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The shape of a sequence of dual random triangles

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Using the concept of the convex hull of a set of lines, a dual random triangle is defined by selecting three lines from a parent triangle of lines. The angles of the constructed triangle define the shape; calculation of the shape distribution is described. For a sequence of nested triangles constructed in this way it is shown that there is convergence to collinearity and to the collinear shape distribution derived by Mannion for a sequence of vertex-generated triangles.

Keywords: convergence of shape; convexity; dual triangles; shape distribution

1. Introduction

Mannion (1990a; 1990b) examined the stochastic properties of a nested sequence of triangles in $\mathbb{R}^2$. In that sequence each daughter triangle is constructed as the convex hull of three points chosen at random within the previous parent triangle. Mannion showed that, ultimately, the triangles tend to degenerate collinear triangles and he determined the degenerate shape distribution, where the shape of a collinear triangle just corresponds to the ratio in which the longest side is divided by the opposite vertex. In this paper we study the dual process of a sequence of triangles of oriented lines each of which is constructed as the convex hull of three lines chosen randomly within the previous triangle. This dual process is a different process but it leads to the same limiting degenerate collinear shape distribution.

We shall be working with triples $\{g_1, g_2, g_3\}$ of directed lines in $\mathbb{R}^2$. The construction here depends upon the definition of the convex hull of a set of directed lines; this is repeated here for convenience. For a set of directed lines in $\mathbb{R}^2$ we have L, the intersection of their left-hand sides, and R, the intersection of their right-hand sides. The convex hull is then defined as the set of directed lines that contain L in their left-hand side and R in their right-hand side. If the directions of the lines are contained in a semicircle then the convex hull is compact. The reader is referred to Gates (1994) for a general discussion of triangle shape and to Gates (1993) for discussion of convexity and convex hull for sets of lines. For triangles constructed by lines as sides it is much easier to measure shape directly by angles; transformation to Kendall's shape coordinates is given in Gates (1994).

A directed line is often specified by its orientation ϕ in $[0, 2\pi)$ and the distance, p, of the origin to the left of the line, with $-\infty < p < \infty$. The standard representation of the set of directed lines in the plane is the cylinder $\mathbb{Z} = S^1 \times \mathbb{R}^1$ in $\mathbb{R}^3$, in which the line with orientation ϕ at distance p is represented as the point $(\cos\phi, \sin\phi, p)$ of $\mathbb{Z}$. The set of all

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lines passing through a point (a, b) of $\mathbb{R}^2$ is represented by the intersection of $\mathbb{Z}$ with the plane through the origin orthogonal to the vector $(b, -a, 1)$. The set of oriented lines has a natural measure and convexity structure, but no natural metric. The natural measure is the unique (up to proportionality) Haar measure invariant under the action of the group of planar motions (see Santaló 1976). In the cylinder representation the invariant measure is just cylindrical surface area. Gates *et al*. (2005) show that familiar convexity properties of planar sets of points have analogues for convexity of lines. We can think of three lines in the plane as three points on the manifold $\mathbb{Z}$; the process described here is the dual of the process of sequentially choosing three vertices in the plane. On $\mathbb{Z}$, as the triangles of the nested sequence become smaller, the patch of the cylinder more closely resembles a planar patch and leads us to suspect that the limiting behaviour of the dual process is the same as the classic case of vertices from $\mathbb{R}^2$. Of course the three lines can also be regarded as the sides of a triangle in $\mathbb{R}^2$.

In this paper we will be concerned with convergence to collinearity and we start with a parent triangle $\{g_1, g_2, g_3\}$; the orientation ϕ_i of g_i will be taken in $[0, \pi]$ and the lines labelled so that $0 \leqslant \phi_2 \leqslant \phi_1 \leqslant \phi_3 \leqslant \pi$; the triangle shape is defined as (B, C), where $B = \phi_3 - \phi_1$ and $C = \phi_1 - \phi_2$. In vertex-generated shape theory it is common to construct a standard representative of a shape class; we employ a corresponding convention here. Thus given angles (B, C) we construct a line triple (shown in Figure 1) with g_1 crossing the x-axis at $\mathbf{b}$ and having orientation $\phi_1 = \pi - B$, with g_2 passing through the origin, $\mathbf{o}$,

