Bernoulli **12**(1), 2006, 1–33

Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations

OLE E. BARNDORFF-NIELSEN[1], MAKOTO MAEJIMA[2] and KEN-ITI SATO[3]

[1]*Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark. E-mail: oebn@imf.au.dk*
[2]*Department of Mathematics, Keio University, Hiyoshi, Yokohama, 223-8522 Japan. E-mail: maejima@math.keio.ac.jp*
[3]*Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya 468-0074, Japan. E-mail: ken-iti.sato@nifty.ne.jp*

The class of distributions on $\mathbb{R}$ generated by convolutions of Γ-distributions and the class generated by convolutions of mixtures of exponential distributions are generalized to higher dimensions and denoted by $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$. From the Lévy process $\{X_t^{(\mu)}\}$ on $\mathbb{R}^d$ with distribution μ at $t=1$, $\Upsilon(\mu)$ is defined as the distribution of the stochastic integral $\int_0^1 \log(1/t)\mathrm{d}X_t^{(\mu)}$. This mapping is a generalization of the mapping Υ introduced by Barndorff-Nielsen and Thorbjørnsen in one dimension. It is proved that $\Upsilon(ID(\mathbb{R}^d)) = B(\mathbb{R}^d)$ and $\Upsilon(L(\mathbb{R}^d)) = T(\mathbb{R}^d)$, where $ID(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$ are the classes of infinitely divisible distributions and of self-decomposable distributions on $\mathbb{R}^d$, respectively. The relations with the mapping Φ from μ to the distribution at each time of the stationary process of Ornstein–Uhlenbeck type with background driving Lévy process $\{X_t^{(\mu)}\}$ are studied. Developments of these results in the context of the nested sequence $L_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, are presented. Other applications and examples are given.

Keywords: Goldie–Steutel–Bondesson class; infinite divisibility; Lévy measure; Lévy process; self-decomposability; stochastic integral; Thorin class

1. Introduction

For distributions on the positive real line, Thorin (1977a; 1977b) introduced the smallest class that contains all Γ-distributions and is closed under convolution and convergence, where convergence of distributions means weak convergence. He called distributions of this class generalized Γ-convolutions. This was in connection with his proof of infinite divisibility of Pareto and lognormal distributions. In Bondesson's (1992) monograph the class is denoted by $\mathcal{T}$. Subsequently Thorin (1978) considered the smallest class on the real line $\mathbb{R}$ which contains all generalized Γ-convolutions and is closed under convolution, convergence and reflection. We denote this class by $T(\mathbb{R})$. Based on the work of Goldie (1967) and Steutel (1967, 1970), Bondesson (1981) studied the smallest class which

1350–7265

contains all mixtures of exponential distributions and is closed under convolution and convergence. He called distributions of this class generalized convolutions of mixtures of exponential distributions. It is similarly extended to a class on $\mathbb{R}$, and we denote the extension by $B(\mathbb{R})$. In Bondesson (1992) the class $T(\mathbb{R})$ and the class of generalized convolutions of mixtures of exponential distributions are denoted by $\mathcal{T}_{\mathrm{e}}$ and $\mathcal{T}_2$, respectively; the class $B(\mathbb{R})$ should not be confused with the class $\mathcal{B}$ there.

We study multidimensional analogues of the classes $T(\mathbb{R})$ and $B(\mathbb{R})$. We define them as subclasses of the class $ID(\mathbb{R}^d)$ of infinitely divisible distributions on $\mathbb{R}^d$ such that their Lévy measures have radial components having the same property as the part on $\mathbb{R}_+ = [0, \infty)$ of the Lévy measures of distributions in $T(\mathbb{R})$ and $B(\mathbb{R})$, respectively. The class $T(\mathbb{R}^d)$ is included in the class $L(\mathbb{R}^d)$ of self-decomposable distributions on $\mathbb{R}^d$, but the class $B(\mathbb{R}^d)$ is not. Precise definitions will be given in Section 2. The class $T(\mathbb{R}^d)$ is duly called the Thorin class, as it is the analogue of $T(\mathbb{R})$. Historically, Goldie (1967) proved the infinite divisibility of mixtures of exponential distributions and Steutel (1967) found the description of their Lévy measures. So it would be appropriate to call B $(\mathbb{R}^d)$ the Goldie–Steutel–Bondesson class. We give a probabilistic characterization of these classes on $\mathbb{R}^d$ by using a mapping Υ defined by a stochastic integral; $\Upsilon(\mu)$ is the distribution of $\int_0^1 \log(1/t)\mathrm{d}X_t^{(\mu)}$, where $\{X_t^{(\mu)}\}$ is the Lévy process on $\mathbb{R}^d$ with distribution μ at $t = 1$. In one dimension this is the mapping introduced by Barndorff-Nielsen and Thorbjørnsen (2002a; 2002b, 2004; 2005) in relation to the Bercovici–Pata bijection between free infinite divisibility and classical infinite divisibility. We will prove that $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ are the images by Υ of $ID(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$, respectively. We will further investigate the relation with the mapping Φ which is defined for μ in $ID_{\log}(\mathbb{R}^d)$, the class of distributions in $ID(\mathbb{R}^d)$ with finite log-moment, and which gives the distribution $\Phi(\mu)$ of $\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t^{(\mu)}$. Both $\Phi\Upsilon$ and $\Upsilon\Phi$ are defined on $ID_{\log}(\mathbb{R}^d)$; they coincide and give another stochastic integral representation of $T(\mathbb{R}^d)$. By analogy with the construction of the well-known nested sequence of subclasses $L_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, of $L(\mathbb{R}^d) = L_0(\mathbb{R}^d)$, we define a new nested sequence of subclasses $T_m(\mathbb{R}^d)$, $m = 0, 1, \ldots, \infty$, beginning with $T_0(\mathbb{R}^d) = T(\mathbb{R}^d)$. Alternatively, the former sequence extended by adding $ID(\mathbb{R}^d)$ at the top and the latter sequence extended by adding $B(\mathbb{R}^d)$ at the top can be generated from the top members by iterating the mapping Φ each time after restriction to $ID_{\log}(\mathbb{R}^d)$. We will show that the latter extended sequence is the image by the mapping Υ of the former extended sequence. Further, we will describe $T_m(\mathbb{R}^d)$ by specifying the Lévy measures. A characterization of $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ by using elementary Γ-variables and elementary mixed-exponential variables in $\mathbb{R}^d$, respectively, will also be given.

2. Main results

For any $\mathbb{R}^d$-valued random variable X, we denote its distribution by $\mathcal{L}(X)$. The characteristic function and the cumulant function of a distribution μ on $\mathbb{R}^d$ are denoted by $\hat{\mu}(z)$ and $C_\mu(z)$, respectively. That is, $C_\mu(z)$ is a continuous function with $C_\mu(0) = 0$ such that $\hat{\mu}(z) = \exp(C_\mu(z))$, $z \in \mathbb{R}^d$; such a function $C_\mu(z)$ exists and is unique if $\hat{\mu}(z) \neq 0$ for all $z \in \mathbb{R}^d$. If $\mu = \mathcal{L}(X)$, then $C_\mu(z)$ is also written as $C_X(z)$.

Any Lévy process $\{X_t^{(\mu)} : t \geqslant 0\}$ on $\mathbb{R}^d$ uniquely induces an $\mathbb{R}^d$-valued independently scattered random measure $\{M^{(\mu)}(B) : B \in \mathcal{B}^0_{[0,\infty)}\}$ such that $M^{(\mu)}([0, t]) = X_t^{(\mu)}$ almost surely, where $\mathcal{B}^0_{[0,\infty)}$ is the class of bounded Borel sets in $[0, \infty)$. Let $f(t)$ be a real-valued measurable function on $[0, \infty)$, $M^{(\mu)}$-integrable (also called $\{X_t^{(\mu)}\}$-integrable) in the sense of Urbanik and Woyczynski (1967) and Rajput and Rosinski (1989) for $d = 1$ and of Sato (2004) for general d. Then $M^{(f,\mu)}(B) = \int_B f(t) M^{(\mu)}(\mathrm{d}t)$ (also written as $\int_B f(t)\mathrm{d}X_t^{(\mu)}$) is defined a.s. for each $B \in \mathcal{B}^0_{[0,\infty)}$ and $\{M^{(f,\mu)}(B) : B \in \mathcal{B}^0_{[0,\infty)}\}$ is again an $\mathbb{R}^d$-valued independently scattered random measure; furthermore, we have $\int_B |C_\mu(f(t)z)|\mathrm{d}t < \infty$ and

$$C_{M^{(f,\mu)}(B)}(z) = \int_B C_\mu(f(t)z)\mathrm{d}t, \qquad z \in \mathbb{R}^d. \tag{2.1}$$

For $B = (a, b)$, $(a, b]$, $[a, b)$, and $[a, b]$, the expressions $\int_B f(t)\mathrm{d}X_t^{(\mu)}$ coincide a.s.; we write them as $\int_a^b f(t)\mathrm{d}X_t^{(\mu)}$. On $[0, \infty)$ or $(0, \infty)$ the stochastic integral of f with respect to $X_t^{(\mu)}$ is defined as the limit in probability of $\int_0^s f(t)\mathrm{d}X_t^{(\mu)}$ as $s \to \infty$ and written as $\int_0^\infty f(t)\mathrm{d}X_t^{(\mu)}$, whenever the limit exists. Let

$$\begin{aligned} ID_{\log}(\mathbb{R}^d) &= \left\{\mu \in ID(\mathbb{R}^d) : \int_{|x|>2} \log|x|\mu(\mathrm{d}x) < \infty\right\} \\ &= \left\{\mu \in ID(\mathbb{R}^d) : \int_{|x|>2} \log|x|\nu^{(\mu)}(\mathrm{d}x) < \infty\right\}, \end{aligned}$$

where $\nu^{(\mu)}$ is the Lévy measure of μ. It is known (Jurek and Vervaat 1983; Sato and Yamazato 1983; Sato 1999) that $\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t^{(\mu)}$ is definable if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$, and that

$$L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d)), \tag{2.2}$$

where

$$\Phi(\mu) = \mathcal{L}\left(\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t^{(\mu)}\right). \tag{2.3}$$

The domain of definition of the mapping Φ is $ID_{\log}(\mathbb{R}^d)$, and Φ is one-to-one. Another characterization of $\Phi(\mu)$ is given in relation to the Langevin equation

$$\mathrm{d}Y_t = \mathrm{d}X_t^{(\mu)} - Y_t\,\mathrm{d}t. \tag{2.4}$$

Equation (2.4) has a stationary solution $\{Y_t : t \geqslant 0\}$ if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$. If $\mu \in ID_{\log}(\mathbb{R}^d)$, then a stationary solution $\{Y_t\}$ is unique, and $\mathcal{L}(Y_t) = \Phi(\mu)$ for all $t \geqslant 0$. The process $\{Y_t\}$ is called a stationary process of Ornstein–Uhlenbeck type. For a historical account of the connection of $L(\mathbb{R}^d)$, Φ, and processes of Ornstein–Uhlenbeck type, see Rocha-Arteaga and Sato (2003, pp. 54–55). Steutel and van Harn (1979) should also be cited, as they mentioned the possibility of expressing $C_{\tilde{\mu}}(z)$ for $\tilde{\mu} \in L(\mathbb{R})$ in the form equivalent to the right-hand side of (4.1) in Section 4 with some μ. For further developments and extensions, see Maejima and Sato (2003) and the references therein.

For any Borel set E in $\mathbb{R}^d$, the class of Borel subsets of E is denoted by $\mathcal{B}(E)$. A function defined on E is called measurable if it is $\mathcal{B}(E)$-measurable. The unit sphere in $\mathbb{R}^d$ is denoted by $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$.

We use the Lévy–Khintchine triplet, or simply the triplet, (A, ν, γ) of $\mu \in ID(\mathbb{R}^d)$ in the sense that

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx) + i\langle \gamma, z\rangle, \tag{2.5}$$

where A is a $d \times d$ symmetric non-negative definite matrix, ν is a measure on $\mathbb{R}^d$ called the Lévy measure of μ, and $\gamma \in \mathbb{R}^d$. A measure ν is the Lévy measure of some $\mu \in ID(\mathbb{R}^d)$ if and only if $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We sometimes denote an infinitely divisible distribution μ with triplet (A, ν, γ) by $\mu_{(A,\nu,\gamma)}$. We also denote the Lévy measure of μ by $\nu^{(\mu)}$.

We use the following polar decomposition of Lévy measures.

Lemma 2.1. *Let $\nu = \nu^{(\mu)}$ for some $\mu \in ID(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leqslant \infty$. Then there exist a measure λ on S with $0 < \lambda(S) \leqslant \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that*

$$\nu_\xi(B) \text{ is } \textit{measurable in } \xi \textit{ for each } B \in \mathcal{B}((0, \infty)); \tag{2.6}$$

$$0 < \nu_\xi((0, \infty)) \leqslant \infty, \qquad \textit{for each } \xi \in S; \tag{2.7}$$

$$\nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)\nu_\xi(dr), \qquad \textit{for } B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\}). \tag{2.8}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν in the following sense: if λ, $\{\nu_\xi\}$ and λ', $\{\nu'_\xi\}$ both have properties (2.6)–(2.8)*, then there is a measurable function $c(\xi)$ on S such that*

$$0 < c(\xi) < \infty; \tag{2.9}$$

$$\lambda'(d\xi) = c(\xi)\lambda(d\xi); \tag{2.10}$$

$$c(\xi)\nu'_\xi(dr) = \nu_\xi(dr), \qquad \textit{for } \lambda \textit{ almost every } \xi \in S. \tag{2.11}$$

Rosinski (1990) has the same result, but the uniqueness is not mentioned. Sometimes we call λ and ν_ξ in Lemma 2.1 *the spherical component* and *the radial component* of ν respectively, as they are uniquely determined in the sense given above. The following description of the Lévy measures of $L(\mathbb{R}^d)$ is well known (see Sato 1999, Theorem 15.10).

Proposition 2.2. *Let $\mu \in ID(\mathbb{R}^d)$ and let $\nu = \nu^{(\mu)}$. Then $\mu \in L(\mathbb{R}^d)$ if and only if either $\nu = 0$ or $\nu \neq 0$ with a polar decomposition (λ, ν_ξ) such that there is a non-negative function $k_\xi(r)$ measurable in ξ and decreasing, right-continuous in r, satisfying*

$$\nu_\xi(dr) = k_\xi(r)r^{-1}\,dr, \qquad \textit{for } \lambda\textit{-a.e. } \xi \in S. \tag{2.12}$$

We call $k_\xi(r)$ *the k-function* of $\mu \in L(\mathbb{R}^d)$ or of its Lévy measure ν, as it is determined by μ λ-a.e. up to multiplication of functions of ξ. The right-continuous modification of

$k_\xi(\mathrm{e}^{-u})$ is denoted by $h_\xi(u)$ and called *the h-function* of $\mu \in L(\mathbb{R}^d)$ or of its Lévy measure ν.

Let us define $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$.

Definition 2.1. *The class $T(\mathbb{R}^d)$ is the collection of $\mu \in L(\mathbb{R}^d)$ with $\nu = \nu^{(\mu)}$ such that either $\nu = 0$ or $\nu \neq 0$, having k-function $k_\xi(r)$ completely monotone in r for λ-a.e. ξ, where λ is the spherical component of ν.*

Definition 2.2. *The class $B(\mathbb{R}^d)$ is the collection of $\mu \in ID(\mathbb{R}^d)$ with $\nu = \nu^{(\mu)}$ such that either $\nu = 0$ or $\nu \neq 0$, having polar decomposition (λ, ν_ξ) satisfying*

$$\nu_\xi(\mathrm{d}r) = l_\xi(r)\mathrm{d}r, \qquad \text{for } \lambda\text{-a.e. } \xi \in S, \tag{2.13}$$

where $l_\xi(r)$ is measurable in ξ and completely monotone in r for λ-a.e. ξ.

We call $l_\xi(r)$ *the l-function* of $\mu \in B(\mathbb{R}^d)$ or of its Lévy measure ν. We can prove that

$$B(\mathbb{R}^d) \cap L(\mathbb{R}^d) \underset{\neq}{\supset} T(\mathbb{R}^d). \tag{2.14}$$

Except for the strictness, this is clear; the strictness will be proved in Section 3.

We introduce a mapping Υ.

Proposition 2.3. *If $f(t)$ is a function on $[0, \infty)$ given by $f(t) = \log(1/t)$ for $0 < t \leqslant 1$ and $f(t) = 0$ otherwise, then $f(t)$ is $\{X_t^{(\mu)}\}$-integrable for every $\mu \in ID(\mathbb{R}^d)$.*

Definition 2.3. *For any $\mu \in ID(\mathbb{R}^d)$, define*

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{t}\, \mathrm{d}X_t^{(\mu)}\right). \tag{2.15}$$

We often write $\Upsilon\mu$ or $\Phi\mu$ for $\Upsilon(\mu)$ or $\Phi(\mu)$, respectively. We now state our main results on $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$.

