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Some measure-valued Markov processes attached to occupation times of Brownian motion

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We study the positive random measure $\Pi_t(\omega, \mathrm{d}y) = l_t^{B_t - y}\,\mathrm{d}y$, where $(l_t^a;\ a \in \mathbb{R},\ t > 0)$ denotes the family of local times of the one-dimensional Brownian motion B. We prove that the measure-valued process $(\Pi_t;\ t \geqslant 0)$ is a Markov proces. We give two examples of functions $(f_i)_{i=1,\ldots,n}$ for which the process $(\Pi_t(f_i)_{i=1,\ldots,n};\ t \geqslant 0)$ is a Markov process.

Keywords: Brownian motion; local times; Markov processes

1. Introduction

Let $(B_t,\ t \geqslant 0)$ denote a one-dimensional Brownian motion, starting from 0, and $(l_t^y;\ y \in \mathbb{R},\ t \geqslant 0)$ its family of local times. We denote by $\mathscr{F}_t$ the natural filtration of B.

Recently, a better understanding of an identity in law, originally due to Bougerol (1983), which involves an exponential functional of Brownian motion, was obtained by Alili *et al.* (1997) using the observation that if $(\xi_t;\ t \geqslant 0)$ and $(\eta_t;\ t \geqslant 0)$ are two independent Lévy processes, starting from 0, then for any fixed $t \geqslant 0$,

$$\int_0^t \mathrm{d}\eta_s \exp(\xi_s) \stackrel{\text{(law)}}{=} \exp(\xi_t) \int_0^t \mathrm{d}\eta_s \exp(-\xi_s), \tag{1.1}$$

and, moreover, the process

$$Y_t^{(\xi,\eta)} \stackrel{\text{(def)}}{=} \exp(\xi_t) \int_0^t \mathrm{d}\eta_s \exp(-\xi_s), \qquad t \geqslant 0, \tag{1.2}$$

is a Markov process. Equation (1.1) follows from the invariance by time reversal of the law of a Lévy process, and the Markov property of $Y^{(\xi,\eta)}$ is simply a consequence of the independence of the increments of ξ and η. The importance of these generalized Ornstein–Uhlenbeck processes was noticed and discussed in depth by de Haan and Karandikar (1989). Bougerol's identity,

$$\text{for fixed } t, \qquad \sinh(B_t) \stackrel{\text{(law)}}{=} \int_0^t \mathrm{d}C_s \exp(B_s),$$

where B and C are independent Brownian motions, is then deduced easily taking in (1.2) for ξ and η two independent Brownian motions.

Here, we shall consider mainly the particular case where $\xi = B$ is a Brownian motion and $\eta_t \equiv t$, and pushing the preceding remark to the level of occupation times, we consider the positive random measure on $\mathbb{R}$

$$\Pi_t(\omega, \mathrm{d}y) = l_t^{B_t - y}\,\mathrm{d}y,$$

which integrates positive functions $f : \mathbb{R} \to \mathbb{R}_+$ as

$$\Pi_t(f) = \int_{\mathbb{R}} f(B_t - z) l_t^z \,\mathrm{d}z = \int_0^t f(B_t - B_s)\,\mathrm{d}s. \tag{1.3}$$

In the case of an exponential function $f_a(x) = \exp(ax)$, the $\mathbb{R}_+$-valued process $(\Pi_t(f_a), t \geqslant 0)$ is a Markov process (see (1.2)) which has been studied in Alili *et al.* (1997) and Carmona *et al.* (1997). On the other hand, for $f_+(x) = 1_{x \geqslant 0}$, the process $\Pi_t(f_+)$ was studied by Walsh (1993), where it is shown that $(\Pi_t(f_+), t \geqslant 0)$ is a Dirichlet process. Our aim here is to study the measure-valued process $(\Pi_t; t \geqslant 0)$.