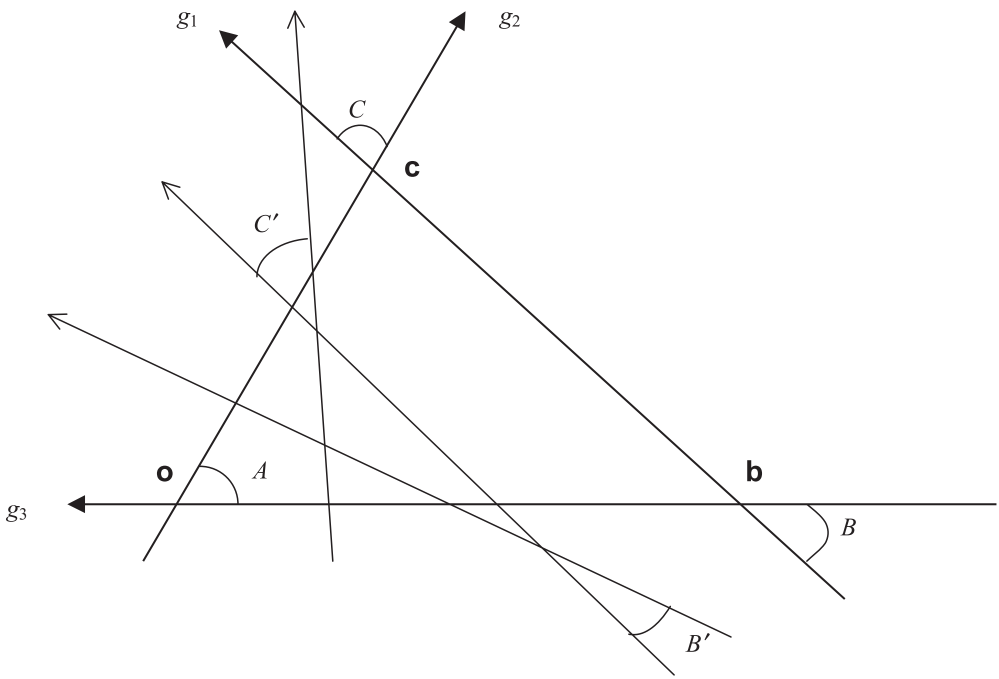

Figure 1. Construction of a daughter triangle.

having orientation $\phi_2 = \pi - B - C$ ($= A$, say) and with g_3 as the negative x-axis. Let g_1 and g_2 meet at $\mathbf{c}$; then the line triple shown has its left-hand side with a single vertex, $\mathbf{o}$, and with its right-hand side having two vertices, $\mathbf{b}$ and $\mathbf{c}$. The standard line measure of the convex hull is $\mathbf{ob} + \mathbf{oc} - \mathbf{bc}$; we normalize the scale to give unit line measure by taking $\mathbf{ob} = h^* \sin(C)$, where

$$h^* = (\sin(B) + \sin(C) - \sin(A))^{-1}, \tag{1}$$

so that $h^* \sin(B)\sin(C)$ is the altitude length from $\mathbf{o}$ to g_1. The angular range of lines in this convex set is $[A, \pi]$.

In selecting a line G' uniformly randomly the orientation Φ' has probability density function given by the internal width function (see Gates 1994):

$$w(\phi'|B, C) = h^* \min\{\sin(B)\sin(\phi' - A), \sin(C)\sin(\phi')\}, \tag{2}$$

where $A \leqslant \phi' \leqslant \pi$. This will be termed the *sine-arch* density. The process of selecting a daughter triangle shape is to choose three angles $(\Phi'_1, \Phi'_2, \Phi'_3)$ independently according to (2), order these, and then calculate the new shape angles (B', C') by

$$B' = \Phi'_{(3)} - \Phi'_{(2)}, \qquad C' = \Phi'_{(2)} - \Phi'_{(1)}. \tag{3}$$

We note that, trivially, $B' + C' \leqslant B + C$. This construction is illustrated in Figure 1.