Theorem A. (i) *The total image of the mapping Υ equals $B(\mathbb{R}^d)$. That is,*

$$B(\mathbb{R}^d) = \Upsilon(ID(\mathbb{R}^d)). \tag{2.16}$$

(ii) *Let $\mu \in ID(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon\mu$, and let $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. Then*

$$\tilde{\nu}(B) = \int_0^\infty \mathrm{e}^{-s}\nu(s^{-1}B)\mathrm{d}s, \qquad \text{for } B \in \mathcal{B}(\mathbb{R}^d). \tag{2.17}$$

If $\nu \neq 0$ and ν has polar decomposition (λ, ν_ξ), then a polar decomposition of $\tilde{\nu}$ is given by $\tilde{\lambda} = \lambda$ and $\tilde{\nu}_\xi(\mathrm{d}r) = \tilde{l}_\xi(r)\mathrm{d}r$, with

$$\tilde{l}_\xi(r) = \int_0^\infty s^{-1}\mathrm{e}^{-r/s}\nu_\xi(\mathrm{d}s). \tag{2.18}$$

Theorem B. (i) *The image of the class $L(\mathbb{R}^d)$ by the mapping Υ equals $T(\mathbb{R}^d)$. That is,*

$$T(\mathbb{R}^d) = \Upsilon(L(\mathbb{R}^d)). \tag{2.19}$$

(ii) *Let $\mu \in L(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon\mu$ with $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. If $\nu \neq 0$ and ν has spherical component λ and k-function $k_\xi(r)$, then $\tilde{\nu}$ has spherical component $\tilde{\lambda} = \lambda$ and k-function*

$$\tilde{k}_\xi(r) = \int_0^\infty k_\xi(rs^{-1})\mathrm{e}^{-s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-ru}\,\mathrm{d}k_\xi^{\#}(u). \tag{2.20}$$

Here $k_\xi^{\#}(u)$ is the right-continuous modification of $k_\xi(u^{-1})$.

In the one-dimensional case ($d = 1$), (2.19) was discovered by Barndorff-Nielsen and Thorbjørnsen who also, in effect, noted that $\Upsilon(ID(\mathbb{R})) \subset B(\mathbb{R})$, but without being aware of the connection to the class $B(\mathbb{R})$; see Barndorff-Nielsen and Thorbjørnsen (2004; 2005).

In proving Theorems A and B, we will show the following properties of the mapping Υ.

Proposition 2.4. (i) *For any $\mu \in ID(\mathbb{R}^d)$, $\int_0^1 |C_\mu(z\log(1/t))|\mathrm{d}t < \infty$ and*

$$C_{\Upsilon\mu}(z) = \int_0^1 C_\mu\left(z\log\frac{1}{t}\right)\mathrm{d}t, \qquad z \in \mathbb{R}^d. \tag{2.21}$$

(ii) *The mapping Υ is one-to-one from $ID(\mathbb{R}^d)$ into $ID(\mathbb{R}^d)$.*

(iii) *$\Upsilon(\mu_1 * \mu_2) = \Upsilon\mu_1 * \Upsilon\mu_2$ for $\mu_1, \mu_2 \in ID(\mathbb{R}^d)$.*

(iv) *For $\mu \in ID(\mathbb{R}^d)$ with triplet (A, ν, γ), $\Upsilon\mu$ has triplet $(\tilde{A}, \tilde{\nu}, \tilde{\gamma})$ with expressions*

$$\tilde{A} = 2A; \tag{2.22}$$

$$\tilde{\nu}(B) = \int_0^\infty \nu(s^{-1}B)\mathrm{e}^{-s}\,\mathrm{d}s, \qquad \text{for } B \in \mathcal{B}(\mathbb{R}^d); \tag{2.23}$$

$$\begin{aligned}\tilde{\gamma} &= \gamma + \int_0^\infty \mathrm{e}^{-s}s\,\mathrm{d}s\int_{\mathbb{R}^d} x\left(\frac{1}{1+s^2|x|^2} - \frac{1}{1+|x|^2}\right)\nu(\mathrm{d}x)\\ &= \gamma + \int_{\mathbb{R}^d}\frac{x|x|^2}{1+|x|^2}\nu(\mathrm{d}x)\int_0^\infty \frac{\mathrm{e}^{-s}s(1-s^2)}{1+s^2|x|^2}\,\mathrm{d}s.\end{aligned} \tag{2.24}$$

(v) *Let $\mu_n \in ID(\mathbb{R}^d)$, $n = 1, 2, \ldots$. If $\mu_n \to \mu$, then $\mu \in ID(\mathbb{R}^d)$ and $\Upsilon\mu_n \to \Upsilon\mu$. Conversely, if $\Upsilon\mu_n \to \tilde{\mu}$ for some distribution $\tilde{\mu}$, then $\tilde{\mu} = \Upsilon\mu$ for some $\mu \in ID(\mathbb{R}^d)$ and $\mu_n \to \mu$.*

(vi) *The mapping Υ has the following alternative expressions:*

$$\Upsilon\mu = \mathcal{L}\left(\int_0^1 \log\frac{1}{1-t}\,\mathrm{d}X_t^{(\mu)}\right), \tag{2.25}$$

$$\Upsilon\mu = \mathcal{L}\left(\lim_{s\downarrow 0}\int_s^1 \frac{X_t^{(\mu)}}{t}\,\mathrm{d}t\right), \tag{2.26}$$

where the limit in (2.26) is almost sure.

For another expression for $T(\mathbb{R}^d)$, we use the function $e_1(u) = \int_u^\infty e^{-s}s^{-1}\,ds$ and the function $e_1^*(t)$ inverse to $e_1(u)$, that is, $t = e_1(u)$ if and only if $u = e_1^*(t)$.

Theorem C. (i) *Let $\mu \in ID(\mathbb{R}^d)$. Then $\Upsilon\mu \in ID_{\log}(\mathbb{R}^d)$ if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$.*

(ii) *The integral $\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}$ exists if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$. If $\mu \in ID_{\log}(\mathbb{R}^d)$, then*

$$\Phi\Upsilon\mu = \Upsilon\Phi\mu = \mathcal{L}\left(\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}\right), \tag{2.27}$$

where $\Phi\Upsilon\mu = \Phi(\Upsilon(\mu))$ and $\Upsilon\Phi\mu = \Upsilon(\Phi(\mu))$.

(iii) *We have*

$$T(\mathbb{R}^d) = \Phi(B(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) \tag{2.28}$$

and

$$T(\mathbb{R}^d) = \left\{\mathcal{L}\left(\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}\right) : \mu \in ID_{\log}(\mathbb{R}^d)\right\}. \tag{2.29}$$

Let us recall the definition of self-decomposability. A distribution μ on $\mathbb{R}^d$ is said to be self-decomposable, or $\mu \in L(\mathbb{R}^d)$, if for each $b > 1$ there is a distribution $\rho_b^{(\mu)}$ such that

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\widehat{\rho_b^{(\mu)}}(z). \tag{2.30}$$

Note that $\rho_b^{(\mu)}$ is uniquely determined by μ and b and that $\rho_b^{(\mu)} \in ID(\mathbb{R}^d)$. We define $L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$ and then, for $m = 1, 2, \ldots,$ define

$$L_m(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) : \rho_b^{(\mu)} \in L_{m-1}(\mathbb{R}^d) \text{ for all } b > 1\}. \tag{2.31}$$

Let $L_\infty(\mathbb{R}^d) = \cap_{0\leqslant m<\infty} L_m(\mathbb{R}^d)$ and let $\mathfrak{S}(\mathbb{R}^d)$ be the class of stable distributions on $\mathbb{R}^d$. Thus we obtain the nested sequence studied by Urbanik (1972), Sato (1980) and others:

$$ID(\mathbb{R}^d) \supset L_0(\mathbb{R}^d) \supset L_1(\mathbb{R}^d) \supset L_2(\mathbb{R}^d) \supset \ldots \supset L_\infty(\mathbb{R}^d) \supset \mathfrak{S}(\mathbb{R}^d) \tag{2.32}$$

The class $L_\infty(\mathbb{R}^d)$ is the smallest class that contains $\mathfrak{S}(\mathbb{R}^d)$ and is closed under convolution and convergence.

Corollary to Theorem C. *We have*

$$T(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) : \rho_b^{(\mu)} \in B(\mathbb{R}^d) \textit{ for all } b > 1\}. \tag{2.33}$$

We now define the classes $T_m(\mathbb{R}^d)$, letting $T_0(\mathbb{R}^d) = T(\mathbb{R}^d)$ and, for $m = 1, 2, \ldots,$

$$T_m(\mathbb{R}^d) = \{\mu \in L(\mathbb{R}^d) : \rho_b^{(\mu)} \in T_{m-1}(\mathbb{R}^d) \text{ for every } b > 1\}. \tag{2.34}$$

Let $T_\infty(\mathbb{R}^d) = \cap_{0\leqslant m<\infty} T_m(\mathbb{R}^d)$. In this way we obtain a decreasing sequence

$$B(\mathbb{R}^d) \supset T_0(\mathbb{R}^d) \supset T_1(\mathbb{R}^d) \supset T_2(\mathbb{R}^d) \supset \ldots \supset T_\infty(\mathbb{R}^d) \supset \mathfrak{S}(\mathbb{R}^d) \tag{2.35}$$

The last inclusion is clear because, for any Gaussian distribution μ, $\rho_b^{(\mu)}$ is Gaussian, and because, for any α-stable distribution μ with $0 < \alpha < 2$, $\mu \in L(\mathbb{R}^d)$ with k-function $r^{-\alpha}$, and thus μ is in $T(\mathbb{R}^d)$ and has α-stable $\rho_b^{(\mu)}$.

Theorem D. *The sequence* (2.32) *is transformed into the sequence* (2.35) *by the mapping* Υ, *that is,* (2.16) *and*

$$T_m(\mathbb{R}^d) = \Upsilon(L_m(\mathbb{R}^d)), \qquad \text{for } m = 0, 1, \ldots, \infty, \tag{2.36}$$

$$\mathfrak{S}(\mathbb{R}^d) = \Upsilon(\mathfrak{S}(\mathbb{R}^d)). \tag{2.37}$$

Moreover, we have

$$T_m(\mathbb{R}^d) \subsetneqq L_m(\mathbb{R}^d), \qquad \textit{for } m = 0, 1, \ldots, \tag{2.38}$$

$$T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d), \tag{2.39}$$

$$T_{m+1}(\mathbb{R}^d) = \Phi(T_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)), \qquad \text{for } m = 0, 1, \ldots, \infty, \tag{2.40}$$

where we understand $m+1 = \infty$ *for* $m = \infty$.

Relation (2.37) was shown in Barndorff-Nielsen and Thorbjørnsen (2002b) for $d = 1$. It is known that

$$L_{m+1}(\mathbb{R}^d) = \Phi(L_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)), \qquad \text{for } m = 0, 1, \ldots, \infty. \tag{2.41}$$

Assertion (2.40) is analogous to this. Thus $L_m(\mathbb{R}^d)$ and $T_m(\mathbb{R}^d)$ are the images of $ID(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$, respectively, by Φ^{m+1}, the $(m+1)$th iteration of Φ. A description of the domain of definition of Φ^{m+1} and a stochastic integral representation of Φ^{m+1} are known. See Jurek (1983), Sato and Yamazato (1983), also Rocha-Arteaga and Sato (2003) Theorems 46 and 49 and Remark 58.[1]

The Lévy measures of $T_m(\mathbb{R}^d)$ are characterized as follows.

Theorem E. *Let* $m \in \{0, 1, \ldots\}$. *Let* $\mu \in ID(\mathbb{R}^d)$. *Then* $\mu \in T_m(\mathbb{R}^d)$ *if and only if* $\mu \in L(\mathbb{R}^d)$ *and* $\nu = \nu^{(\mu)}$ *is either* $\nu = 0$ *or* $\nu \neq 0$, *having infinitely differentiable h-function* $h_\xi(u)$ *such that*

$$\begin{gathered} h_\xi^{(j)}(u) \geqslant 0 \text{ for } u \in \mathbb{R},\ 0 \leqslant j < m, \text{ and } h_\xi^{(m)}(-\log r) \\ \text{is completely monotone in } r > 0,\ \lambda\text{-a.e. } \xi, \end{gathered} \tag{2.42}$$

where $h_\xi^{(j)}$ *is the jth derivative of* h_ξ *and* λ *is the spherical component of* ν.

A characterization of $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ usign mixed-exponential distributions and Γ-distributions is as follows.

Definition 2.4. *Call Ux an elementary mixed-exponential variable in* $\mathbb{R}^d$ *(an elementary* Γ*-variable in* $\mathbb{R}^d$*) if x is a non-random non-zero vector in* $\mathbb{R}^d$ *and U is a real random variable whose distribution is a mixture of a finite number of exponential distributions (a real* Γ*-distributed random variable).*

[1] In line 4 of the remark, μ_m should be replaced by μ.

Theorem F. *The class $B(\mathbb{R}^d)$ ($T(\mathbb{R}^d)$) is the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary mixed-exponential variables in $\mathbb{R}^d$ (of all elementary Γ-variables in $\mathbb{R}^d$). Actually, μ is in $B(\mathbb{R}^d)$ ($T(\mathbb{R}^d)$) if and only if there are μ_n, $n = 1, 2, \ldots,$ with $\mu_n \to \mu$ such that each μ_n is the distribution of the sum of a finite number of independent elementary mixed-exponential variables in $\mathbb{R}^d$ (elementary Γ-variables in $\mathbb{R}^d$).*

Many examples of distributions in $T(\mathbb{R})$ supported on $\mathbb{R}_+$ are given in Bondesson (1992) and Steutel and van Harn (2004). As shown by Bondesson (1992, Theorem 7.3.1), all normal variance mixtures where the law of the variance is a generalized Γ-convolution belong to $T(\mathbb{R})$. (Any such mixture equals the law at time 1 of a subordination of Brownian motion by a generalized Γ-convolution subordinator.) We also note that if $X_1, \ldots, X_d$ are independent real random variables with $\mathcal{L}(X_j)$ in $T(\mathbb{R})$ ($B(\mathbb{R})$) for each j, then $\mathcal{L}(X) \in T(\mathbb{R}^d)$ ($B(\mathbb{R}^d)$) for $X = (X_j)_{1 \leqslant j \leqslant d}$.

We will prove the results above in the sections that follow. In the final section we will discuss several examples.

3. Proof of Theorems A and B

We prove Theorems A and B on the relationship of the classes $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ with the mapping Υ. We also show relation (2.14), Lemma 2.1, and Propositions 2.3 and 2.4.

Proof of Lemma 2.1. Let $c = \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x)$ and let N be a random variable on $\mathbb{R}^d$ with distribution $c^{-1}(|x|^2 \wedge 1)\nu(\mathrm{d}x)$. Let $R = |N|$ and $\Xi = N/|N|$. Define $\lambda^0 = \mathcal{L}(\Xi)$ and $\nu^0_\xi(B) = c\int_B (r^2 \wedge 1)^{-1} P(R \in \mathrm{d}r \,|\, \Xi = \xi)$, using the conditional distribution. Then λ^0 and $\{\nu^0_\xi\}$ satisfy (2.6)–(2.8) with the additional properties that $\lambda^0(S) = 1$ and $\int_0^\infty (r^2 \wedge 1)\nu^0_\xi(\mathrm{d}r) = 1$ for all $\xi \in S$.

The proof of the uniqueness is as follows. Let λ, $\{\nu_\xi\}$ and λ', $\{\nu'_\xi\}$ both satisfy (2.6)–(2.8). Define $a(\xi) = \int_0^\infty (r^2 \wedge 1)\nu_\xi(\mathrm{d}r)$ and $a'(\xi) = \int_0^\infty (r^2 \wedge 1)\nu'_\xi(\mathrm{d}r)$. By (2.7), $a(\xi)$ and $a'(\xi)$ are positive for all ξ. We have $a(\xi) < \infty$ for λ-a.e. ξ and $a'(\xi) < \infty$ for λ'-a.e. ξ, since $\int_S a(\xi)\lambda(\mathrm{d}\xi) = \int_S a'(\xi)\lambda'(\mathrm{d}\xi) = c < \infty$. For any $B \in \mathcal{B}(S)$,

$$c\lambda^0(B) = \int_{\{x:|x|^{-1}x \in B\}} (|x|^2 \wedge 1)\nu(\mathrm{d}x) = \int_B a(\xi)\lambda(\mathrm{d}\xi) = \int_B a'(\xi)\lambda'(\mathrm{d}\xi).$$

Hence λ^0, λ, and λ' are mutually absolutely continuous. By the uniqueness of the conditional distribution $P(R \in \mathrm{d}r \,|\, \Xi = \xi)$, we obtain $ca(\xi)^{-1}\nu_\xi(\mathrm{d}r) = \nu^0_\xi(\mathrm{d}r)$ and $ca'(\xi)^{-1}\nu'_\xi(\mathrm{d}r) = \nu^0_\xi(\mathrm{d}r)$ for λ^0-a.e. ξ. Letting $c(\xi) = a(\xi)/a'(\xi)$ with appropriate modification on a set of λ^0-measure 0, we obtain (2.9)–(2.11). □

Remark 3.1. By the uniqueness of a polar decomposition of ν in the sense of Lemma 2.1, the properties of μ in Definitions 2.1 and 2.2 of $T(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ do not depend on the choice of polar decompositions.