2. A stochastic differential equation satisfied by $(\Pi_t, t \geqslant 0)$

Proposition 2.1. *The process* $(\Pi_t, t \geqslant 0)$ *is the unique solution (in the space* $\mathscr{M}_\mathrm{b}(\mathbb{R})$ *of bounded measures on* $\mathbb{R}$*) of the following SDE: for every f in* C_b^2,

$$\Pi_t(f) = tf(0) + \int_0^t \mathrm{d}B_s \Pi_s(f') + \tfrac{1}{2}\int_0^t \mathrm{d}s \Pi_s(f''). \tag{2.1}$$

Proof. An application of Itô's formula to $f(B_t - B_s)$, $t \geqslant s$, and Fubini's theorem show that $(\Pi_t, t \geqslant 0)$ solves the above stochastic differential equation (SDE). To prove uniqueness of the solutions of (2.1), we consider, for each $x \in \mathbb{R}$, the Fourier transform $(\Pi_t^{(x)}, x \in \mathbb{R})$ of Π_t, that is, $\Pi_t^{(x)} = \int \Pi_t(\mathrm{d}y)\exp(\mathrm{i}xy)$. Now, $\Pi_t^{(x)}$ solves a linear SDE; hence, (Π_t) is the unique solution of the equation (2.1). □

In the next corollary, we give some examples of functions f (or $f_1, f_2, \ldots, f_n$) for which the process $(\Pi_t(f); t \geqslant 0)$ (or $(\Pi_t(f_i)_{i=1,\ldots,n}; t \geqslant 0)$) is a Markov process.

Corollary 2.1. (a) *For* $f_a(x) = \exp(ax)$, *the process* $(\Pi_t(f_a); t \geqslant 0)$ *is an* $\mathbb{R}_+$*-valued Markov process (see (1.2)). More generally, for any* $n \in \mathbb{N}$, *and* $a_1, \ldots, a_n$, *the process* $(\Pi_t(f_{a_i}); i \leqslant n)$ *is an n-dimensional Markov process, whose infinitesimal generator coincides on* $C^2(\mathbb{R}_+^n)$ *with*

$$L = \frac{1}{2}\left(\sum_{i=1}^n a_i^2 y_i^2 \frac{\partial^2}{\partial y_i^2} + 2\sum_{i<j} a_i a_j y_i y_j \frac{\partial^2}{\partial y_i\,\partial y_j}\right) + \sum_{i=1}^n \left(\left(\frac{a_i^2}{2} + b_i\right) y_i + 1\right)\frac{\partial}{\partial y_i}.$$

(b) *We set* $\Pi_t^{(n)} = \Pi_t(P_n)$, *where* $P_n(x) = x^n$. *Then, for every* $n \in \mathbb{N}$, $(\Pi_t^{(0)}, \ldots, \Pi_t^{(n)})_{t \geqslant 0}$

constitutes an $\mathbb{R}^{n+1}$*-valued Markov process, whose infinitesimal generator coincides on* $C^2(\mathbb{R}^{n+1})$ *with*

$$L^{(n)} = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 x_{i-1}^2 \frac{\partial^2}{\partial x_i^2} + 2\sum_{1\leqslant i<j\leqslant n} ij x_{i-1}x_{j-1}\frac{\partial^2}{\partial x_i \partial x_j}\right) + \left(\frac{\partial}{\partial x_0} + \sum_{i=2}^{n}\frac{i(i-1)}{2}x_{i-2}\frac{\partial}{\partial x_i}\right).$$

Proof. This is just a consequence of formula (2.1). □

Remarks. (a) We can write (2.1) formally as

$$\begin{cases} \mathrm{d}\Pi_t = \nabla^*\Pi_t\,\mathrm{d}B_t + \left(\frac{1}{2}\Delta^*\Pi_t + \delta_0\right)\mathrm{d}t, \\ \Pi_0 = 0, \end{cases} \tag{2.2}$$

where ∇ is the operator $\partial/\partial x$ and $\Delta = \partial^2/\partial x^2$; that is, Π_t solves a stochastic partial differential equation driven by a one-dimensional Brownian motion. This type of equation is well known and appears in filtering theory. We refer to Pardoux (1993) and Kallianpur (1996) for a review on stochastic partial differential equations and filtering theory.