It will be convenient to denote $B + C$ by r and $B' + C'$ by R. If we start with an obtuse triangle ($r < \pi/2$), then the marginal daughter density function for (B', C') is

$$h(u, v) = 6\int_A^{\pi-u-v} w(\phi)w(\phi + v)w(\phi + u + v)\mathrm{d}\phi, \tag{4}$$

where the dependence on (B, C) has been suppressed. In specific cases analytic evaluation of (4) is elementary, but requires substitutions of the appropriate trigonometric component of the w function. The result is a piecewise linear trigonometric function.

2. Internal ratio and range distribution

Before our application we develop some formulae in general notation. Suppose that $X_{(1)}, X_{(2)}, X_{(3)}$ are the order statistics of a random sample of size 3 from a continuous distribution on $[0, L]$ with distribution function F and concave probability density function f. Define the *internal ratio* as $S = (X_{(2)} - X_{(1)})/(X_{(3)} - X_{(1)})$. The density, IRf, of S is given by

$$IRf(t) = 6\iint(x_3 - x_1)f(x_1)f(x_3)f(x_1 + t(x_3 - x_1))\mathrm{d}x_1\,\mathrm{d}x_3, \tag{5}$$

with domain of integration $0 \leqslant x_1 \leqslant x_3 \leqslant L$. A case of particular interest is when f is the *triangle-arch* density (with parameter s),

$$w_0(x|s) = \min\{2xs^{-1}, 2(1-x)(1-s)^{-1}\}, \qquad 0 \leqslant x \leqslant 1, \tag{6}$$

which has $L = 1$ and is a limiting case of the scaled sine-arch density. The internal ratio density for w_0 was evaluated by Mannion (1990b), who also demonstrated that it is uniformly bounded below by $\frac{2}{3}$.

Lemma 1. *If f is continuous and concave on* $[0, L]$ *then* $IRf(t) \geqslant 1/12$.

Proof. We can assume that $L = 1$ and that f attains its maximum f^* at s^* in [0, 1]. Then, by concavity, $f(x) \geqslant \frac{1}{2}f^* w_0(x|s^*)$. From (5) we see that

$$IRf(t) \geqslant \left(\tfrac{1}{2}f^*\right)^3 IRw_0(t|s^*) \geqslant \left(\tfrac{1}{2}f^*\right)^3 \tfrac{2}{3},$$

using the Mannion result. As f is a density on [0, 1], $f^* \geqslant 1$ and so the lower bound follows. □

This result will be ergodically significant in the next section where the internal ratio density will be a stochastic transition kernel.

The following lemma is useful when approximating sampled distributions.

Lemma 2. *If f and g are concave densities on* $[0, L]$ *with* $|f(x) - g(x)| \leqslant k$, *then*

$$|IRf(t) - IRg(t)| \leqslant 28k.$$

Proof. Assume that $L = 1$. Then by direct substitution of the bounds $f(x) \pm k$ for $g(x)$ in (5) and using the facts that $f(x) \leqslant 2$ and $k \leqslant 2$, we obtain

$$|IRf(t) - IRg(t)| \leqslant 12k + 6k^2 + k^3 \leqslant 28k. \qquad \square$$

The internal ratio is the shape of a sample of size 3; complementary to that is the range $R = X_{(3)} - X_{(1)}$. Using results in Exercise 14.19 and equation (14.82) of Kendall and Stuart (1977), we can say that

$$\mathrm{E}(R) = 3\int_0^L (F(x)(1 - F(x)))\mathrm{d}x \leqslant 3\frac{L}{4}. \tag{7}$$

If X_1, X_2, X_3 is an initial sample from a distribution on $[0, L]$ and we repeatedly take nested samples, so that $X_1^{(k)}, X_2^{(k)}, X_3^{(k)}$ is a sample from the interval $[X_{(1)}^{(k-1)}, X_{(3)}^{(k-1)}]$, then (7) implies that, with R_k the range of the kth nested sample,

$$\mathrm{E}(R_k) \leqslant (3L/4)^k. \tag{8}$$

In our application to a sequence of nested dual triangles (8) will ensure a geometric rate of convergence to collinearity.

Next we need some detail on the approximation of the scaled triangle-arch density to the sine-arch density to establish a bound for the use of Lemma 2.