Remark 3.2. By an extension of Bernstein's theorem to the case with a parameter, for each $\mu \in B(\mathbb{R}^d)$ there exists a unique family $\{Q_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $Q_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0, \infty))$ and

$$l_\xi(r) = \int_0^\infty \mathrm{e}^{-ru} Q_\xi(\mathrm{d}u); \tag{3.1}$$

see the proof of Lemma 3.3 of Sato (1980) for the details. Here we have used $l_\xi(\infty) = 0$. We have $\int_0^\infty (r^2 \wedge 1) l_\xi(r)\mathrm{d}r = \int_0^\infty a(u) Q_\xi(\mathrm{d}u)$, where

$$a(u) = u^{-3} \int_0^u r^2 \mathrm{e}^{-r}\, \mathrm{d}r + u^{-1}\mathrm{e}^{-u}. \tag{3.2}$$

Since $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$,

$$\int_S \lambda(\mathrm{d}\xi) \int_0^\infty a(u) Q_\xi(\mathrm{d}u) < \infty. \tag{3.3}$$

Noting that $a(u) \sim u^{-1}$ as $u \downarrow 0$ and $a(u) \sim 2u^{-3}$ as $u \uparrow \infty$, we see that (3.3) is equivalent to

$$\int_S \lambda(\mathrm{d}\xi) \int_0^\infty (u^{-1} \wedge u^{-3}) Q_\xi(\mathrm{d}u) < \infty. \tag{3.4}$$

Similarly, for each $\mu \in T(\mathbb{R}^d)$ there exists a unique family $\{R_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $R_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0, \infty))$, and

$$k_\xi(r) = \int_0^\infty \mathrm{e}^{-ru} R_\xi(\mathrm{d}u). \tag{3.5}$$

This time we have $\int_0^\infty (r \wedge r^{-1}) k_\xi(r)\mathrm{d}r = \int_0^\infty b(u) R_\xi(\mathrm{d}u)$, with

$$b(u) = u^{-2} \int_0^u r\mathrm{e}^{-r}\, \mathrm{d}r + \int_u^\infty r^{-1}\mathrm{e}^{-r}\, \mathrm{d}r. \tag{3.6}$$

Thus we have

$$\int_S \lambda(\mathrm{d}\xi) \int_0^\infty b(u) R_\xi(\mathrm{d}u) < \infty, \tag{3.7}$$

which is equivalent to

$$\int_S \lambda(\mathrm{d}\xi) \left(\int_0^{1/2} \log \frac{1}{u} R_\xi(\mathrm{d}u) + \int_{1/2}^\infty u^{-2} R_\xi(\mathrm{d}u) \right) < \infty, \tag{3.8}$$

since $b(u) \sim \log(1/u)$ as $u \downarrow 0$ and $b(u) \sim u^{-2}$ as $u \uparrow \infty$.

Proof of (2.14). The inclusion $T(\mathbb{R}^d) \subset L(\mathbb{R}^d)$ is evident from Proposition 2.2 and Definition 2.1. If $k_\xi(r)$ is completely monotone, then so is $k_\xi(r) r^{-1}$, since the product of completely monotone functions is completely monotone. Hence $T(\mathbb{R}^d) \subset B(\mathbb{R}^d)$.

For $d = 1$, let us construct $\mu \in B(\mathbb{R}) \cap L(\mathbb{R})$ such that $\mu \notin T(\mathbb{R})$. Let

$$k(r) = \mathrm{e}^{-a_1 r} - \mathrm{e}^{-b_1 r} + \mathrm{e}^{-a_2 r}, \qquad r > 0,$$

with $0 < a_1 < b_1 < a_2$, and let $l(r) = k(r)r^{-1}$. Then $k(r)$ is not completely monotone, since $k(r) = \int_0^\infty \mathrm{e}^{-ru} Q(\mathrm{d}u)$ with a signed measure Q such that $Q(\{b_1\}) < 0$. But $l(r)$ is completely monotone, since

$$l(r) = \frac{\mathrm{e}^{-a_1 r} - \mathrm{e}^{-b_1 r}}{r} + \frac{\mathrm{e}^{-a_2 r}}{r} = \int_{a_1}^{b_1} \mathrm{e}^{-ru}\,\mathrm{d}u + \int_{a_2}^{\infty} \mathrm{e}^{-ru}\,\mathrm{d}u.$$

Hence the distribution μ given by $\hat{\mu}(z) = \exp \int_0^\infty (\mathrm{e}^{\mathrm{i}zr} - 1) l(r) \mathrm{d}r$ is in $B(\mathbb{R}) \setminus T(\mathbb{R})$; μ is in fact a mixture of exponential distributions with parameters a_1 and a_2 by Steutel's theorem (see Sato 1999, Lemma 51.14; or Steutel and van Harn 2004, Chapter VI, Proposition 3.4). We claim that for some choice of a_1, b_1, and a_2, $k(r)$ is decreasing so that $\mu \in L(\mathbb{R})$. Indeed, let $a_1 = 1 - \varepsilon$, $b_1 = 1$, and $a_2 = 1 + \varepsilon$ with $0 < \varepsilon < 1$. Then $k'(r) = \mathrm{e}^{-r}(1 - f(r))$, with $f(r) = (1-\varepsilon)\mathrm{e}^{\varepsilon r} + (1+\varepsilon)\mathrm{e}^{-\varepsilon r}$. We have $f(r_0) = \min_{r>0} f(r)$ when $\mathrm{e}^{2\varepsilon r_0} = (1+\varepsilon)/(1-\varepsilon)$. Hence $f(r_0) = 2(1-\varepsilon^2)^{1/2} \to 2$ as $\varepsilon \downarrow 0$. It follows that $k'(r) < 0$ for all $r > 0$ if ε is small enough. A d-dimensional example is given by taking this $k(r)$ for the radial component of a Lévy measure. □

Proof of Proposition 2.3. Let $\mu = \mu_{(A,\nu,\gamma)}$. We use a general result (an analogue of Theorem 2.7 of Rajput and Rosinski 1989) for integrability of functions with respect to an $\mathbb{R}^d$-valued independently scattered random measure. In order to show that a measurable function $f(t)$ is $\{X_t^{(\mu)}\}$-integrable, it suffices to show that, for any $0 < t_0 < \infty$,

$$\int_0^{t_0} \langle z, Az\rangle f(t)^2\,\mathrm{d}t < \infty,$$

$$\int_0^{t_0} \mathrm{d}t \int_{\mathbb{R}^d} (|f(t)x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty,$$

$$\int_0^{t_0} \left| \langle \gamma, f(t)z\rangle + \int_{\mathbb{R}^d} (g(f(t)z, x) - g(z, f(t)x))\nu(\mathrm{d}x) \right| \mathrm{d}t < \infty,$$

where

$$g(z, x) = \mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle/(1+|x|^2). \tag{3.9}$$

(The first condition is equivalent to $\int_0^{t_0} f(t)^2\,\mathrm{d}t < \infty$ if $A \neq 0$.) Hence, for our proof, it suffices to show that

$$\int_0^\infty \langle z, Az\rangle s^2 \mathrm{e}^{-s}\,\mathrm{d}s < \infty, \tag{3.10}$$

$$\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(\mathrm{d}x) < \infty, \tag{3.11}$$

$$\int_0^\infty \mathrm{e}^{-s} \left| \langle \gamma, sz\rangle + \int_{\mathbb{R}^d} (g(sz, x) - g(z, sx))\nu(\mathrm{d}x) \right| \mathrm{d}s < \infty. \tag{3.12}$$

Among these (3.10) is evident; (3.11) holds since it is

$$\int_{\mathbb{R}^d} |x|^2 \nu(\mathrm{d}x) \int_0^{1/|x|} s^2 \mathrm{e}^{-s}\, \mathrm{d}s + \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_{1/|x|}^{\infty} \mathrm{e}^{-s}\, \mathrm{d}s < \infty.$$

Since $\int_0^\infty s\mathrm{e}^{-s}\, \mathrm{d}s < \infty$, (3.12) follows from

$$\int_0^\infty \mathrm{e}^{-s} \left| \int_{\mathbb{R}^d} (g(sz, x) - g(z, sx)) \nu(\mathrm{d}x) \right| \mathrm{d}s < \infty. \tag{3.13}$$

Indeed, we can show (3.13) as

$$\begin{aligned} &\int_0^\infty s\mathrm{e}^{-s}\, \mathrm{d}s \int_{\mathbb{R}^d} \left| \langle z, x \rangle \left(\frac{1}{1 + s^2|x|^2} - \frac{1}{1+|x|^2} \right) \right| \nu(\mathrm{d}x) \\ &\leqslant |z| \int_{\mathbb{R}^d} \frac{|x|^3}{1+|x|^2} \nu(\mathrm{d}x) \int_0^\infty \frac{\mathrm{e}^{-s} s |1 - s^2|}{1 + s^2|x|^2}\, \mathrm{d}s \\ &\leqslant |z| I_1 \int_{|x| \leqslant 1} |x|^3 \nu(\mathrm{d}x) + |z| \int_{|x|>1} |x| I_2(x) \nu(\mathrm{d}x), \end{aligned}$$

where

$$I_1 = \int_0^\infty \mathrm{e}^{-s} s(1 + s^2) \mathrm{d}s,$$

$$I_2(x) = \int_0^1 \frac{s\, \mathrm{d}s}{1 + s^2|x|^2} + \int_1^\infty \frac{\mathrm{e}^{-s} s^3\, \mathrm{d}s}{1+|x|^2} = \frac{\log(1+|x|^2)}{2|x|^2} + \frac{\int_1^\infty \mathrm{e}^{-s} s^3\, \mathrm{d}s}{1+|x|^2}.$$

No restriction on $\mu = \mu_{(A,\nu,\gamma)}$ is needed. □

Proof of Proposition 2.4. (i) This is a consequence of Proposition 4.3 of Sato (2004), since we have Proposition 2.3. A direct check of the integrability asserted is also possible, as we have $|\mathrm{Re}\, C_\mu(z)| + |\mathrm{Im}\, C_\mu(z)| \leqslant c_1 + c_2|z|^2$ with constants c_1, c_2 depending on μ.

(ii) We have $\Upsilon\mu \in ID(\mathbb{R}^d)$ from Proposition 4.3 of Sato (2004). It follows from (i) that

$$C_{\Upsilon\mu}(z) = \int_0^\infty C_\mu(sz) \mathrm{e}^{-s}\, \mathrm{d}s \tag{3.14}$$

and hence, for $u > 0$, $C_{\Upsilon\mu}(u^{-1}z) = u\int_0^\infty C_\mu(vz)\mathrm{e}^{-uv}\, \mathrm{d}v$. That is, for each $z \in \mathbb{R}^d$, $u^{-1}C_{\Upsilon\mu}(u^{-1}z)$, $u > 0$, is the Laplace transform of $C_\mu(vz)$, $v > 0$. Therefore $C_\mu(vz)$ is determined by $C_{\Upsilon\mu}$ for almost every $v > 0$. Since $C_\mu(vz)$ is continuous in v, it is determined for all $v > 0$. Now let $v = 1$ to obtain our assertion.

(iii) Use $\{X_t^{(\mu_1)} + X_t^{(\mu_2)}\}$, where $\{X_t^{(\mu_1)}\}$ and $\{X_t^{(\mu_2)}\}$ are independent.

(iv) By a general result (see Lemma 2.7 and Corollary 4.4 of Sato 2004),

$$\tilde{A} = \int_0^1 (\log(1/t))^2 \, \mathrm{d}t A,$$

$$\tilde{\nu}(B) = \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d} 1_B(x \log(1/t)) \nu(\mathrm{d}x), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

$$\tilde{\gamma} = \int_0^1 \log(1/t) \left(\gamma - \int_{\mathbb{R}^d} x \left(\frac{1}{1+|x|^2} - \frac{1}{1+|(\log(1/t))x|^2} \right) \nu(\mathrm{d}x) \right) \mathrm{d}t.$$

These imply (2.22)–(2.24) by change of variables.

(v) Assume that $\mu_n = \mu_{(A_n, \nu_n, \gamma_n)} \to \mu = \mu_{(A,\mu,\nu)}$ as $n \to \infty$. Then $C_{\mu_n}(z) \to C_\mu(z)$ and $\operatorname{tr} A_n$, $\int (|x|^2 \wedge 1) \nu_n(\mathrm{d}x)$, and $|\gamma_n|$ are bounded. Since $\Upsilon \mu_n$ and $\Upsilon \mu$ have cumulant functions expressed as in (2.21) or (3.14) and since we have (iv), we can use the dominated convergence theorem to obtain $C_{\Upsilon \mu_n}(z) \to C_{\Upsilon \mu}(z)$, that is, $\Upsilon \mu_n \to \Upsilon \mu$.

Conversely, assume that $\tilde{\mu}_n = \Upsilon \mu_n \to \tilde{\mu}$. Let $(\tilde{A}_n, \tilde{\nu}_n, \tilde{\gamma}_n)$ and (A_n, ν_n, γ_n) be the triplets of $\tilde{\mu}_n$ and μ_n. We claim that $\{\mu_n\}$ is precompact. The following four conditions[2] combined are necessary and sufficient for precompactness of $\{\mu_n\}$: (a) $\sup_n \operatorname{tr} A_n < \infty$; (b) $\sup_n \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu_n(\mathrm{d}x) < \infty$; (c) $\lim_{l \to \infty} \sup_n \int_{|x|>l} \nu_n(\mathrm{d}x) = 0$; (d) $\sup_n |\gamma_n| < \infty$. Since $\{\tilde{\mu}_n\}$ is precompact, these four relations already hold with (A_n, ν_n, γ_n) replaced by $(\tilde{A}_n, \tilde{\nu}_n, \tilde{\gamma}_n)$. We denote them by (ã)–($\tilde{\text{d}}$). Then (a) follows from (2.22) and (ã); (b) follows from ($\tilde{\text{b}}$) since, by (2.23),

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \tilde{\nu}_n(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s} \, \mathrm{d}s \int_{\mathbb{R}^d} (|sx|^2 \wedge 1) \nu_n(\mathrm{d}x)$$

$$\geqslant \int_{|x| \leqslant 1} |x|^2 \nu_n(\mathrm{d}x) \int_0^1 s^2 \mathrm{e}^{-s} \, \mathrm{d}s + \int_{|x|>1} \nu_n(\mathrm{d}x) \int_1^\infty \mathrm{e}^{-s} \, \mathrm{d}s;$$

and (c) is obtained from (c̃) because

$$\int_{|x|>l} \tilde{\nu}_n(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s} \, \mathrm{d}s \int_{|x|>l/s} \nu_n(\mathrm{d}x) \geqslant \int_1^\infty \mathrm{e}^{-s} \, \mathrm{d}s \int_{|x|>l} \nu_n(\mathrm{d}x).$$

To see (d), use ($\tilde{\text{d}}$) and the estimate

$$\sup_n \left| \int_{\mathbb{R}^d} \frac{x|x|^2}{1+|x|^2} \nu_n(\mathrm{d}x) \int_0^\infty \frac{\mathrm{e}^{-s} s (1-s^2)}{1+s^2|x|^2} \, \mathrm{d}s \right| < \infty,$$

which is a consequence of (b) as in the proof of (3.13). This finishes the proof of precompactness of $\{\mu_n\}$. We can now choose a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ convergent to some μ. Thus $\Upsilon \mu_{n'} \to \Upsilon \mu$ and $\Upsilon \mu = \tilde{\mu}$. It follows from (ii) that μ does not depend on the choice of the subsequence. Hence $\mu_n \to \mu$.

(vi) Let $X_t = X_t^{(\mu)}$. Let $X'_t = X_1 - X_{(1-t)-}$ for $0 \leqslant t < 1$. Then $\{X'_t : 0 \leqslant t < 1\}$ is a process identical in law with $\{X_t : 0 \leqslant t < 1\}$ (Proposition 41.8 of Sato 1999). Let $\tilde{f}(t)$ be a function on $[0, \infty)$ equal to $\log(1/(1-t))$ for $0 \leqslant t < 1$ and 0 otherwise. Then $\tilde{f}(t)$ is

[2]There is an error in E12.5 of Sato (1999); a condition corresponding to (c) should be added.

$\{X_t\}$-integrable similarly to $f(t)$ of Proposition 2.3, and we have $\int_s^1 \log(1/t)\mathrm{d}X'_t = \int_0^{1-s} \tilde{f}(t)\mathrm{d}X_t$. Hence (2.25).

In order to show (2.26), first note that $\int_s^1 \log(1/t)\mathrm{d}X_t$ tends to $\int_0^1 \log(1/t)\mathrm{d}X_t$ a.s. as $s \downarrow 0$, since $\int_B f(t)\mathrm{d}X_t$, $B \in \mathcal{B}^0_{[0,\infty)}$, is an independently scattered random measure. By Theorem 4.7 of Sato (2004),

$$\int_s^1 \log\frac{1}{t}\,\mathrm{d}X_t = \int_s^1 \mathrm{d}X_t \int_t^1 \frac{1}{u}\,\mathrm{d}u = \int_s^1 \frac{\mathrm{d}u}{u}\int_s^u \mathrm{d}X_t = \int_s^1 \frac{X_u}{u}\,\mathrm{d}u - X_s \log\frac{1}{s}.$$

We have $X_s \log(1/s) \to 0$ a.s. as $s \downarrow 0$ by Proposition 47.11 of Sato (1999). Therefore $\lim_{s\downarrow 0}\int_s^1 (X_u/u)\mathrm{d}u$ exists a.s and (2.26) holds. □

Proof of Theorem A. Let $\mu \in ID(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon\mu$. Let $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. Then (2.17) holds. Thus, if $\nu = 0$, then $\tilde{\nu} = 0$ and $\tilde{\mu} \in B(\mathbb{R}^d)$. Assume that ν is non-zero and has polar decomposition (λ, ν_ξ). Then, for any non-negative measurable function f,

$$\int_{\mathbb{R}^d} f(x)\tilde{\nu}(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_{\mathbb{R}^d} f(sx)\nu(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_S \lambda(\mathrm{d}\xi)\int_0^\infty f(sr\xi)\nu_\xi(\mathrm{d}r)$$

$$= \int_S \lambda(\mathrm{d}\xi)\int_0^\infty \nu_\xi(\mathrm{d}r)r^{-1}\int_0^\infty \mathrm{e}^{-s/r} f(s\xi)\mathrm{d}s = \int_S \lambda(\mathrm{d}\xi)\int_0^\infty f(s\xi)\tilde{l}_\xi(s)\mathrm{d}s,$$

where $\tilde{l}_\xi(s)$ is defined by (2.18). Define a measure $\tilde{Q}_\xi$ on $(0, \infty)$ by

$$\tilde{Q}_\xi(B) = \int_0^\infty 1_B(r^{-1})r^{-1}\nu_\xi(\mathrm{d}r), \qquad B \in \mathcal{B}((0, \infty)).$$

Then $\tilde{Q}_\xi(B)$ is measurable in ξ and $\tilde{l}_\xi(s)$ is the Laplace transform of $\tilde{Q}_\xi$. Hence $\tilde{l}_\xi$ is completely monotone. Letting $\tilde{\lambda} = \lambda$ and $\tilde{\nu}_\xi(\mathrm{d}r) = \tilde{l}_\xi(r)\mathrm{d}r$, we see that $(\tilde{\lambda}, \tilde{\nu}_\xi)$ is a polar decomposition of $\tilde{\nu}$ and that $\tilde{\mu} \in B(\mathbb{R}^d)$.