(b) We can consider, more generally, the process $(\Pi_t^A;\ t\geqslant 0)$ defined as

$$\Pi_t^A(f) = \int_0^t \mathrm{d}A_s f(B_t - B_s), \tag{2.3}$$

where $(A_t;\ t\geqslant 0)$ is a semimartingale, which is assumed to be independent of the Brownian motion B. Equation (2.1) becomes

$$\Pi_t^A(f) = A_t f(0) + \int_0^t \mathrm{d}B_s \Pi_s^A(f') + \frac{1}{2}\int_0^t \mathrm{d}s \Pi_s^A(f''). \tag{2.1$'$}$$

The simplest situation is $\mathrm{d}A_t^{(0)} = \delta_0(\mathrm{d}t)$ which yields: $\Pi_t^{A^{(0)}}(f) = f(B_t)$. Note that all processes Π_t^A satisfy the SDE:

$$\sigma_t(f) = \sigma_t(1)f(0) + \int_0^t \sigma_s(f')\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \sigma_s(f'')\,\mathrm{d}s.$$

(c) Proposition 2.1 extends to the case where B is a Brownian motion in $\mathbb{R}^d$ and $(\Pi_t;\ t\geqslant 0)$ solves the follwing SDE: for $f\in C^2(\mathbb{R}^d)$,

$$\Pi_t(f) = tf(0) + \int_0^t \Pi_s(\nabla f)\,\mathrm{d}B_s + \tfrac{1}{2}\int_0^t \mathrm{d}s\Pi_s(\Delta f).$$

More generally, we can extend Proposition 2.1 to the $\mathbb{R}\times\mathscr{M}_{\mathrm{b}}(\mathbb{R})$-valued process (B_t, Π_t).

Proposition 2.2. $(B_t, \Pi_t;\ t\geqslant 0)$ *is a continuous Markov process, with state space* $\mathbb{R}\times\mathscr{M}_{\mathrm{b}}(\mathbb{R})$ *and is the solution of the following SDE: for* $f\in C^2(\mathbb{R})$, $g\in C^2(\mathbb{R})$,

$$g(B_t)\Pi_t(f) = \int_0^t \mathrm{d}B_s(\Pi_s(f')g(B_s) + \Pi_s(f)g'(B_s))$$

$$+ \int_0^t \left(\tfrac{1}{2}\Pi_s(f)g''(B_s) + \tfrac{1}{2}\Pi_s(f'')g(B_s) + \Pi_s(f')g'(B_s) + f(0)g(B_s)\right) \mathrm{d}s.$$

This is immediate, using (2.1) and Itô's formula.

3. Semigroup and generator of the process $(\Pi_t;\ t \geqslant 0)$

First, we introduce some notation. If $f : \mathbb{R} \to \mathbb{R}^+$ is measurable, we denote by Φ_f the function on $\mathscr{M}_\mathrm{b}^+(\mathbb{R})$ defined by

$$\Phi_f(\nu) = \exp(-\langle \nu,\ f\rangle).$$

If $\Phi : \mathscr{M}_\mathrm{b}^+(\mathbb{R}) \to \mathbb{R}$, we set

$$D\Phi(\nu) = \lim_{x\to 0} \frac{1}{x}(\Phi(\tau_x\nu) - \Phi(\nu))$$

when the limit exists and $\langle \tau_x\nu,\ f\rangle = \langle \nu,\ f(x + \cdot)\rangle$. Finally, Λ_t denotes the occupation measure of B, that is,

$$\Lambda_t(f) = \int_0^t f(B_s)\,\mathrm{d}s.$$

Proposition 3.1. $(\Pi_t;\ t \geqslant 0)$ *is a homogeneous Markov process with state space* $\mathscr{M}_\mathrm{b}^+(\mathbb{R})$ *whose semigroup* $(Q_t;\ t \geqslant 0)$ *is given by:*

$$Q_t(\mu;\ \mathrm{d}\nu) = P(\tau_{B_t}\mu + \Lambda_t \in \mathrm{d}\nu). \tag{3.1}$$

The generator $\mathscr{L}$ *of* $(\Pi_t;\ t \geqslant 0)$ *coincides, on the functions* Φ_f*, with*

$$\mathscr{L}\Phi_f(\mu) = \tfrac{1}{2}D^2(\Phi_f)(\mu) - f(0)\Phi_f(\mu). \tag{3.2}$$

The resolvent of the semigroup Q_t *satisfies*

$$U_p(\Phi_f)(\mu) = \int_0^\infty \exp(-pt)Q_t\Phi_f(\mu)\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} \exp(-\langle \mu,\ f(x + \cdot)\rangle)U^f(p;\ x)\,\mathrm{d}x, \tag{3.3}$$

where the function $U(x) := U^f(p;\ x)$ *is the unique solution of the differential equation*

$$\tfrac{1}{2}U''(x) = (p + f(x))U(x),$$

subject to the condition that $U'(x)$ *exists for* $x \neq 0$ *and is bounded, that* U *vanishes at* $\pm\infty$ *and that* $U'(0+) - U'(0-) = -2$.