A triangle shape can be represented by (B, C) or by (s, r), where $r = B + C$ and $s = B/r$. From (2) the daughter triangle is chosen from the distribution with density

$$a(x|s, r) = w(\pi - x|sr, (1-s)r) = h^* \min\{\sin((1-s)r)\sin(x), \sin(sr)\sin(r - x)\}, \tag{9}$$

where $h^* = [\sin(sr) + \sin((1-s)r) - \sin(r)]^{-1}$.

Lemma 3.

(i) $0 \leqslant \left(\frac{1}{2}\right) r^3 - (s(1-s)h^*)^{-1} \leqslant Kr^5$, *where* $K = 1/24$.
(ii) $|a(x|s, r) - r^{-1}w_0(xr^{-1}|s)| \leqslant r/3$, *for* $0 \leqslant x \leqslant r$.

Proof. From (1), using trigonometric identities, we can write

$$(h^*)^{-1} = 4\sin\left(\frac{r}{2}\right)\sin\left(\frac{rs}{2}\right)\sin\left(\frac{r(1-s)}{2}\right).$$

The left-hand inequality in (i) is obtained by using the upper bound $\sin(t) \leqslant t$; using $t - t^3/6 \leqslant \sin(t)$, we have

$$\begin{aligned}(h^*)^{-1} &\geqslant \frac{r^3 s(1-s)}{2}\left(1 - \frac{r^2}{24}\right)\left(1 - \frac{r^2 s^2}{24}\right)\left(1 - \frac{r^2(1-s)^2}{24}\right) \\ &\geqslant \frac{r^3 s(1-s)}{2}\left(1 - \frac{r^2}{24}\right)^2 \geqslant \frac{r^3 s(1-s)}{2}\left(1 - \frac{r^2}{12}\right).\end{aligned}$$

from which the right-hand inequality of (i) follows.

To prove (ii) we start by rearranging (i):

$$2(s(1-s)r^3)^{-1} \leqslant h^* \leqslant 2(s(1-s)r^3)^{-1}(1 + 2Kr^2).$$

For $0 \leqslant x \leqslant sr$, using the upper bound for $\sin(\cdot)$, we have

$$\begin{aligned}a(x|s, r) &\leqslant 2xr(1-s)(s(1-s)r^3)^{-1}(1 + 2Kr^2) \\ &= r^{-1}w_0(xr^{-1})(1 + 2Kr^2) \leqslant r^{-1}w_0(xr^{-1}) + 4Kr.\end{aligned}$$

Using the lower bound, we have

$$2(s(1-s)r^3)^{-1}[(1-s)r - \tfrac{1}{6}((1-s)r)^3][x - \tfrac{1}{6}x^3] \leqslant a(x|s, r),$$

leading to

$$2x(sr^2)^{-1}[1 - \tfrac{1}{6}((1-s)r)^2 - \tfrac{1}{6}x^2] \leqslant a(x|s, r)$$

and then, as $x \leqslant sr$, to

$$r^{-1}w_0(xr^{-1}|s) - \tfrac{1}{3}r \leqslant a(x|s, r).$$

For $sr \leqslant x \leqslant r$, we have

$$\begin{aligned}a(x|s, r) &\leqslant 2sr(r-x)(s(1-s)r^3)^{-1}(1 + 2Kr^2) \\ &= r^{-1}w_0(xr^{-1}|s)(1 + 2Kr^2) \leqslant r^{-1}w_0(xr^{-1}|s) + 4Kr.\end{aligned}$$

Similarly, using the lower bound for $\sin(\cdot)$ in this case we have

$$r^{-1}w_0(xr^{-1}|s) - \tfrac{1}{3}r \leqslant a(x|s,\ r).$$

As $4K \leqslant 1/3$ we see that $r/3$ is a bound on the difference in either case. □

Combining Lemmas 2 and 3, using the fact that $r^{-1}w_0(xr^{-1}|s)$ is a density on $[0,\ r]$ with the same internal ratio density as w_0, we have the following corollary.

Corollary 1. $|IRa(t|s,\ r) - IRw_0(t|s)| \leqslant 28r/3$.