Conversely, suppose that $\tilde{\mu} \in B(\mathbb{R}^d)$ with triplet $(\tilde{A}, \tilde{\nu}, \tilde{\gamma})$. If $\tilde{\nu} = 0$, then $\tilde{\mu} = \Upsilon\mu$ with $\mu = \mu_{(\tilde{A}/2, 0, \tilde{\gamma})}$ by Proposition 2.4(iv). Suppose that $\tilde{\nu} \neq 0$. Then, in a decomposition $(\tilde{\lambda}, \tilde{\nu}_\xi)$ of $\tilde{\nu}$, we have $\tilde{\nu}_\xi(\mathrm{d}r) = \tilde{l}_\xi(r)\mathrm{d}r$, where $\tilde{l}_\xi(r)$ is completely monotone in r and measurable in ξ. Thus there are measures $\tilde{Q}_\xi$ like the measures Q_ξ in Remark 3.2. Now define

$$\nu_\xi(B) = \int_0^\infty 1_B(u^{-1})u^{-1}\tilde{Q}_\xi(\mathrm{d}u).$$

Then ν_ξ is a measure on $(0, \infty)$ for each ξ and

$$\int_0^\infty f(r)\nu_\xi(\mathrm{d}r) = \int_0^\infty f(u^{-1})u^{-1}\tilde{Q}_\xi(\mathrm{d}u)$$

for all non-negative measurable functions f on $(0, \infty)$. Notice that it follows that

$$\int_0^\infty f(r)\tilde{Q}_\xi(\mathrm{d}r) = \int_0^\infty f(u^{-1})u^{-1}\nu_\xi(\mathrm{d}u)$$

for all non-negative measurable functions f on $(0, \infty)$. Hence we have (2.18). Let $\lambda = \tilde{\lambda}$. Then

$$\int_S \lambda(\mathrm{d}\xi)\int_0^\infty (r^2 \wedge 1)\nu_\xi(\mathrm{d}r) = \int_S \tilde{\lambda}(\mathrm{d}\xi)\int_0^\infty (u^{-2} \wedge 1)u^{-1}\tilde{Q}_\xi(\mathrm{d}u)$$

$$= \int_S \tilde{\lambda}(\mathrm{d}\xi)\left(\int_0^1 u^{-1}\tilde{Q}_\xi(\mathrm{d}u) + \int_1^\infty u^{-3}\tilde{Q}_\xi(\mathrm{d}u)\right) < \infty$$

by (3.4). Define ν by (2.8). Then ν is the Lévy measure of an infinitely divisible distribution and we can verify that

$$\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_{\mathbb{R}^d} f(sx)\nu(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\tilde{\nu}(\mathrm{d}x)$$

for all non-negative measurable functions f on $\mathbb{R}^d$. Define A and γ by (2.22) and (2.24) and let $\mu = \mu_{(A,\nu,\gamma)}$. Then $\tilde{\mu} = \Upsilon\mu$. □

Proof of Theorem B. Let $\mu \in L(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon\mu$. Let $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. If $\nu = 0$, then $\tilde{\nu} = 0$ and $\tilde{\mu} \in T(\mathbb{R}^d)$. Assume that $\nu \neq 0$. Then ν has a polar decomposition λ, $\nu_\xi(\mathrm{d}r) = k_\xi(r)r^{-1}\,\mathrm{d}r$ with decreasing $k_\xi(r)$. We claim that $\tilde{\mu} \in T(\mathbb{R}^d)$. For any non-negative measurable f,

$$\int f(x)\tilde{\nu}(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int f(sx)\nu(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_S \lambda(\mathrm{d}\xi)\int_0^\infty f(sr\xi)k_\xi(r)r^{-1}\,\mathrm{d}r$$

$$= \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_S \lambda(\mathrm{d}\xi)\int_0^\infty f(r\xi)k_\xi(rs^{-1})r^{-1}\,\mathrm{d}r$$

$$= \int_S \lambda(\mathrm{d}\xi)\int_0^\infty f(r\xi)r^{-1}\,\mathrm{d}r\int_0^\infty k_\xi(rs^{-1})\mathrm{e}^{-s}\,\mathrm{d}s.$$

Define $\tilde{k}_\xi(r)$ by the first equality in (2.20). Recall that $k_\xi(r)$ tends to 0 as $r \to \infty$. Let $k_\xi^\#(u)$ be the right-continuous modification of $k_\xi(1/u)$. Then

$$-\int 1_{[a,\infty)}(v)\mathrm{d}k_\xi(v) = \lim_{a'\uparrow a} k_\xi(a') = k_\xi^\#(a^{-1}) = \int 1_{(0,a^{-1}]}(u)\mathrm{d}k_\xi^\#(u)$$

$$= \int 1_{[a,\infty)}(u^{-1})\mathrm{d}k_\xi^\#(u)$$

for all $a > 0$. More generally,

$$-\int_0^\infty g(v)\mathrm{d}k_\xi(v) = \int_0^\infty g(u^{-1})\mathrm{d}k_\xi^\#(u)$$

for any non-negative measurable function g on $(0, \infty)$. Then

$$\tilde{k}_\xi(r) = -\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s\int_{r/s}^\infty \mathrm{d}k_\xi(v) = -\int_0^\infty \mathrm{d}k_\xi(v)\int_{r/v}^\infty \mathrm{e}^{-s}\,\mathrm{d}s$$

$$= -\int_0^\infty \mathrm{e}^{-r/v}\,\mathrm{d}k_\xi(v) = \int_0^\infty \mathrm{e}^{-ru}\,\mathrm{d}k_\xi^\#(u). \tag{3.15}$$

It follows that $\tilde{k}_\xi(r)$ is completely monotone. Hence $\tilde{\mu} \in T(\mathbb{R}^d)$.

Conversely, suppose $\tilde{\mu} \in T(\mathbb{R}^d)$ with triplet $(\tilde{A}, \tilde{\nu}, \tilde{\gamma})$. If $\tilde{\nu} = 0$, then $\tilde{\mu} = \Upsilon\mu$ with μ Gaussian and hence $\tilde{\mu} \in \Upsilon(L(\mathbb{R}^d))$. Suppose $\tilde{\nu} \neq 0$. Then we have a decomposition $(\tilde{\lambda}, \tilde{\nu}_\xi)$ of $\tilde{\nu}$ with $\tilde{\nu}_\xi(\mathrm{d}r) = \tilde{k}_\xi(r) r^{-1}\,\mathrm{d}r$, where $\tilde{k}_\xi(r)$ is completely monotone. Thus $\tilde{k}_\xi(r)$ is the Laplace transform of some $\tilde{R}_\xi(\mathrm{d}u)$. Let $k^\#_\xi(u) = \tilde{R}_\xi((0, u])$ and let $k_\xi(v)$ be the right-continuous modification of $k^\#_\xi(1/v)$. The calculation in (3.15) shows the first equality in (2.20). Hence we have

$$\int_{\mathbb{R}^d} f(x)\tilde{\nu}(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_S \tilde{\lambda}(\mathrm{d}\xi) \int_0^\infty f(sr\xi) k_\xi(r) r^{-1}\,\mathrm{d}r$$

for all non-negative measurable functions $f(x)$. Define $\lambda = \tilde{\lambda}$ and let ν be the measure with polar decomposition λ, $k_\xi(r) r^{-1}\,\mathrm{d}r$. Then we have (2.23) and

$$\begin{aligned}
\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(\mathrm{d}x) &= \int_S \lambda(\mathrm{d}\xi) \int_0^\infty (r^2 \wedge 1) k_\xi(r) r^{-1}\,\mathrm{d}r \\
&= \int_S \tilde{\lambda}(\mathrm{d}\xi) \int_0^\infty (r^2 \wedge 1) k^\#_\xi(r^{-1}) r^{-1}\,\mathrm{d}r = \int_S \tilde{\lambda}(\mathrm{d}\xi) \int_0^\infty (r \wedge r^{-1})\mathrm{d}r \int_0^{1/r} \tilde{R}_\xi(\mathrm{d}u) \\
&= \int_S \tilde{\lambda}(\mathrm{d}\xi) \left(\int_0^\infty \tilde{R}_\xi(\mathrm{d}u) \int_0^{1\wedge(1/u)} r\,\mathrm{d}r + \int_0^1 \tilde{R}_\xi(\mathrm{d}u) \int_1^{1/u} r^{-1}\,\mathrm{d}r \right) \\
&= \int_S \tilde{\lambda}(\mathrm{d}\xi) \left(\frac{1}{2}\int_0^1 \tilde{R}_\xi(\mathrm{d}u) + \frac{1}{2}\int_1^\infty u^{-2}\tilde{R}_\xi(\mathrm{d}u) + \int_0^1 \log\frac{1}{u}\,\tilde{R}_\xi(\mathrm{d}u) \right) < \infty
\end{aligned}$$

by (3.8) for $\tilde{R}_\xi$ in place of R_ξ. Hence, ν is the Lévy measure of some $\mu = \mu_{(A,\nu,\gamma)}$ in $L(\mathbb{R}^d)$. Here we choose A and γ to satisfy (2.22) and (2.24). Thus $\tilde{\mu} = \Upsilon\mu$. □

4. Proof of Theorems C and D

We give the proofs of Theorems C and D together with some general results on complete closedness in the strong sense.

Proof of Theorem C. (i) Let $\mu \in ID(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon\mu$. Let $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. We have

$$\begin{aligned}
\int_{|x|>2} \log|x|\tilde{\nu}(\mathrm{d}x) &= \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_{|x|>2/s} \log(s|x|)\nu(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \left(\int_{2/|x|}^\infty \mathrm{e}^{-s}\log s\,\mathrm{d}s + \mathrm{e}^{-2/|x|}\log|x| \right) \nu(\mathrm{d}x) = \int_{\mathbb{R}^d} h(x)\nu(\mathrm{d}x),
\end{aligned}$$

say. Then $h(x) = o(|x|^2)$ as $|x| \downarrow 0$ and $h(x) \sim \log|x|$ as $|x| \to \infty$. Thus, $\int_{|x|>2} \log |x|\tilde{\nu}(\mathrm{d}x) < \infty$ if and only if $\int_{|x|>2} \log|x|\nu(\mathrm{d}x) < \infty$.

(ii) If $\mu \in ID_{\log}(\mathbb{R}^d)$, then $\int_0^\infty |C_\mu(\mathrm{e}^{-t}z)|\mathrm{d}t < \infty$ and

$$C_{\Phi\mu}(z) = \int_0^\infty C_\mu(\mathrm{e}^{-t}z)\mathrm{d}t \tag{4.1}$$

(see the references given for (2.2) and (2.3)). If $\mu \in ID(\mathbb{R}^d)$, then $\int_0^\infty \mathrm{e}^{-s}|C_\mu(sz)|\mathrm{d}s < \infty$ and

$$C_{\Upsilon\mu}(z) = \int_0^\infty \mathrm{e}^{-s} C_\mu(sz)\mathrm{d}s \tag{4.2}$$

by Proposition 2.4(i). Let $\mu \in ID_{\log}(\mathbb{R}^d)$. Using $\Upsilon\mu \in ID_{\log}(\mathbb{R}^d)$ in (i), we have

$$C_{\Phi\Upsilon\mu}(z) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{e}^{-s} C_\mu(\mathrm{e}^{-t}sz)\mathrm{d}s, \tag{4.3}$$

$$C_{\Upsilon\Phi\mu}(z) = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^\infty C_\mu(\mathrm{e}^{-t}sz)\mathrm{d}t. \tag{4.4}$$

We claim that

$$\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^\infty |C_\mu(\mathrm{e}^{-t}sz)|\mathrm{d}t < \infty, \qquad \text{for } z \in \mathbb{R}^d. \tag{4.5}$$

If this is proved, then we can interchange the order of the integrations in (4.3) and (4.4) and obtain $\Phi\Upsilon\mu = \Upsilon\Phi\mu$.

The proof of (4.5) is as follows. Let $\mu = \mu_{(A,\nu,\gamma)}$. Then

$$|C_\mu(z)| \leqslant \frac{1}{2}(\mathrm{tr}\,A)|z|^2 + |\gamma|\,|z| + \int |g(z, x)|\nu(\mathrm{d}x),$$

where $g(z, x)$ is given by (3.9). Hence

$$|C_\mu(\mathrm{e}^{-t}sz)| \leqslant \frac{1}{2}(\mathrm{tr}\,A)\mathrm{e}^{-2t}s^2|z|^2 + |\gamma|\mathrm{e}^{-t}s|z| + \int_{\mathbb{R}^d} |g(z, \mathrm{e}^{-t}sx)|\nu(\mathrm{d}x)$$
$$+ \int |g(\mathrm{e}^{-t}sz, x) - g(z, \mathrm{e}^{-t}sx)|\nu(\mathrm{d}x) = I_1 + I_2 + I_3 + I_4,$$

say. Finiteness of $\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^\infty (I_1(s, t) + I_2(s, t))\mathrm{d}t$ is straightforward. Noting that $|g(z, x)| \leqslant c_z|x|^2/(1 + |x|^2)$ with a constant c_z depending on z, we have

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^\infty I_3(s, t)\mathrm{d}t &\leqslant c_z \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^\infty \frac{(\mathrm{e}^{-t}s|x|)^2}{1 + (\mathrm{e}^{-t}s|x|)^2}\,\mathrm{d}t \\
&= c_z \int \nu(\mathrm{d}x) \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s \int_0^{s|x|} \frac{u}{1 + u^2}\,\mathrm{d}u \\
&= \frac{c_z}{2} \int \nu(\mathrm{d}x) \int_0^\infty \mathrm{e}^{-s} \log(1 + s^2|x|^2)\mathrm{d}s \\
&\leqslant \frac{cc_z}{2} \int_{\mathbb{R}^d} |x|^2\nu(\mathrm{d}x) \int_0^{\sqrt{2}/|x|} \mathrm{e}^{-s}s^2\,\mathrm{d}s + cc_z \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_{\sqrt{2}/|x|}^\infty \mathrm{e}^{-s}(\log s + \log|x|)\mathrm{d}s,
\end{aligned}$$

which is finite since $\int_{|x|\leqslant 2}|x|^2\nu(\mathrm{d}x)<\infty$ and $\int_{|x|>2}\log|x|\nu(\mathrm{d}x)<\infty$. Here c is a constant such that $\log(1+v)\leqslant c(v1_{(0,2]}(v)+(\log v)1_{(2,\infty)}(v))$ for $v>0$. Concerning I_4, note that

$$|g(az,x)-g(z,ax)|=|\langle az,x\rangle|\frac{|x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)}\leqslant|z|\frac{|x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)}$$

for $a\in\mathbb{R}$. Then

$$\begin{aligned}\int_0^\infty I_4(s,t)\mathrm{d}t&\leqslant|z|\int_{\mathbb{R}^d}\frac{|x|^3\nu(\mathrm{d}x)}{1+|x|^2}\int_0^\infty\frac{\mathrm{e}^{-t}s+\mathrm{e}^{-3t}s^3}{1+\mathrm{e}^{-2t}s^2|x|^2}\,\mathrm{d}t\\&=|z|\int_{\mathbb{R}^d}\frac{|x|^3\nu(\mathrm{d}x)}{1+|x|^2}\int_0^{s|x|}\frac{u|x|^{-1}+u^3|x|^{-3}}{(1+u^2)u}\,\mathrm{d}u\\&\leqslant\frac{\pi}{2}|z|\int_{\mathbb{R}^d}\frac{|x|^2\nu(\mathrm{d}x)}{1+|x|^2}+|z|\int_{\mathbb{R}^d}\frac{\nu(\mathrm{d}x)}{1+|x|^2}\int_0^{s|x|}\frac{u^2\,\mathrm{d}u}{1+u^2}=J_1+J_2,\end{aligned}$$

say. Here J_1 does not depend on s and

$$\begin{aligned}\int_0^\infty\mathrm{e}^{-s}J_2(s)\mathrm{d}s&=|z|\int_{\mathbb{R}^d}\frac{\nu(\mathrm{d}x)}{1+|x|^2}\int_0^\infty\frac{u^2\,\mathrm{d}u}{1+u^2}\int_{u/|x|}^\infty\mathrm{e}^{-s}\,\mathrm{d}s\\&\leqslant|z|\int_{\mathbb{R}^d}\frac{\nu(\mathrm{d}x)}{1+|x|^2}\left(\int_0^1u^2\mathrm{e}^{-u/|x|}\,\mathrm{d}u+\int_1^\infty\mathrm{e}^{-u/|x|}\,\mathrm{d}u\right)\\&=|z|\int_{\mathbb{R}^d}\frac{|x|^3\nu(\mathrm{d}x)}{1+|x|^2}\int_0^{1/|x|}u^2\mathrm{e}^{-u}\,\mathrm{d}u+|z|\int_{\mathbb{R}^d}\frac{|x|\mathrm{e}^{-1/|x|}}{1+|x|^2}\nu(\mathrm{d}x)<\infty.\end{aligned}$$

This completes the proof of (4.5).