For fixed t, the law of Π_t, or equivalently the law of the process $\{l_t^{B_t - y};\ y \in \mathbb{R}\}$, has been described by Leuridan (1998); see also related work by Pitman (1998; 1999) who concentrates on the law of Π_t, conditionally on $B_t = b$, that is, the law of Brownian bridge local times. We note that although the equation satisfied by (Π_t) is quite simple, the law of its marginal for fixed time t is quite complicated, as shown in these papers.

Proof of Proposition 3.1. We note that the natural filtration $\mathscr{F}_t^{\Pi}$ of (Π_t) is equal to the filtration of B, since for $f_1(x) = x$,

$$\Pi_t(f_1) = \int_0^t s\,\mathrm{d}B_s, \quad \text{and thus} \quad B_t = \int_0^t \frac{\mathrm{d}\Pi_s(f_1)}{s}.$$

Furthermore,

$$\begin{aligned}
&\mathrm{E}(\Phi_f(\Pi_{t+s})|\mathscr{F}_s)\\
&\quad = \mathrm{E}\left(\exp\left(-\int_0^{t+s} f(B_{t+s} - B_u)\,\mathrm{d}u\right)\middle|\mathscr{F}_s\right)\\
&\quad = \mathrm{E}\left(\exp\left(-\int_0^{s} f(B_{t+s} - B_s + B_s - B_u)\,\mathrm{d}u\right)\exp\left(-\int_s^{t+s} f(B_{t+s} - B_u)\,\mathrm{d}u\right)\middle|\mathscr{F}_s\right).
\end{aligned}$$

We introduce $\hat{B}_v = B_{v+s} - B_s$. $(\hat{B}_v;\ v \geqslant 0)$ is a Brownian motion independent of $\mathscr{F}_s$. Thus,

$$\mathrm{E}(\Phi_f(\Pi_{t+s})|\mathscr{F}_s) = \hat{\mathrm{E}}\left(\exp\left(-\int_0^s f(\hat{B}_t + B_s(\omega) - B_u(\omega))\,\mathrm{d}u\right)\exp\left(-\int_0^t f(\hat{B}_t - \hat{B}_u)\,\mathrm{d}u\right)\right),$$

where the expectation is taken with respect to $\hat{B}$. Therefore,

$$\begin{aligned}
Q_t(\Phi_f)(\mu) &= \mathrm{E}\left(\exp(-\langle\mu, f(B_t + \cdot)\rangle)\exp\left(-\int_0^t f(B_s)\,\mathrm{d}s\right)\right)\\
&= \mathrm{E}(\exp(-\langle\tau_{B_t}\mu + \Lambda_t, f\rangle)) = \mathrm{E}(\Phi_f(\tau_{B_t}\mu + \Lambda_t)).
\end{aligned}$$

This gives formula (3.1).

By definition of $\mathscr{L}$,

$$\mathscr{L}(\Phi_f)(\mu) = \lim_{t\to 0}\frac{1}{t}(Q_t(\Phi_f)(\mu) - \Phi_f(\mu)).$$

By (3.1), $\mathscr{L}(\Phi_f)(\mu) = I + J$, with

$$I = \lim_{t\to 0}\frac{1}{t}\mathrm{E}\left((\exp(-\langle\mu, f(B_t + \cdot)\rangle) - \exp(-\langle\mu, f\rangle)) \times \exp\left(-\int_0^t f(B_u)\,\mathrm{d}u\right)\right)$$

$$J = \lim_{t\to 0}\frac{1}{t}\exp(-\langle\mu, f\rangle)\mathrm{E}\left(\exp\left(-\int_0^t f(B_u)\,\mathrm{d}u\right) - 1\right).$$

It follows that:

$$I = \tfrac{1}{2}e''_f(0), \qquad \text{where } e_f(x) = \exp(-\langle \mu, f(x+\cdot)\rangle)$$

and

$$J = \exp(-\langle \mu, f\rangle) f(0).$$

Now, by an easy computation, we verify that

$$e''_f(0) = \exp(-\langle \mu, f\rangle)((\langle \mu, f'\rangle)^2 - \langle \mu, f''\rangle) = D^2(\Phi_f)(\mu),$$

proving formula (3.2).