3. Convergence of a nested sequence of triangles

In this section we consider the stochastic convergence of the shapes of a nested sequence of dual random triangles. Let $(s_0,\ r_0)$ be an initial triangle shape (with $r_0 < \pi/2$). We construct a standard representative dual triangle as described in Section 1; three lines are selected at random to produce a daughter triangle shape $(S_1,\ R_1)$, and the process is repeated. Removing the geometry, we sample X_1, X_2, X_3 from $a(\cdot|s_0,\ r_0)$ and let $S_1 = (X_{(2)} - X_{(1)})/(X_{(3)} - X_{(1)})$ and $R_1 = X_{(3)} - X_{(1)}$. The process is repeated to produce a sequence $\{(S_n,\ R_n)\}$. This sequence is a discrete-time continuous state space Markov chain, but we shall instead consider $\{S_n\}$ as a non-homogeneous but asymptotically homogeneous Markov chain. For large n, R_n should be small and we shall be nearly sampling from the triangle-arch distribution. We have to prove that the limiting distribution of $\{S_n\}$ is that of the collinear shape process determined by Mannion. We shall formulate versions of the results given by Seneta (1973) for inhomogeneous finite-state Markov chains.

Notationally define the *total variation* norm of an integrable function h on $[0,\ 1]$ by

$$|h|_1 = \int_0^1 |h(x)|\mathrm{d}x,$$

so that $|h|_1 = 1$ if h is a probability density function. Let $f_{n|m}$ denote the density of the shape S_n after n selections given the shape and range after m selections ($n > m$), so that $f_{m+1|m}(\cdot) = IRa(\cdot|s_m,\ r_m)$; let $f_n = f_{n|0}$ denote the unconditional density after n selections starting with an initial parent triangle and F_n the corresponding distribution function of S_n.

For any integrable function, h, on $[0,\ 1]$ let $h\mathbf{P}_m$ denote the function on $[0,\ 1]$ defined by

$$(h\mathbf{P}_m)(x) = \int_0^1 h(s)f_{m+1|m}(x)\mathrm{d}s = \int_0^1 h(s)IRa(x|s,\ r_m)\mathrm{d}s,$$

which also depends on r_m. If h is a density then so is $h\mathbf{P}_m$. Also let $\mathbf{P}_0$ denote the transition kernel defined by $IRw_0(\cdot|\cdot)$.

Lemma 4. *For any two densities h_1 and h_2 on* $[0,\ 1]$,

$$|(h_1 - h_2)\mathbf{P}_m|_1 \leqslant (1-\lambda)|h_1 - h_2|_1,$$

where $\lambda = 1/12$.

Proof. For any s, r the function $(IRa(\cdot|s, r)) - \lambda)$ is non-negative and has integral $1 - \lambda$, hence

$$|(h_1 - h_2)\mathbf{P}_m|_1 = \int_0^1 \left| \int_0^1 (h_1(s) - h_2(s)(IRa(x|s, r_m))\mathrm{d}s \right| \mathrm{d}x$$

$$= \int_0^1 \left| \int_0^1 (h_1(s) - h_2(s)(IRa(x|s, r_m) - \lambda)\mathrm{d}s \right| \mathrm{d}x$$

$$\leqslant \int_0^1 \int_0^1 |h_1(s) - h_2(s)|(IRa(x|s, r_m) - \lambda)\mathrm{d}x\,\mathrm{d}s$$

$$= (1 - \lambda)\int_0^1 |h_1(s) - h_2(s)|\mathrm{d}s = (1 - \lambda)|h_1 - h_2|_1.$$

□

Let G (or g) denote the invariant distribution (or density) for the collinear Markov process with transition kernel $\mathbf{P}_0$, from the triangle-arch distribution, so that $g\mathbf{P}_0 = g$. Mannion (1990b) gave

$$g(t) = (3\pi^{-2})\{1 + \alpha(t) + \alpha(1 - t)\}, \qquad 0 \leqslant t \leqslant 1,$$

where

$$\alpha(t) = t^{-2} + \tfrac{1}{2}t^{-3}\{2 + 3t + 2t^2\}(1 - t)^2 \ln(1 - t).$$

The following lemma allows us to show a contraction of $f_{n|m}$ to g.