It follows from (4.3) and (4.5) that

$$\begin{aligned}C_{\Phi\Upsilon\mu}(z)&=\int_0^\infty\mathrm{d}t\int_0^\infty C_\mu(uz)\mathrm{e}^{t-u\mathrm{e}^t}\,\mathrm{d}u=\int_0^\infty C_\mu(uz)\mathrm{e}^{-u}u^{-1}\,\mathrm{d}u\\&=-\int_0^\infty C_\mu(uz)\mathrm{d}e_1(u)=\int_0^\infty C_\mu(e_1^*(t)z)\mathrm{d}t\end{aligned}\tag{4.6}$$

for $\mu\in ID_{\log}(\mathbb{R}^d)$. For such μ, we have from (4.5) that

$$\int_0^\infty|C_\mu(e_1^*(t)z)|\mathrm{d}t<\infty.\tag{4.7}$$

The function $e_1^*(t)$ is $\{X_t^{(\mu)}\}$-integrable for every $\mu=\mu_{(A,\nu,\gamma)}\in ID_{\log}(\mathbb{R}^d)$. This is because, for each $t_0\in(0,\infty)$, the integrals

$$\int_0^{t_0} \langle z, Az \rangle e_1^*(t)^2\, \mathrm{d}t, \qquad \int_0^{t_0} \mathrm{d}t \int_{\mathbb{R}^d} (|xe_1^*(t)|^2 \wedge 1)\nu(\mathrm{d}x),$$

$$\int_0^{t_0} |\langle \gamma, z \rangle| e_1^*(t) \mathrm{d}t, \qquad \int_0^{t_0} \mathrm{d}t \int_{\mathbb{R}^d} |g(e_1^*(t)z, x) - g(z, e_1^*(t)x)|\nu(\mathrm{d}x)$$

are finite. Indeed,

$$\int_0^{t_0} \mathrm{d}t \int_{\mathbb{R}^d} (|e_1^*(t)x|^2 \wedge 1)\nu(\mathrm{d}x) = \int_{e_1^*(t_0)}^{\infty} \mathrm{e}^{-s} s^{-1}\, \mathrm{d}s \int_{\mathbb{R}^d} (|sx|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$$

like (3.11), and finiteness of the other integrals is shown similarly. It follows from (4.3) and (4.4) that, if $\mu \in ID_{\log}(\mathbb{R}^d)$, then $\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}$ exists and equals $\Phi\Upsilon\mu$ in distribution.

It remains to show that $\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}$ does not exist if $\mu \in ID(\mathbb{R}^d)\backslash ID_{\log}(\mathbb{R}^d)$. It is easy to see that

$$e_1(s) \sim \mathrm{e}^{-s}s^{-1} \quad (s \to \infty), \qquad e_1(s) \sim \log(1/s) \quad (s \downarrow 0).$$

It follows that

$$e_1^*(t) \sim c\mathrm{e}^{-t} \quad (t \to \infty), \qquad e_1^*(t) \sim \log(1/t) \quad (t \downarrow 0)$$

with some constant $c > 0$, for we have

$$\lim_{t\to\infty} \frac{e_1^*(t)}{\mathrm{e}^{-t}} = \lim_{s\downarrow 0} \frac{s}{\mathrm{e}^{-e_1(s)}} = \lim_{s\downarrow 0} \mathrm{e}^{e_1(s)+\log s} = \exp\left(\int_1^\infty \mathrm{e}^{-u}u^{-1}\, \mathrm{d}u - \int_0^1 (1-\mathrm{e}^{-u})u^{-1}\, \mathrm{d}u\right),$$

$$\lim_{t\downarrow 0} \frac{e_1^*(t)}{\log(1/t)} = \lim_{s\to\infty} \frac{s}{-\log e_1(s)} = \lim_{s\to\infty} \frac{1}{-e_1'(s)/e_1(s)} = \lim_{s\to\infty} \frac{e_1(s)}{\mathrm{e}^{-s}s^{-1}} = 1.$$

Let $\mu \in ID(\mathbb{R}^d)$ and suppose that $\int_0^\infty e_1^*(t)\mathrm{d}X_t^{(\mu)}$ exists and has distribution $\tilde{\mu}$. Let $t_n \to \infty$ and denote $\tilde{\mu}_n = \mathcal{L}(\int_0^{t_n} e_1^*(t)\mathrm{d}X_t^{(\mu)})$. Then $\tilde{\mu}_n \to \tilde{\mu}$. Let $\tilde{\nu}_n = \nu^{(\tilde{\mu}_n)}$, $\tilde{\nu} = \nu^{(\tilde{\mu})}$, and $\nu = \nu^{(\mu)}$. Then $\int f(x)\tilde{\nu}_n(\mathrm{d}x) \to \int f(x)\tilde{\nu}(\mathrm{d}x)$ for all bounded continuous functions f vanishing on a neighbourhood of 0 (Sato 1999, Theorem 8.7). Choose $t_0 > 0$ such that $e_1^*(t) \geqslant c\mathrm{e}^{-t}/2$ for $t > t_0$. Since $\tilde{\nu}_n(B) = \int_0^{t_n} \mathrm{d}t \int 1_B(e_1^*(t)x)\nu(\mathrm{d}x)$, we obtain

$$\int_{|x|>1} \tilde{\nu}_n(\mathrm{d}x) = \int_0^{t_n} \mathrm{d}t \int 1_{\{|x|>1/e_1^*(t)\}}(x)\nu(\mathrm{d}x) \geqslant \int_{t_0}^{t_n} \mathrm{d}t \int 1_{\{|x|>2\mathrm{e}^t/c\}}(x)\nu(\mathrm{d}x)$$

$$= \int_{\mathbb{R}^d} \nu(\mathrm{d}x) \int_{(t_0,t_n]\cap(0,\log(c|x|/2))} \mathrm{d}t \to \int_{\{\log(c|x|/2)>t_0\}} (\log(c|x|/2) - t_0)\nu(\mathrm{d}x).$$

Hence $\int_{|x|>a} \log|x|\nu(\mathrm{d}x) < \infty$ for some a, that is, $\mu \in ID_{\log}(\mathbb{R}^d)$.

(iii) is a consequence of (i) and (ii) combined with Theorems A and B. □

As in Maejima *et al.* (1999), a class M of distributions on $\mathbb{R}^d$ is said to be *completely closed in the strong sense* if the following are satisfied:

(ccs1) M is a subclass of $ID(\mathbb{R}^d)$.
(ccs2) $\mu_1, \mu_2 \in M$ implies $\mu_1 * \mu_2 \in M$.

(ccs3) $\mu_n \in M$ $(n = 1, 2, \ldots)$ and $\mu_n \to \mu$ imply $\mu \in M$.

(ccs4) If X is an $\mathbb{R}^d$-valued random variable with $\mathcal{L}(X) \in M$, then $\mathcal{L}(aX + b) \in M$ for any $a > 0$ and $b \in \mathbb{R}^d$.

(ccs5) $\mu \in M$ implies $\mu^{s*} \in M$ for any $s > 0$.

In the following we sometimes omit $\mathbb{R}^d$ in writing $ID_{\log}(\mathbb{R}^d)$, $L_m(\mathbb{R}^d)$ or $T_m(\mathbb{R}^d)$.

Lemma 4.1. *Let M be a class of distributions on $\mathbb{R}^d$, completely closed in the strong sense. Then the following statements are true.*

(i) *The classes $\Upsilon(M)$ and $\Phi(M \cap ID_{\log}(\mathbb{R}^d))$ are subclasses of M.*

(ii) $\Phi(M \cap ID_{\log}(\mathbb{R}^d)) = \{\sigma \in L(\mathbb{R}^d) : \rho_b^{(\sigma)} \in M \text{ for all } b > 1\}$, *where $\rho_b^{(\sigma)}$ is defined by (2.30) with σ in place of μ.*

(iii) *The classes $\Upsilon(M)$ and $\Phi(M \cap ID_{\log}(\mathbb{R}^d))$ are completely closed in the strong sense.*

Proof. (i) Let $\mu \in M$ and $X_t = X_t^{(\mu)}$. Let $\sigma = \Upsilon\mu = \mathcal{L}(I)$, where $I = \int_0^1 \log(1/t)\mathrm{d}X_t$. For any $s_n \downarrow 0$ let $\sigma_n = \mathcal{L}(I_n)$, where $I_n = \int_{s_n}^1 \log(1/t)\mathrm{d}X_t$. By Proposition 4.5 of Sato (2004), I_n is the limit in probability of a sequence $\int_{s_n}^1 f_m(t)\mathrm{d}X_t$ as $m \to \infty$, where $f_m(t)$ is a non-negative step function for each m. We see that $\mathcal{L}(\int_{s_n}^1 f_m(t)\mathrm{d}X_t) \in M$ from (ccs2), (ccs4) and (ccs5). Thus $\sigma_n \in M$ by (ccs3). As $n \to \infty$, I_n tends to I in probability and thus $\sigma_n \to \sigma$. Hence $\sigma \in M$. The proof that $\Phi\mu \in M$ for $\mu \in M \cap ID_{\log}$ is similar, using (2.3).

(ii) Suppose that $\mu \in M \cap ID_{\log}$ and $\sigma = \Phi\mu$. Then $\sigma \in L$ by (2.2). Notice that

$$b^{-1}\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t = \int_0^\infty \mathrm{e}^{-(t+\log b)}\,\mathrm{d}X_t \overset{\mathrm{d}}{=} \int_{\log b}^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t,$$

$$\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t = \int_{\log b}^\infty \mathrm{e}^{-t}\,\mathrm{d}X_t + \int_0^{\log b} \mathrm{e}^{-t}\,\mathrm{d}X_t,$$

for $X_t = X_t^{(\mu)}$ and $b > 1$, and thus $\rho_b^{(\sigma)} = \mathcal{L}(\int_0^{\log b}\mathrm{e}^{-t}\,\mathrm{d}X_t)$. Hence $\rho_b^{(\sigma)} \in M$ as in the proof of (i).

Conversely, suppose that $\sigma \in L$ and $\rho_b^{(\sigma)} \in M$ for all $b > 1$. Choosing $\mu \in ID_{\log}$ with $\Phi\mu = \sigma$, we see that $C_{\rho_b^{(\sigma)}}(z) = \int_0^{\log b} C_\mu(\mathrm{e}^{-t}z)\mathrm{d}t$. Let $g_b(z)$ be the cumulant function of $(\rho_b^{(\sigma)})^{(1/\log b)*} \in M$. Then $g_b(z) = (1/\log b)\int_0^{\log b} C_\mu(\mathrm{e}^{-t}z)\mathrm{d}t$, which tends to $C_\mu(z)$ as $b \downarrow 1$. Hence $(\rho_b^{(\sigma)})^{(1/\log b)*} \to \mu$ and $\mu \in M$.

(iii) Properties (ccs1)–(ccs3) for $\Upsilon(M)$ follow from Proposition 2.4. To see (ccs4), note that $\int_0^1 \log(1/t)\mathrm{d}t = 1$ and $a\int_0^1 \log(1/t)\mathrm{d}X_t^{(\mu)} + b = \int_0^1 \log(1/t)dX'_t$, where $X'_t = aX_t^{(\mu)} + tb$. For (ccs5), note that $sC_{\Upsilon\mu}(z) = s\int_0^1 C_\mu(z\log(1/t))\mathrm{d}t = \int_0^1 C_{\mu^{s*}}(z\log(1/t))\mathrm{d}t$.

Similarly, we can prove (ccs1), (ccs2), (ccs4) and (ccs5) for $\tilde{M} = \Phi(M \cap ID_{\log})$. To prove (ccs3), suppose that $\sigma_n \in \tilde{M}$ and $\sigma_n \to \sigma$ as $n \to \infty$. Then, by (ii), $\rho_b^{(\sigma_n)} \in M$. Since the characteristic function of $\rho_b^{(\sigma_n)}$ equals $\hat{\sigma}_n(z)/\hat{\sigma}_n(b^{-1}z)$, which tends to a continuous function $\hat{\sigma}(z)/\hat{\sigma}(b^{-1}z)$ as $n \to \infty$, $\rho_b^{(\sigma_n)}$ tends to some $\rho \in M$. We have $\hat{\sigma}(z) = \hat{\sigma}(b^{-1}z)\widehat{\rho}(z)$. Hence $\sigma \in \tilde{M}$ again by (ii). $\square$

Proof of Corollary to Theorem C. By Lemma 4.1(iii) it follows from Theorem A(i) that $B(\mathbb{R}^d)$ is completely closed in the strong sense. Hence, by Lemma 4.1(ii), we obtain (2.33) from (2.28) of Theorem C. □

Proof of Theorem D. Let us prove (2.36). Although (2.41) and the complete closedness in the strong sense of $L_m(\mathbb{R}^d)$ are known facts, it is more natural to reprove them and to prove the complete closedness in the strong sense of $T_m(\mathbb{R}^d)$, together with the proof of (2.36). For $m = 0$, (2.36) has already been proved in Theorem B. To prove it for $m = 1$, first note that L_0 is completely closed in the strong sense by Lemma 4.1(iii) and (2.2). Hence, so is T_0 by (2.36) for $m = 0$. Lemma 4.1(ii) says that $L_1 = \Phi(L_0 \cap ID_{\log})$. Now we have

$$T_1 = \Phi(T_0 \cap ID_{\log}) = \Phi(\Upsilon(L_0) \cap ID_{\log}) = \Phi(\Upsilon(L_0 \cap ID_{\log}))$$
$$= \Upsilon\Phi(L_0 \cap ID_{\log}) = \Upsilon(L_1),$$

using definition (2.34) of T_1, Lemma 4.1(ii), Theorem C(i) and Theorem C(ii) consecutively. This is (2.36) for $m = 1$. Continuing this procedure, we obtain (2.36), (2.40), (2.41), and the complete closedness in the strong sense of $L_m(\mathbb{R}^d)$ and $T_m(\mathbb{R}^d)$ for all finite m. It follows that (2.36) also holds for $m = \infty$. Moreover, (2.40) and (2.41) also hold for $m = \infty$, since $T_\infty = \{\mu \in L : \rho_b^{(\mu)} \in T_\infty \text{ for every } b > 1\}$ from (2.34), and similarly for L_∞.

Let us show (2.37). Denote by $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbb{R}^d)$ the class of α-stable distributions on $\mathbb{R}^d$. It is enough to show that $\Upsilon(\mathfrak{S}_\alpha) = \mathfrak{S}_\alpha$. This is evident in the case $\alpha = 2$ (Gaussian). Let $\mu \in \mathfrak{S}_\alpha$ with $0 < \alpha < 2$. Then it has k-function $k_\xi(r) = r^{-\alpha}$. Thus by (2.20), $\Upsilon\mu$ has k-function $\int_0^\infty r^{-\alpha}s^\alpha \mathrm{e}^{-s}\,\mathrm{d}s = \Gamma(\alpha+1)r^{-\alpha}$. Thus $\Upsilon\mu \in \mathfrak{S}_\alpha$. On the other hand, this shows that, for any $\tilde{\mu} \in \mathfrak{S}_\alpha$, there is a $\mu \in \mathfrak{S}_\alpha$ such that $\tilde{\mu} = \Upsilon\mu$.

The assertion $T_m \subset L_m$ for all finite m is a consequence of (2.36) and Lemma 4.1(i). But we have to show that the inclusion is strict. Define

$$ID_{\log^n}(\mathbb{R}^d) = \left\{\mu \in ID(\mathbb{R}^d) : \int_{|x|>2} (\log|x|)^n \mu(\mathrm{d}x) < \infty\right\}$$

for $n = 1, 2, \ldots$. The condition here is equivalent to finiteness of $\int_{|x|>2}(\log|x|)^n \nu^{(\mu)}(\mathrm{d}x)$. Let $ID_{\log^0}(\mathbb{R}^d) = ID(\mathbb{R}^d)$. It is known that

$$\Phi(ID_{\log^{n+1}}(\mathbb{R}^d)) = L(\mathbb{R}^d) \cap ID_{\log^n}(\mathbb{R}^d), \qquad \text{for } n = 0, 1, \ldots, \tag{4.8}$$

$$L_m(\mathbb{R}^d) = \Phi^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d)), \qquad \text{for } m = 0, 1, \ldots \tag{4.9}$$

(see the references given after (2.41)). The proof of (2.14) actually showed that $B \cap L_0 \cap ID_{\log^n} \underset{\neq}{\supset} T_0 \cap ID_{\log^n}$ for $n = 0, 1, \ldots$. Hence $L_0 \cap ID_{\log^n} \underset{\neq}{\supset} T_0 \cap ID_{\log^n}$ for $n = 0, 1, \ldots$. Applying Φ and using (2.40) and (2.41), we obtain $L_1 \cap ID_{\log^n} \underset{\neq}{\supset} T_1 \cap ID_{\log^n}$ for $n = 0, 1, \ldots$. Repeating this, we have $L_m \cap ID_{\log^n} \underset{\neq}{\supset} T_m \cap ID_{\log^n}$ for $m = 0, 1, \ldots$ and $n = 0, 1, \ldots$. For $n = 0$ this is (2.38).

