(CN^2) \to 1.$$

This, together with (11), implies that $\liminf_{N\to\infty} P\big(\sigma_N \leqslant C_N N^2\big) \geqslant 1 - \varepsilon$. Let $\varepsilon \to 0$ and the result follows. $\square$

Acknowledgement

Supported in part by grant G1999075106 from the Ministry of Science and Technology of China.

References

Durrett, R. and Liu, X.F. (1988) The contact process on a finite set. *Ann. Probab.*, **16**, 1158–1173.

Durrett, R. and Schonmann, R.H. (1988) The contact process on a finite set II. *Ann. Probab.*, **16**, 1570–1583.

Durrett, R., Schonmann, R.H. and Tanaka, N.I. (1989) The contact process on a finite set III: The critical case. *Ann. Probab.*, **17**, 1303–1321.

Liggett, T.M. (1985) *Interacting Particle Systems.* New York: Springer-Verlag.

Mountford, T.S. (1992) A critical value for the uniform nearest particle system. *Ann. Probab.*, **20**, 2031–2042.

Mountford, T.S. (2003) Critical reversible attractive nearest particle systems. In E. Merzbach (ed.), *Topics in Spatial Stochastic Processes*, Lecture Notes in Math. 1802. Berlin: Springer-Verlag.

Schinazi, R. (1992) Brownian fluctuations of the edge for critical reversible nearest particle systems. *Ann. Probab.*, **20**, 194–205.

Wang Z.K. (1980) *Birth and Death Processes and Markov Chains* (in Chinese). Beijing: Science Publishing House.

Received October 2004 and revised May 2005