Lemma 5. *For* $k > 0$,

$$|f_{k+m|m} - g|_1 \leqslant |f_m - g|_1(1 - \lambda)^k + r_m\left(\frac{28}{3}\right)\sum_{j=0}^{k-1}(1 - \lambda)^j.$$

Proof. We have

$$\begin{aligned}|f_{m+1|m} - g|_1 &\leqslant |(f_m - g)\mathbf{P}_m|_1 + |g\mathbf{P}_m - g\mathbf{P}_0|_1 \\ &\leqslant (1 - \lambda)|f_m - g|_1 + \int_0^1\int_0^1 g(s)|IRa(x|s, r_m) - IRw_0(x|s)|\mathrm{d}s\,\mathrm{d}x \\ &\leqslant (1 - \lambda)|f_m - g|_1 + \int_0^1\int_0^1 g(s)\left(\frac{28r_m}{3}\right)\mathrm{d}s\,\mathrm{d}x = (1 - \lambda)|f_m g|_1 + r_m\left(\frac{28}{3}\right),\end{aligned}$$

using Corollary 1 and proving Lemma 5 for $k = 1$.

Suppose it is true for $k = i$. Then for $k = i + 1$ we have

$$
\begin{aligned}
|f_{m+i+1|m} - g|_1 &\leqslant |(f_{m+i|m} - g)\mathbf{P}_{m+i}|_1 + |g\mathbf{P}_{m+i} - g\mathbf{P}_0|_1 \\
&\leqslant (1-\lambda)|f_{m+i|m} - g|_1 + \int_0^1\int_0^1 g(s)\left(\frac{28r_{m+i}}{3}\right)\mathrm{d}s\,\mathrm{d}x \\
&\leqslant (1-\lambda)|f_{m+i|m} - g|_1 + r_m\left(\frac{28}{3}\right),
\end{aligned}
$$

as $r_{m+i} \leqslant r_m$. Using the induction assumption, we can say that

$$
\begin{aligned}
|f_{m+i+1|m} - g|_1 &\leqslant (1-\lambda)\left\{|f_m - g|_1(1-\lambda)^i + r_m\left(\frac{28}{3}\right)\sum_{j=0}^{i-1}(1-\lambda)^j\right\} + r_m\left(\frac{28}{3}\right) \\
&= |f_m - g|_1(1-\lambda)^{i+1} + r_m\left(\frac{28}{3}\right)\sum_{j=0}^{i}(1-\lambda)^j,
\end{aligned}
$$

confirming the result at $k = i+1$ and hence for all k. □

Corollary 2. $|f_{k+m|m} - g|_1 \leqslant (11/12)^k|f_m - g|_1 + 112r_m$.

Lemma 5 allows us to establish convergence in distribution of the law of S_n to G. For any n, let $m = [n/2]$ and let $k = n - [n/2]$. Then

$$
\begin{aligned}
\Pr(S_n \leqslant t) - G(t) &= \mathrm{E}(\Pr(S_n \leqslant t|R_m) - G(t)) \\
&= \mathrm{E}\left(\int_0^t (f_{n|m}(u) - g(u))\mathrm{d}u\right) \leqslant \mathrm{E}\left(\int_0^t |(f_{n|m}(u) - g(u))|\mathrm{d}u\right) \\
&\leqslant \mathrm{E}\big(|f_{n|m} - g|\big) \leqslant \mathrm{E}\left(\left(\frac{11}{12}\right)^{k+1} + 112R_m\right) \leqslant \left(\frac{11}{12}\right)^{n/2} + 112\left(\frac{3}{4}\right)^{n/2},
\end{aligned}
$$

where we have used (8). Essentially the same steps also show that $G(t) - \Pr(S_n \leqslant t)$ has the same bound and so

$$
|\Pr(S_n \leqslant t) - G(t)| \leqslant \left(\frac{11}{12}\right)^{n/2} + 112\left(\frac{3}{4}\right)^{n/2},
$$

an error bound clearly uniform (and geometrically decreasing) in t.

Theorem 1. *$F_n(t) \to G(t)$, uniformly in t, as $n \to \infty$.*

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