Finally, we prove (2.39). It follows from $T_m \subset L_m$ for finite m that $T_\infty \subset L_\infty$. On the other hand, we know that $\mathfrak{S} \subset T_\infty$ and that T_∞ is completely closed in the strong sense. Since L_∞ is the smallest class containing $\mathfrak{S}$ and closed under convolution and convergence, we have $T_\infty \supset L_\infty$. □

5. Proof of Theorem E

For $a > 0$, let Δ_a be the difference operator $\Delta_a f(u) = f(u+a) - f(u)$, $u \in \mathbb{R}$, and let Δ_a^n be its nth iteration. Clearly

$$\Delta_a^n f(u) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(u + ja)$$

for $n = 0, 1, \ldots$. We say that a function $f(u)$ is *monotone of order* n if $\Delta_a^j f(u) \geqslant 0$, $j = 0, 1, \ldots, n$, for any $a > 0$ and $u \in \mathbb{R}$. When f is monotone of order n for all $n = 0, 1, \ldots,$ f is called *absolutely monotone*. A characterization of distributions in the class $L_m(\mathbb{R}^d)$ in terms of Lévy measures is given as follows (Sato 1980).

Proposition 5.1. *Let $\mu \in L_0(\mathbb{R}^d)$ with $\nu = \nu^{(\mu)}$ such that $\nu = 0$ or $\nu \neq 0$ with spherical component λ and h-function $h_\xi(u)$.*

(i) *Let $m \in \{1, 2, \ldots\}$. Then $\mu \in L_m(\mathbb{R}^d)$ if and only if either $\nu = 0$, or $\nu \neq 0$ with $h_\xi(u)$ being monotone of order $m+1$ in u for λ-a.e. ξ.*
(ii) *We have $\mu \in L_\infty(\mathbb{R}^d)$ if and only if either $\nu = 0$, or $\nu \neq 0$ with $h_\xi(u)$ being absolutely monotone in u for λ-a.e. ξ.*

Proof of Theorem E. Let us denote by T'_m the class of $\mu \in L_0$ such that either $\nu = 0$ or $\nu \neq 0$ with h-function satisfying (2.42). First, notice that condition (2.42) is equivalent to the condition that

$$h_\xi^{(j)}(-\log r) \text{ is completely monotone in } r > 0, \text{ for } j = 0, 1, \ldots, m, \lambda\text{-a.e. } \xi. \tag{5.1}$$

Indeed, this clearly implies (2.42). On the other hand, if (2.42) holds, then $-(\mathrm{d}/\mathrm{d}r)(h^{(m-1)}(-\log r)) = h^{(m)}(-\log r) r^{-1}$ is completely monotone, and thus, since $h^{(m-1)} \geqslant 0$, $h^{(m-1)}(-\log r)$ is itself completely monotone, and so on. Since $h_\xi(-\log r) = k_\xi(r)$, we have $T'_0 = T_0$ by Definition 2.1. Let us prove $T'_m = T_m$ for all finite m.

Part 1 (Proof that $T_m \subset T'_m$). Assume that $1 \leqslant m < \infty$ and let $\tilde{\mu} \in T_m$. By virtue of (2.36) of Theorem D, there is $\mu \in L_m$ such that $\tilde{\mu} = \Upsilon\mu$. Let $\nu = \nu^{(\mu)}$ and $\tilde{\nu} = \nu^{(\tilde{\mu})}$. If $\nu = 0$, then $\tilde{\nu} = 0$ and $\tilde{\mu} \in T'_m$. Assume that $\nu \neq 0$ and let k_ξ and h_ξ ($\tilde{k}_\xi$ and $\tilde{h}_\xi$) be the k-function and h-function of ν ($\tilde{\nu}$). For notational simplicity, we omit ξ in writing these functions. By Proposition 5.1, h is monotone of order $m+1$, and by Lemma 3.2 of Sato (1980), h is $m-1$ times continuously differentiable, $h^{(j)}$ is non-negative for $j = 0, 1, \ldots, m-1$, and $h^{(m-1)}$ is increasing and convex. Thus there exists the Radon–Nikodym derivative $h^{(m)}$ of $h^{(m-1)}$ such that $h^{(m)}$ is non-negative and increasing. We take $h^{(m)}$ as right-continuous. We see that, for $j = 1, \ldots, m$, $h^{(j)}$ is non-negative, increasing, and satisfies $h^{(j)}(-\infty) = 0$ and $h^{(j-1)}(u) = \int_{-\infty}^u h^{(j)}(v)\mathrm{d}v$. The function $\tilde{h}$ is of class C^∞, because $\tilde{\mu} \in T_m \subset T_0$. We claim that

$$\tilde{h}^{(j)}(-\log r) = \int_0^\infty \mathrm{e}^{-ru}\, \mathrm{d}h^{(j)}(\log u), \qquad \text{for } j = 0, 1, \ldots, m. \tag{5.2}$$

Since $\tilde{h}(-\log r) = \tilde{k}(r)$, (5.2) holds for $j = 0$ by virtue of (2.20). Assume that (5.2) is true for a given $j < m$. Then, for any $0 < r_1 < r_2$,

$$\begin{aligned}
\int_{r_1}^{r_2} \tilde{h}^{(j)}(-\log r)\frac{\mathrm{d}r}{r} &= -(\tilde{h}^{(j)}(-\log r_2) - \tilde{h}^{(j)}(-\log r_1)) \\
&= -\int_0^\infty (\mathrm{e}^{-r_2 u} - \mathrm{e}^{-r_1 u})\mathrm{d}h^{(j)}(\log u) = \int_0^\infty h^{(j+1)}(\log u)\frac{\mathrm{d}u}{u}\int_{r_1}^{r_2} \mathrm{e}^{-ru} u\, \mathrm{d}r \\
&= \int_0^\infty \mathrm{d}u \int_0^u \mathrm{d}h^{(j+1)}(\log v)\int_{r_1}^{r_2} \mathrm{e}^{-ru}\, \mathrm{d}r \\
&= \int_0^\infty \mathrm{d}h^{(j+1)}(\log v)\int_v^\infty \mathrm{d}u \int_{r_1}^{r_2} \mathrm{e}^{-ru}\, \mathrm{d}r = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r}\int_0^\infty \mathrm{e}^{-rv}\, \mathrm{d}h^{(j+1)}(\log v)
\end{aligned}$$

since $\int_v^\infty \mathrm{d}u \int_{r_1}^{r_2} \mathrm{e}^{-ru}\, \mathrm{d}r = \int_{r_1}^{r_2} \mathrm{e}^{-rv} r^{-1}\, \mathrm{d}r$. Hence (5.2) holds with $j+1$ in place of j for almost all r. As both sides of (5.2) are continuous, it follows that (5.2) holds for all $r > 0$. This completes the proof of (5.2). Hence (5.1) holds, that is, $\tilde{\mu} \in T'_m$.

Part 2 (Proof that $T'_m \subset T_m$). We use induction in m. We already know that $T'_0 = T_0$. Given $1 \leqslant m < \infty$, assume that $T'_{m-1} \subset T_{m-1}$. Let $\tilde{\mu} \in T'_m$. Then $\tilde{\mu} \in T_0$ and we can find $\mu \in L_0$ such that $\tilde{\mu} = \Upsilon \mu$. In order to show $\tilde{\mu} \in T_m$, it is enough to show $\mu \in L_m$, again by Theorem D. Let $\tilde{\nu} = \nu^{(\tilde{\mu})}$ and $\nu = \nu^{(\mu)}$. If $\tilde{\nu} = 0$, then $\tilde{\mu}$ and μ are Gaussian and $\mu \in L_m$. Suppose $\tilde{\nu} \neq 0$. Omitting ξ in the subscript again, let k, h, $\tilde{k}$, and $\tilde{h}$ be as in Part 1. Using Part 1, we have $T'_{m-1} = T_{m-1}$. Since $\tilde{\mu} \in T'_m \subset T'_{m-1}$, we have $\mu \in L_{m-1}$ and thus h is monotone of order m. Moreover, the equality in (5.2) holds for $j = 0, 1, \ldots, m-1$. It follows from $\tilde{\mu} \in T'_m$ that, for $j = 0, \ldots, m$, not only $\tilde{h}^{(j)}(-\log r)$ is completely monotone but also $\tilde{h}^{(j)}(-\infty) = 0$. Indeed, $\tilde{h}(-\infty) = 0$ since $\tilde{k}(\infty) = 0$, $\tilde{h}'(-\infty) = 0$ since $\tilde{h}(u_2) - \tilde{h}(u_1) = \int_{u_1}^{u_2} \tilde{h}'(u)\mathrm{d}u$, and so on. Therefore, $\tilde{h}^{(m)}(-\log r)$ is the Laplace transform of a measure σ on $(0, \infty)$. Now

$$\begin{aligned}
\tilde{h}^{(m-1)}(-\log r) &= \int_r^\infty \tilde{h}^{(m)}(-\log u)u^{-1}\, \mathrm{d}u = \int_r^\infty u^{-1}\, \mathrm{d}u \int_0^\infty \mathrm{e}^{-uv}\sigma(\mathrm{d}v) \\
&= \int_0^\infty \sigma(\mathrm{d}v)\int_r^\infty \mathrm{e}^{-uv}u^{-1}\, \mathrm{d}u = \int_0^\infty \sigma(\mathrm{d}v)\int_v^\infty \mathrm{e}^{-ru}u^{-1}\, \mathrm{d}u \\
&= \int_0^\infty \mathrm{e}^{-ru}u^{-1}\sigma((0, u])\mathrm{d}u.
\end{aligned}$$

This, together with the equality in (5.2) with $j = m-1$, implies that $\mathrm{d}h^{(m-1)}(\log u) = u^{-1}\sigma((0, u])\mathrm{d}u$. It follows that the Radon–Nikodym derivative of $h^{(m-1)}(u)$ exists and has a non-negative increasing version $h^{(m)}(u)$. Indeed, $h^{(m)}(\log u) = \sigma((0, u])$. Hence h is monotone of order $m+1$. Thus $\mu \in L_m$, completing the proof. $\square$

Remark 5.1. It follows from Theorem E and (5.1) that $\mu \in T_\infty(\mathbb{R}^d)$ if and only if $\mu \in L(\mathbb{R}^d)$ and $\nu = \nu^{(\mu)}$ is either $\nu = 0$, or $\nu \neq 0$ having h-function $h_\xi(u)$ such that $h_\xi^{(j)}(-\log r)$ is completely monotone in $r > 0$ for all $j = 0, 1, \ldots,$ λ-a.e. ξ, where λ is the spherical component of ν. This property of the h-function is equivalent to the absolute monotonicity of $h_\xi(u)$ in u, λ-a.e. ξ. We can prove this directly, but this is also a consequence of $T_\infty = L_\infty$ in (2.39) and of Proposition 5.1(ii).

6. Proof of Theorem F

We prove the characterization of $B(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ by elementary mixed-exponential variables and elementary Γ-variables in $\mathbb{R}^d$.

Part 1 (Characterization of $B(\mathbb{R}^d)$). Let B^0 be the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary mixed-exponential variables in $\mathbb{R}^d$. In order to prove $B^0 = B(\mathbb{R}^d)$, it is enough to check the following facts:

(a) $B(\mathbb{R}^d)$ is closed under convolution and convergence.
(b) $\mathcal{L}(Ux) \in B(\mathbb{R}^d)$ for all elementary mixed-exponential variables Ux in $\mathbb{R}^d$.
(c) $\delta_x \in B^0$ for all $x \in \mathbb{R}^d$.
(d) If $\mu = \mu_{(0,\nu,0)} \in B(\mathbb{R}^d)$, then $\mu \in B^0$.
(e) If $\mu = \mu_{(A,0,0)}$, then $\mu \in B^0$.

Indeed, (a) and (b) imply $B(\mathbb{R}^d) \supset B^0$; (c)–(e) imply $B(\mathbb{R}^d) \subset B^0$.

Proof of (a). Closedness under convolution is evident. Since $B(\mathbb{R}^d) = \Upsilon(ID(\mathbb{R}^d))$, closedness under convergence is proved in Proposition 2.4(v).

Proof of (b). Let

$$P(U \in B) = \sum_{j=1}^{n} c_j \int_{B\cap(0,\infty)} a_j e^{-a_j s}\, ds, \qquad B \in \mathcal{B}(\mathbb{R}),$$

with $c_j > 0$, $\sum_{j=1}^n c_j = 1$, and $0 < a_1 < \ldots < a_n < \infty$. Then, by Lemma 51.14 of Sato (1999),

$$Ee^{ivU} = \exp\int_0^\infty (e^{ivr} - 1)l(r)dr, \qquad \text{with } l(r) = \int_0^\infty e^{-ru} \sum_{j=1}^{n} 1_{(a_j,a_j')}(u)du,$$

for $v \in \mathbb{R}$ with $a_1 < a_1' < a_2 < a'_2 < a_3 < \ldots < a_n < a_n' = \infty$. Hence, for $x \neq 0$,

$$C_{Ux}(z) = \int_0^\infty (e^{i\langle z,x\rangle r} - 1)l(r)dr = \int_S \delta_{x/|x|}(d\xi)\int_0^\infty (e^{i\langle z,\xi\rangle |x| r} - 1)l(r)dr$$

$$= \int_S \delta_{x/|x|}(d\xi)\int_0^\infty (e^{i\langle z,\xi\rangle r} - 1)l(r/|x|)dr/|x|, \qquad z \in \mathbb{R}^d.$$

Therefore $\mathcal{L}(Ux) \in B(\mathbb{R}^d)$.

Proof of (c) *and* (d). Let $B^0(\mathbb{R}_+)$ be the smallest class closed under convolution and convergence and containing all finite mixtures of exponential distributions. Then $\mu^0 \in B^0(\mathbb{R}_+)$ if and only if

$$C_{\mu^0}(v) = \int_0^\infty (e^{ivr} - 1)l(r)dr + ir^0 v, \qquad v \in \mathbb{R},$$

with $r^0 \geqslant 0$ and with $l(r)$ being completely monotone and satisfying $\int_0^\infty (r \wedge 1)l(r)dr < \infty$ (Theorem 51.10 of Sato 1999). Therefore, if $l(r)$ is such a function and if $\mu \in ID(\mathbb{R}^d)$ satisfying

$$C_\mu(z) = \int_0^\infty (e^{i\langle z, \xi^0\rangle r} - 1)l(r)dr + i\langle z, \xi^0\rangle r^0, \qquad z \in \mathbb{R}^d,$$

with some $\xi^0 \in S$ and $r^0 \geqslant 0$, then $\mu \in B^0$. Choosing $l(r) = 0$, we obtain (c).

Consider $\mu \in ID(\mathbb{R}^d)$ such that

$$C_\mu(z) = \int_S \lambda(d\xi)\int_0^\infty (e^{i\langle z,\xi\rangle r} - 1)l_\xi(r)dr + i\langle \gamma^0, z\rangle, \qquad z \in \mathbb{R}^d, \tag{6.1}$$

with $\gamma^0 \in \mathbb{R}^d$, $l_\xi(r)$ completely monotone, $\int_S \lambda(d\xi)\int_0^\infty (r \wedge 1)l_\xi(r)dr < \infty$, and Supp($\lambda$) being a finite set. Then $\mu \in B^0$ by the discussion above.

Next, consider $\mu \in ID(\mathbb{R}^d)$ such that

$$C_\mu(z) = \int_S \lambda(d\xi)\int_0^\infty g(z, r\xi)l_\xi(r)dr,$$

with g as in (3.9) and with $l_\xi(r)$ being completely monotone and $\int_S \lambda(d\xi)\int_0^\infty (r^2 \wedge 1)l_\xi(r)dr < \infty$. This is a general form of $\mu = \mu_{(0,\nu,0)} \in B(\mathbb{R}^d)$. Using Remark 3.2, write

$$C_\mu(z) = \int_S \lambda(d\xi)\int_{(0,\infty)} Q_\xi(du)\int_0^\infty g(z, r\xi)e^{-ru}\,dr,$$

where we have (3.3) with $a(u)$ of (3.2). We can choose finite measures λ_n and $Q_{n,\xi}$ ($n = 1, 2, \ldots$) such that Supp(λ_n) is a finite set for each n, Supp($Q_{n,\xi}$) is a finite set for each n and ξ, and

$$\int_S \lambda_n(d\xi)\int_{(0,\infty)} a(u)f(u, \xi)Q_{n,\xi}(du) \to \int_S \lambda(d\xi)\int_{(0,\infty)} a(u)f(u, \xi)Q_\xi(du)$$

for any bounded continuous function $f(u, \xi)$ on $(0, \infty) \times S$. Using the measure ν_n corresponding to λ_n and $Q_{n,\xi}$, let μ_n be such that $C_{\mu_n}(z) = \int g(z, x)\nu_n(dx)$. Then, noticing that $\int_S \lambda(d\xi)\int_0^\infty (r \wedge 1)l_\xi(r)dr < \infty$ is equivalent to $\int_S \lambda(d\xi)\int_{(0,\infty)} a_0(u)Q_\xi(du) < \infty$ with $a_0(u) = u^{-2}\int_0^u ve^{-v}\,dv + u^{-1}e^{-u}$ (thus $a_0(u) \sim u^{-1}$ as $u \downarrow 0$ and $a_0(u) \sim u^{-2}$ as $u \to \infty$), we see that $C_{\mu_n}(z)$ is of the form (6.1). Hence $\mu_n \in B^0$. Denote

$$f_z(u, \xi) = a(u)^{-1}\int_0^\infty g(z, r\xi)e^{-ru}\,dr. \tag{6.2}$$

Then $f_z(u, \xi)$ is bounded and continuous in $(u, \xi) \in (0, \infty) \times S$, since

$$\int_0^\infty |g(z,\, r\xi)|\mathrm{e}^{-ru}\,\mathrm{d}r \leqslant c_z\int_0^\infty r^2(1+r^2)^{-1}\mathrm{e}^{-ru}\,\mathrm{d}r \leqslant c_z\int_0^\infty (r^2\wedge 1)\mathrm{e}^{-ru}\,\mathrm{d}r = c_z a(u),$$

with c_z as in the proof of (4.5). Thus we have $\int g(z,\, x)\nu_n(\mathrm{d}x) \to \int g(z,\, x)\nu(\mathrm{d}x)$, that is, $\mu_n \to \mu$. Hence $\mu \in B^0$.

Proof of (e). Let $\mu = \mu_{(A,0,0)}$, Gaussian with mean 0. We claim that $\mu \in B^0$. For the function $f_z(u,\, \xi)$ in (6.2), let us show that

$$\lim_{u\to\infty} f_z(u,\, \xi) = -\frac{1}{2}\langle z,\, \xi\rangle^2. \tag{6.3}$$

Indeed,

$$\begin{aligned} f_z(u,\, \xi) &= \frac{1}{a(u)}\int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,\xi\rangle r} - 1 - \mathrm{i}\langle z,\, \xi\rangle r)\mathrm{e}^{-ru}\,\mathrm{d}r + \frac{\mathrm{i}\langle z,\, \xi\rangle}{a(u)}\int_0^\infty \frac{r^3}{1+r^2}\mathrm{e}^{-ru}\,\mathrm{d}r \\ &= \frac{1}{ua(u)}\int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,\xi\rangle r/u} - 1 - \mathrm{i}\langle z,\, \xi\rangle r/u)\mathrm{e}^{-r}\,\mathrm{d}r + \frac{\mathrm{i}\langle z,\, \xi\rangle}{ua(u)}\int_0^\infty \frac{(r/u)^3}{1+(r/u)^2}\mathrm{e}^{-r}\,\mathrm{d}r, \end{aligned}$$

and the first term in the last expression tends to $-\frac{1}{2}\langle z,\, \xi\rangle^2$ and the second term tends to 0, since $a(u) \sim 2u^{-3}$ and $\mathrm{e}^{\mathrm{i}\langle z,\xi\rangle r/u} - 1 - \mathrm{i}\langle z,\, \xi\rangle r/u \sim -\frac{1}{2}\langle z,\, \xi\rangle^2 r^2/u^2$, while $(ua(u))^{-1}|\mathrm{e}^{\mathrm{i}\langle z,\xi\rangle r/u} - 1 - \mathrm{i}\langle z,\, \xi\rangle r/u| \leqslant \frac{1}{2}|\langle z,\, \xi\rangle|^2 r^2$ uniformly for large u. In addition to (6.3),

$$\begin{aligned} |f_z(u,\, \xi)| &\leqslant \frac{c_z}{a(u)}\int_0^\infty \frac{r^2}{1+r^2}\mathrm{e}^{-ru}\,\mathrm{d}r \leqslant \frac{c_z}{ua(u)}\int_0^\infty \frac{(r/u)^2}{1+(r/u)^2}\mathrm{e}^{-r}\,\mathrm{d}r \\ &\leqslant c_z u^2\int_0^\infty \frac{r^2}{u^2+r^2}\mathrm{e}^{-r}\,\mathrm{d}r \leqslant 2c_z \end{aligned}$$

for u so large that $a(u) \geqslant u^{-3}$. Let X be a Gaussian random variable on $\mathbb{R}^d$ with distribution μ and let $\lambda(B) = \mathrm{E}(1_B(X/|X|)|X|^2)$ for $B \in \mathcal{B}(S)$. Define μ_n as

$$C_{\mu_n}(z) = \int_S \lambda(\mathrm{d}\xi)\int_{(0,\infty)} \delta_n(\mathrm{d}u) f_z(u,\, \xi),$$

where δ_n is the δ-distribution located at n. Then $\mu_n \in B^0$ by (d) and $C_{\mu_n}(z)$ tends to $-\frac{1}{2}\int_S\langle z,\, \xi\rangle^2\lambda(\mathrm{d}\xi)$. This means $\mu_n \to \mu$, since

$$\int_S \langle z,\, \xi\rangle^2\lambda(\mathrm{d}\xi) = \mathrm{E}(\langle z,\, X/|X|\rangle^2|X|^2) = \mathrm{E}(\langle z,\, X\rangle^2) = \sum_{j,l=1}^d \mathrm{E}(z_j z_l X_j X_l) = \langle z,\, Az\rangle.$$

Thus we have $\mu \in B^0$.

We have shown that $B^0 = B(\mathbb{R}^d)$. The second statement of the theorem follows from this fact. To see this, let B^{00} be the class of all μ for which we can find $\mu_n \to \mu$ such that each μ_n is the distribution of the sum of a finite number of independent elementary mixed-exponential random variables in $\mathbb{R}^d$. Then obviously $B^{00} \subset B^0$. It is also easy to see that B^{00} is closed under convolution. If $\mu^{(n)} \in B^{00}$, $n = 1, 2, \ldots,$ and $\mu^{(n)} \to \mu$, then $\mu \in B^{00}$ since the topology of weak convergence is a metric topology. Thus $B^{00} \supset B^0$ from the definition of B^0.

Part 2 (Characterization of $T(\mathbb{R}^d)$). We can give a proof similar to that for $B(\mathbb{R}^d)$. Let T^0 be the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and convergence and containing the distributions of all elementary Γ-variables in $\mathbb{R}^d$. This time it is enough to prove the statements $(a)_T$– $(e)_T$ which are the statements (a)–(e) in Part 1 with replacement of $B(\mathbb{R}^d)$, B^0, and 'elementary mixed-exponential variables' by $T(\mathbb{R}^d)$, T^0, and 'elementary Γ-variables'.

The proof of $(a)_T$ is from Theorem B and Lemma 4.1. To see $(b)_T$, we have only to note that, for any real Γ-distributed variable U,

$$\mathrm{E}\mathrm{e}^{\mathrm{i}vU} = \exp\int_0^\infty (\mathrm{e}^{\mathrm{i}vr} - 1)a\mathrm{e}^{-br}r^{-1}\,\mathrm{d}r, \qquad v \in \mathbb{R},$$

with some $a > 0$ and $b > 0$, and that, for any $x \neq 0$,

$$C_{Ux}(z) = \int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,x\rangle r} - 1)a\mathrm{e}^{-br}r^{-1}\,\mathrm{d}r = \int_S \delta_{x/|x|}(\mathrm{d}\xi)\int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,\xi\rangle r} - 1)a\mathrm{e}^{-br/|x|}r^{-1}\,\mathrm{d}r.$$

To see $\delta_x \in T^0$ for $x \neq 0$, note that

$$n|x|\int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,x\rangle r} - 1)\mathrm{e}^{-n|x|r}r^{-1}\,\mathrm{d}r = n|x|\int_0^\infty (\mathrm{e}^{\mathrm{i}\langle z,x\rangle r/(n|x|)} - 1)\mathrm{e}^{-r}r^{-1}\,\mathrm{d}r \to \mathrm{i}\langle z, x\rangle$$

as $n \to \infty$, since $n|x|r^{-1}(\mathrm{e}^{\mathrm{i}\langle z,x\rangle r/(n|x|)} - 1)$ tends to $\mathrm{i}\langle z, x\rangle$ boundedly by $|\langle z, x\rangle|$. That is, δ_x is approximated by distributions of elementary Γ-variables if $x \neq 0$. Evidently $\delta_0 \in T^0$, since $Ux_n \to 0$ as $x_n \to 0$. Hence we obtain $(c)_T$.

The proof of $(d)_T$ is similar to that of (d). In this case a general $\mu = \mu_{(0,\nu,0)}$ in $T(\mathbb{R}^d)$ satisfies

$$C_\mu(z) = \int_S \lambda(\mathrm{d}\xi)\int_{(0,\infty)} R_\xi(\mathrm{d}u)\int_0^\infty g(z, r\xi)\mathrm{e}^{-ru}r^{-1}\,\mathrm{d}r,$$

where R_ξ satisfies (3.7) with $b(u)$ of (3.6). Instead of $f_z(u, \xi)$ we use

$$h_z(u, \xi) = b(u)^{-1}\int_0^\infty g(z, r\xi)\mathrm{e}^{-ur}r^{-1}\,\mathrm{d}r,$$

which is bounded and continuous in $(u, \xi) \in (0, \infty) \times S$. The statement $(e)_T$ is proved like (e), by using $\lim_{u\to\infty} h_z(u, \xi) = -\frac{1}{2}\langle z, \xi\rangle^2$. This completes the proof that $T^0 = T(\mathbb{R}^d)$.

The last sentence of the theorem for $T(\mathbb{R}^d)$ is proved as in Part 1.

7. Examples

Example 7.1 *Tempered stable distributions of Rosiński*. Let $0 < \alpha < 2$. Rosiński (2004) calls a distribution $\mu \in ID(\mathbb{R}^d)$ tempered α-stable if $\mu = \mu_{(A,\nu,\gamma)}$ is such that $A = 0$ and ν has polar decomposition

$$\nu(B) = \int_S \lambda(\mathrm{d}\xi)\int_0^\infty 1_B(r\xi)r^{-\alpha-1}q_\xi(r)\mathrm{d}r, \tag{7.1}$$

where $q_\xi(r)$ is completely monotone in r, measurable in ξ, and

$$\lambda(S) < \infty, \qquad q_\xi(0+) = 1, \qquad q_\xi(+\infty) = 0. \tag{7.2}$$

We denote by $\mathfrak{S}^*_\alpha = \mathfrak{S}^*_\alpha(\mathbb{R}^d)$ the class of tempered α-stable distributions on $\mathbb{R}^d$ in Rosiński's sense. Notice that, by the uniqueness of polar decomposition in the sense of Lemma 2.1, $\mathfrak{S}^*_\alpha \cap \mathfrak{S}^*_{\alpha'}$ consists only of δ-distributions if $\alpha \neq \alpha'$. Rosiński (2004) studies Lévy processes $\{X_t\}$ with $\mathcal{L}(X_1) \in \mathfrak{S}^*_\alpha$ and shows their functional limit theorems for small t and for large t, their absolute continuity on path spaces with respect to some α-stable Lévy processes, and their series representations.

Fix the dimension d arbitrarily. Omitting $\mathbb{R}^d$ in $T(\mathbb{R}^d)$, $T_1(\mathbb{R}^d)$, $L_1(\mathbb{R}^d)$ and so on, we make the following statements.

(i) For every $0 < \alpha < 2$, $\mathfrak{S}^*_\alpha \subset T$. This is obvious since $r^{-\alpha} q_\xi(r)$ is completely monotone whenever $q_\xi(r)$ is.
(ii) If $1 \leqslant \alpha < 2$, then $\mathfrak{S}^*_\alpha \subset T_1$.
(iii) If $\frac{2}{3} \leqslant \alpha < 2$, then $\mathfrak{S}^*_\alpha \subset L_2$.
(iv) If $\frac{1}{4} \leqslant \alpha < 2$, then $\mathfrak{S}^*_\alpha \subset L_1$.
(v) Let $0 < \alpha < \frac{1}{4}$. If μ is in $\mathfrak{S}^*_\alpha$ with $q_\xi(r) = c(\xi)\mathrm{e}^{-b(\xi)r}$ for all ξ in a set of positive λ-measure, where $c(\xi)$ and $b(\xi)$ are positive measurable functions of ξ, then $\mu \notin L_1$ and consequently $\mu \notin T_1$.

The proofs are as follows. Let $\mu \in \mathfrak{S}^*_\alpha$. The k-function of μ is $k_\xi(r) = r^{-\alpha} q_\xi(r)$. We suppress the subscript ξ in $k_\xi(r)$, $h_\xi(u)$, $q_\xi(r)$ and $Q_\xi(\mathrm{d}v)$. Then

$$\begin{aligned}
h(u) &= \mathrm{e}^{\alpha u} q(\mathrm{e}^{-u}),\\
h'(u) &= \alpha \mathrm{e}^{\alpha u} q(\mathrm{e}^{-u}) - \mathrm{e}^{(\alpha-1)u} q'(\mathrm{e}^{-u}),\\
h''(u) &= \alpha^2 \mathrm{e}^{\alpha u} q(\mathrm{e}^{-u}) - (2\alpha - 1)\mathrm{e}^{(\alpha-1)u} q'(\mathrm{e}^{-u}) + \mathrm{e}^{(\alpha-2)u} q''(\mathrm{e}^{-u}),\\
h'''(u) &= \alpha^3 \mathrm{e}^{\alpha u} q(\mathrm{e}^{-u}) - (3\alpha^2 - 3\alpha + 1)\mathrm{e}^{(\alpha-1)u} q'(\mathrm{e}^{-u})\\
&\qquad + 3(\alpha - 1)\mathrm{e}^{(\alpha-2)u} q''(\mathrm{e}^{-u}) - \mathrm{e}^{(\alpha-3)u} q'''(\mathrm{e}^{-u}).
\end{aligned}$$

Recall that $q(r)$ is completely monotone. If $1 \leqslant \alpha < 2$, then $h'(-\log r) = \alpha r^{-\alpha} q(r) - r^{1-\alpha} q'(r)$ is completely monotone and hence $\mu \in T_1$ by Theorem E. We have $h'(u) \geqslant 0$ for all $0 < \alpha < 2$ and $h''(u) \geqslant 0$ for $\frac{1}{4} \leqslant \alpha < 2$ since

$$\begin{aligned}
h''(-\log r) &= r^{-\alpha}[\alpha^2 q(r) - (2\alpha - 1) r q'(r) + r^2 q''(r)]\\
&= r^{-\alpha} \int_0^\infty ((rv + \alpha - \tfrac{1}{2})^2 + \alpha - \tfrac{1}{4}) \mathrm{e}^{-rv} Q(\mathrm{d}v),
\end{aligned}$$

where $Q = Q_\xi$ is the probability measure on $(0, \infty)$ satisfying $q(r) = \int_0^\infty \mathrm{e}^{-rv} Q(\mathrm{d}v)$. Thus $\mu \in L_1$ if $\frac{1}{4} \leqslant \alpha < 2$. If $0 < \alpha < \frac{1}{4}$ and if $q(r)$ is as is assumed in (v), then, for ξ in a set of positive λ-measure,

$$h''(-\log r) = cr^{-\alpha}((rb + \alpha - \tfrac{1}{2})^2 + \alpha - \tfrac{1}{4})\mathrm{e}^{-rb} < 0$$

for $r = (\frac{1}{2} - \alpha)/b$ and hence $\mu \notin L_1$. If $\frac{2}{3} \leqslant \alpha < 2$, then

$$h'''(-\log r) = r^{-\alpha}\int_0^\infty [\alpha^3 + (3\alpha^2 - 3\alpha + 1)rv + 3(\alpha - 1)r^2v^2 + r^3v^3]\mathrm{e}^{-rv}Q(\mathrm{d}v) \geqslant 0,$$

since $g(w) = \alpha^3 + (3\alpha^2 - 3\alpha + 1)w + 3(\alpha - 1)w^2 + w^3$ is non-negative for $w \geqslant 0$ (notice that $g'(w) \geqslant 0$ for $w \geqslant 0$ and $g(0) \geqslant 0$).

The simplest case of $\mu \in \mathfrak{S}_\alpha^*(\mathbb{R})$ is given by

$$C_\mu(z) = c\int_0^\infty (\mathrm{e}^{\mathrm{i}zx} - 1)x^{-\alpha-1}\mathrm{e}^{-bx}\,\mathrm{d}x$$

with $0 < \alpha < 1$ and positive constants c and b. This is the distribution of the Esscher transform of an α-stable subordinator at a fixed time. The relation with $L_1(\mathbb{R})$ of this μ was discussed in Maejima *et al.* (2000, p. 397). When $\alpha = \frac{1}{2}$ this gives an inverse Gaussian distribution. Thus $\mu \in L_1(\mathbb{R})$ and $\Upsilon\mu \in T_1(\mathbb{R})$ for an inverse Gaussian μ.

Example 7.2. As mentioned near the end of Section 2, many examples of distributions in $T(\mathbb{R})$ supported on $\mathbb{R}_+$ are given in Bondesson (1992) and Steutel and van Harn (2004). Using Proposition 2.4(iv) for $\mu \in ID(\mathbb{R})$, we can prove that $\Upsilon\mu$ has support equal to $\mathbb{R}$ if and only if μ has support equal to $\mathbb{R}$. Hence, by Theorem B, distributions in $T(\mathbb{R})$ with support $\mathbb{R}$ can be constructed by Υ if we have self-decomposable distributions with support $\mathbb{R}$. For such self-decomposable distributions as well as other examples, see Jurek (1997). Further, using Theorem D, we can construct concrete examples of distributions in $T_m(\mathbb{R})$, $m = 1, 2$, since we have several examples of distributions in $L_m(\mathbb{R})$, $m = 1, 2$, with explicit densities.

Let $\{\Gamma_t^{(a)}\}$ be a Γ-process with scale parameter $a > 0$. We have, for $t > 0$,

$$P(\Gamma_t^{(a)} \in B) = \frac{a^t}{\Gamma(t)}\int_{B\cap(0,\infty)} x^{t-1}\mathrm{e}^{-ax}\,\mathrm{d}x, \qquad B \in \mathcal{B}(\mathbb{R}).$$

Then the distribution of $\log\Gamma_t^{(a)}$ has density

$$\frac{a^t}{\Gamma(t)}\exp(tx - a\mathrm{e}^x), \qquad x \in \mathbb{R},$$

for $t > 0$. Linnik and Ostrovskii (1977, Chapter 2, Section 6, Example 3) shows that this distribution is infinitely divisible with triplet $(0, \nu, \gamma)$ with

$$\nu(\mathrm{d}x) = 1_{(-\infty,0)}(x)|x|^{-1}(1 - \mathrm{e}^x)^{-1}\mathrm{e}^{tx}\,\mathrm{d}x$$

and some γ (see also Jurek 1997; Sato 1999, E 18.19). Thus $\mathcal{L}(\log\Gamma_t^{(a)}) \in L(\mathbb{R})$ for all $t > 0$ and $a > 0$. Let $\{Y_t\}$ be a strictly α-stable subordinator $(0 < \alpha < 1)$, $\mathrm{E}(\mathrm{e}^{-uY_t}) = \exp(-btu^\alpha)$, $u \geqslant 0$, with some $b > 0$, and let $\{Z_t\}$ be a symmetric α'-stable Lévy process $(0 < \alpha' \leqslant 2)$, $\mathrm{E}(\mathrm{e}^{\mathrm{i}zZ_t}) = \exp(-ct|z|^{\alpha'})$, $z \in \mathbb{R}$, with some $c > 0$. Akita and Maejima (2002) showed the following:

(i) $\mathcal{L}(\log\Gamma_t^{(a)}) \in L_1(\mathbb{R})$ for $t \geqslant \frac{1}{2}$.
(ii) $\mathcal{L}(\log\Gamma_t^{(a)}) \in L_2(\mathbb{R})$ for $t \geqslant 1$.

(iii) $\mathcal{L}(\log Y_t) \in L_1(\mathbb{R})$ for $t > 0$.
(iv) $\mathcal{L}(\log|Z_t|) \in L_1(\mathbb{R})$ for $t > 0$.

Applying the mapping Υ to these distributions, we get examples of $T_1(\mathbb{R})$ and $T_2(\mathbb{R})$. In particular, $\Upsilon(\mathcal{L}(\log \Gamma_t^{(a)}))$ has Lévy measure

$$1_{(-\infty,0)}(x)\left(\int_0^\infty \frac{\mathrm{e}^{tx/s-s}}{1-\mathrm{e}^{x/s}}\,\mathrm{d}s\right)\frac{\mathrm{d}x}{|x|}$$

and belongs to $T_1(\mathbb{R})$ for $t \geqslant \frac{1}{2}$ and to $T_2(\mathbb{R})$ for $t \geqslant 1$. The generating triplets of $\mathcal{L}(\log Y_t)$ and $\mathcal{L}(\log|Z_t|)$ can be obtained by the method of the proofs of (iii) and (iv) in Akita and Maejima (2002). They are purely non-Gaussian. The Lévy measure of $\mathcal{L}(\log Y_t)$ is

$$1_{(0,\infty)}(x)\frac{(\mathrm{e}^{-\alpha x}-\mathrm{e}^{-x})\mathrm{d}x}{(1-\mathrm{e}^{-\alpha x})(1-\mathrm{e}^{-x})x}$$

for any $t > 0$ if $b = 1$, and that of $\mathcal{L}(\log|Z_t|)$ is

$$\left(1_{(-\infty,0)}(x)\frac{\mathrm{e}^{x}}{1-\mathrm{e}^{2x}} + 1_{(0,\infty)}(x)\frac{\mathrm{e}^{-\alpha' x}-\mathrm{e}^{-2x}}{(1-\mathrm{e}^{-2x})(1-\mathrm{e}^{-\alpha' x})}\right)\frac{\mathrm{d}x}{|x|}$$

for any $t > 0$ if $c = 1$. The explicit distributions for $\alpha = \frac{1}{2}$ and $\alpha' = 1$ are

$$P(\log Y_t \in B) = \frac{t}{2\pi^{1/2}}\int_B \exp\left(-\frac{1}{2}x - \frac{t^2}{4}\mathrm{e}^{-x}\right)\mathrm{d}x, \qquad \text{for } b = 1,$$

$$P(\log|Z_t| \in B) = \frac{2t}{\pi}\int_B \frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}+t^2}\,\mathrm{d}x, \qquad \text{for } c = 1.$$

Recall that $\mathcal{L}(Y_t) = \mathcal{L}(1/\Gamma_{1/2}^{(t^2/4)})$ for this Y_t with $\alpha = \frac{1}{2}$ and $b = 1$.

Example 7.3. Let $\{X_t\}$ be Brownian motion on $\mathbb{R}^d$ with drift $\gamma \in \mathbb{R}^d$, that is, $\{X_t\}$ is the Lévy process with $\mathcal{L}(X_t) = \mu_{(tI,0,t\gamma)}$, where I is the $d \times d$ unit matrix. Let $\{Z_t\}$ be a subordinator such that $\mathcal{L}(Z_t)$ is a generalized Γ-convolution (equivalently, $\mathcal{L}(Z_t)$ is in $T(\mathbb{R})$ and has support in $\mathbb{R}_+$). Subordination of $\{X_t\}$ by $\{Z_t\}$ gives a Lévy process $\{Y_t\}$ on $\mathbb{R}^d$. That is, $Y_t = X_{Z_t}$, where $\{X_t\}$ and $\{Z_t\}$ are independent. Assume that $\mathcal{L}(Z_t)$ is not a δ-distribution. Let $\mu^t = \mathcal{L}(Y_t)$. In the case $d = 1$, Halgreen (1979) showed that $\mu^t \in L(\mathbb{R})$ for any γ. Then Takano (1989; 1990) showed that in the case $d \geqslant 2$ one had a different phenomenon: if $\gamma = 0$, then $\mu^t \in L(\mathbb{R}^d)$, but, under some additional assumption on the so-called U-measure of the generalized Γ-convolution $\mathcal{L}(Z_1)$, if $\gamma \neq 0$, then $\mu^t \notin L(\mathbb{R}^d)$ for all $t > 0$.

Generalized inverse Gaussian distributions are in the class of generalized Γ-convolutions (Halgreen 1979). If $\mathcal{L}(Z_t)$ is a generalized inverse Gaussian, then the explicit expression for the density of $\mathcal{L}(Y_t)$ using modified Bessel functions is obtained by Barndorff-Nielsen (1977; 1978); the process $\{Y_t\}$ is referred to as a generalized hyperbolic motion, the finite-dimensional laws of $\{Y_t\}$ being of the generalized hyperbolic type.

Let us assume for the rest of this example that $\{Z_t\}$ is the Γ-process with scale parameter 1. This is a special case of the generalized inverse Gaussian. We have

$$\widehat{\mu^t}(z) = (1 + 2^{-1}|z|^2 - \mathrm{i}\langle \gamma, z\rangle)^{-t}.$$

The expression for the density of μ^t, $t > 0$, mentioned above, is in this case

$$c(t, \gamma)|x|^{t-(d/2)} K_{t-(d/2)}((2 + |\gamma|^2)^{1/2}|x|)\mathrm{e}^{\langle \gamma, x\rangle}$$

with $c(t, \gamma) = 2(2\pi)^{-d/2}\Gamma(t)^{-1}(2 + |\gamma|^2)^{-(t-d/2)/2}$. Here $K_{t-(d/2)}$ is the modified Bessel function of the third kind with index $t - d/2$. In particular, $\mu^{(d+1)/2}$ has density

$$c \exp(-\sqrt{2 + |\gamma|^2}|x| + \langle \gamma, x\rangle)$$

with a normalizing constant c. We can prove the following for every $t > 0$.

(i) Let $d = 1$. Then $\mu^t \in T(\mathbb{R})$ and $\mu^t \notin L_1(\mathbb{R})$ (hence $\mu^t \notin T_1(\mathbb{R})$), irrespective of whether $\gamma = 0$ or $\gamma \neq 0$.
(ii) Let $d \geqslant 2$. If $\gamma = 0$, then $\mu^t \in L(\mathbb{R}^d)$, $\mu^t \notin T(\mathbb{R}^d)$, and $\mu^t \notin L_1(\mathbb{R}^d)$. If $\gamma \neq 0$, then $\mu^t \notin L(\mathbb{R}^d)$ (hence $\mu^t \notin T(\mathbb{R}^d)$).

To prove (i), choose $\lambda = \delta_{+1} + \delta_{-1}$. It is known that $\mu^t \in L$ with k-function

$$k_\xi(r) = \begin{cases} t \exp[-(\sqrt{2 + \gamma^2} - \gamma)r], & \text{for } \xi = +1, \\ t \exp[-(\sqrt{2 + \gamma^2} + \gamma)r], & \text{for } \xi = -1. \end{cases}$$

Hence $k_\xi(r)$ is completely monotone and $\mu^t \in T$. The fact that $\mu^t \notin L_1(\mathbb{R})$ is observed by Maejima *et al.* (2000, p. 397).

We now prove (ii). As is shown by Takano (1989), the Lévy measure of μ^t has polar decomposition $\lambda(\mathrm{d}\xi)$, $\nu_\xi(\mathrm{d}r)$ where λ is the Lebesgue measure on the $(d-1)$-dimensional unit sphere S and

$$\nu_\xi(\mathrm{d}r) = 2t\, \mathrm{e}^{\langle \gamma, \xi\rangle r} L_{d/2}(\sqrt{2 + |\gamma|^2}r) r^{-1}\, \mathrm{d}r \tag{7.3}$$

with $L_{d/2}(u) = (2\pi)^{-d/2} u^{d/2} K_{d/2}(u)$.

If $\gamma \neq 0$, then $\mu^t \notin L(\mathbb{R}^d)$, which is a special case of Takano (1990).

Now assume that $\gamma = 0$. Write $p = d/2$ and $k(r) = r^p K_p(r)$. Since $k'(r) = -r^p K_{p-1}(r) < 0$, we have $\mu^t \in L(\mathbb{R}^d)$ (this is also a consequence of a general result; see Sato 2001). Furthermore,

$$k''(r) = r^p K_{p-2}(r) - r^{p-1} K_{p-1}(r) = 2^{-p} r^{2p-2} \int_0^\infty \mathrm{e}^{-s-r^2/(4s)} s^{-p}(2s-1)\mathrm{d}s$$

by the well known integral representation of the modified Bessel function ((30.28) of Sato 1999). Note that $\int_0^{1/2} \mathrm{e}^{-s-r^2/(4s)} s^{-p}(2s-1)\mathrm{d}s \to -\infty$ as $r \downarrow 0$ (here we use the fact that $d \geqslant 2$). Thus $k''(r) < 0$ when r is small enough. Hence $k(r)$ is not completely monotone and $\mu^t \notin T(\mathbb{R}^d)$. For the function $h(u) = k(\mathrm{e}^{-u})$ we have $h''(u) = k''(\mathrm{e}^{-u})\mathrm{e}^{-2u} + k'(\mathrm{e}^{-u})\mathrm{e}^{-u} < 0$ for some u, and hence $\mu^t \notin L_1(\mathbb{R}^d)$.

Acknowledgement

The authors thank Víctor Pérez-Abreu for his stimulating remarks in the course of the preparation of this paper. They are grateful to the referees for valuable comments.

References

Akita, K. and Maejima, M. (2002) On certain self-decomposable self-similar processes with independent increments. *Statist. Probab. Lett.*, **59**, 53–59.

Barndorff-Nielsen, O. (1977) Exponentially decreasing distributions for the logarithm of particle size. *Proc. Roy. Soc. Lond. Ser. A*, **353**, 401–419.

Barndorff-Nielsen, O. (1978) Hyperbolic distributions and distributions on hyperbolae. *Scand. J. Statist.*, **5**, 151–157.

Barndorff-Nielsen, O.E. and Thorbjørnsen, S. (2002a) Lévy laws in free probability. *Proc. Natl. Acad. Sci. USA*, **99**, 16 568–16 575.

Barndorff-Nielsen, O.E. and Thorbjørnsen, S. (2002b) Lévy processes in free probability. *Proc. Natl. Acad. Sci. USA*, **99**, 16 576–16 580.

Barndorff-Nielsen, O.E. and Thorbjørnsen, S. (2004) A connection between free and classical infinite divisibility. *Infin. Dimens. Anal. Quantum Probab. Related Top.*, **7**, 573–590.

Barndorff-Nielsen, O.E., Thorbjørnsen, S. (2005) Regularising mappings of Lévy measures. *Stochastic Process Appl.* To appear.

Bondesson, L. (1981) Classes of infinitely divisible distributions and densities. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **57**, 39–71. Correction and addendum, **59**, (1982), 277–277.

Bondesson, L. (1992) *Generalized Gamma Convolutions and Related Classes of Distribution Densities,* Lecture Notes in Statist. 76. New York: Springer-Verlag.

Goldie, C. (1967) A class of infinitely divisible random variables. *Proc. Cambridge Philos. Soc.*, **63**, 1141–1143.

Halgreen, C. (1979) Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **47**, 13–17.

Jurek, Z.J. (1983) The class $L_m(Q)$ of probability measures on Banach spaces. *Bull. Polish Acad. Sci. Math.*, **31** 51–62.

Jurek, Z.J. (1997) Selfdecomposability: an exception or a rule. *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **51**, 93–107.

Jurek, Z.J. and Vervaat, W. (1983) An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrscheinlichkeitstheorie. Verw. Geb.*, **62**, 247–262.

Linnik, J.V. and Ostrovskii, I.V. (1977) *Decomposition of Random Variables and Vectors*. Providence, RI: American Mathematical Society.

Maejima, M. and Sato, K. (2003) Semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein–Uhlenbeck type processes. *J. Math. Kyoto Univ.*, **43**, 609–639.

Maejima, M., Sato, K. and Watanabe, T. (1999) Operator semi-selfdecomposability, (C, Q)-decomposability and related nested classes. *Tokyo J. Math.*, **22**, 473–509.

Maejima, M., Sato, K. and Watanabe, T. (2000) Distributions of selfsimilar and semi-selfsimilar processes with independent increments. *Statist. Probab. Lett.*, **47**, 395–401.

Rajput, B. and Rosinski, J. (1989) Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields*, **82**, 451–487.

Rocha-Arteaga, A. and Sato, K. (2003) *Topics in Infinitely Divisible Distributions and Lévy Processes*, Aportaciones Mat. Invest. 17. Mexico: Sociedad Matemática Mexicana.

Rosinski, J. (1990) On series representations of infinitely divisible random vectors. *Ann. Probab.*, **18**, 405–430.

Rosiński, J. (2004) Tempering stable processes. Preprint.

Sato, K. (1980) Class L of multivariate distributions and its subclasses. *J. Multivariate Anal.*, **10**, 207–232.

Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Sato, K. (2001) Subordination and self-decomposability. *Statist. Probab. Lett.*, **54**, 317–324.

Sato, K. (2004) Stochastic integrals in additive processes and application to semi-Lévy processes. *Osaka J. Math.*, **41**, 211–236.

Sato, K. and Yamazato, M. (1983) Stationary processes of Ornstein–Uhlenbeck type. In K. Ito and J.V. Prokhorov (eds), *Probability Theory and Mathematical Statistics*, Lecture Notes in Math. 1021, pp. 541–551. Berlin: Springer-Verlag.

Steutel, F.W. (1967) Note on the infinite divisibility of exponential mixtures. *Ann. Math. Statist.*, **38**, 1303–1305.

Steutel, F.W. (1970) *Preservation of Infinite Divisibility under Mixing and Related Topics*, Math. Centre Tracts 33. Amsterdam: Mathematisch Centrum.

Steutel, F.W. and van Harn, K. (1979) Discrete analogues of self-decomposability and stability. *Ann. Probab.*, **7**, 893–899.

Steutel, F.W. and van Harn, K. (2004) *Infinite Divisibility of Probability Distributions on the Real Line*. New York: Marcel Dekker.

Takano, K. (1989) On mixtures of the normal distribution by the generalized gamma convolutions. *Bull. Fac. Sci. Ibaraki Univ.*, **21**, 29–41.

Takano, K. (1990) Correction and addendum to 'On mixtures of the normal distribution by the generalized gamma convolutions'. *Bull. Fac. Sci. Ibaraki Univ.*, **22**, 49–52.

Thorin, O. (1977a) On the infinite divisibility of the Pareto distribution. *Scand. Actuar. J.*, 31–40.

Thorin, O. (1977b) On the infinite divisibility of the lognormal distribution. *Scand. Actuar. J.*, 121–148.

Thorin, O. (1978) An extension of the notion of a generalized Γ-convolution. *Scand. Actuarial J.*, 141–149.

Urbanik, K. (1972) Slowly varying sequences of random variables. *Bull. Acad. Polonaise Sci. Sér. Sci. Math. Astronom. Phys.*, **20**, 679–682.

Urbanik, K. and Woyczynski, W.A. (1967) Random integrals and Orlicz spaces. *Bull. Acad. Polonaise Sci.*, **15**, 161–169.

Received August 2004 and revised April 2005