Equation (3.3) is a consequence of the Feynman–Kac formula (Kac 1949; Jeanblanc *et al.* 1997):

$$\int_0^\infty \exp(-pt)\mathrm{E}\left(q(B_t)\exp\left(-\int_0^t f(B_u)\,\mathrm{d}u\right)\right)\mathrm{d}t = \int_{\mathbb{R}} q(x)U^f(p;\, x)\,\mathrm{d}x$$

(where $U(= U^f)$ is defined as in the proposition) and of equation (3.1). The function U can also be expressed as (see Jeanblanc *et al.* 1997, (3.14)):

$$U^f(p;\, x) = 2\frac{\Phi^{f_+}(p;\, x)1_{x>0} + \Phi^{f_-}(p;\, x)1_{x<0}}{-\Phi^{f_+}(p;\, 0+) - \Phi^{f_-}(p;\, 0+)}, \tag{3.4}$$

where f_+ is the restriction of f to $\mathbb{R}_+$ and $f_-(x) = f(-x)$, $x \geqslant 0$, and for a measurable function $g: \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi^g(p;\, x)$ denotes the unique bounded solution of the Sturm–Liouville equations,

$$\tfrac{1}{2}\Phi'' = (p+g)\Phi, \qquad \Phi(0) = 1.$$

Let θ_p be an exponential variable of parameter p, independent of $(B_t,\, t \geqslant 0)$. Formula (3.4) reflects the path decomposition of $(B_t;\, t \leqslant \theta_p)$ at time g_{θ_p}, the last zero of B before θ_p (see Jeanblanc *et al.* 1997). □

As in the previous section, we can extend Proposition 3.1 to the process $(B_t,\, \Pi_t)$.

Proposition 3.2. $(B_t,\, \Pi_t;\, t \geqslant 0)$ *is a homogeneous Markov process with state space* $\mathbb{R} \times \mathscr{M}_b(\mathbb{R})$ *whose semigroup* R_t *is given by*

$$R_t((x,\, \mu);\, (\mathrm{d}y,\, \mathrm{d}\nu)) = P(x + B_t \in \mathrm{d}y;\, \tau_{B_t}\mu + \Lambda_t \in \mathrm{d}\nu).$$

The proof is similar to the previous one.

4. An intertwining relationship between two measure-valued Markov processes

Many examples of pairs (X_t) and (Y_t) of Markov processes with respect to filtrations $(\mathscr{X}_t)$ and $(\mathscr{Y}_t)$ such that $\mathscr{Y}_t \subset \mathscr{X}_t$ lead to intertwining relationships between the semigroups of X and Y; see for example Pitman and Rogers (1981), Yor (1989), Carmona *et al.* (1998) and, more recently Matsumoto and Yor (1998) in connection with exponential Brownian functionals – in particular $X_t = \int_0^t \exp(B_t - B_s)\,\mathrm{d}C_s$ and $Y_t = \int_0^t \exp(B_t - B_s)\,\mathrm{d}s$, where B and C are two independent Brownian motions satisfy an intertwining relationship.

We are interested in the extension of this result to the Markov processes $(\Pi_t)_t$ and $(\Pi_t^C)_t$, where $\Pi_t^C(f)$ is defined by

$$\Pi_t^C(f) = \int_0^t \mathrm{d}C_s f(B_t - B_s),\ t \geqslant 0.$$

For fixed t, the variable Π_t^C is a random linear functional on $\mathscr{S}$, the Schwartz space of rapidly decreasing functions, that is, for φ, $\psi \in \mathscr{S}$ and a, $b \in \mathbb{R}$,

$$\Pi_t^C(a\varphi + b\psi) = a\Pi_t^C(\varphi) + b\Pi_t^C(\psi) \text{ a.s.}$$

Since Π_t^C is continuous in probability on $\mathscr{S}$ (using $\|\Pi_t^C(f)\|_2 \leqslant C_t\|f\|_{L^2(\mathbb{R})}$), Π_t^C has a version with values in $\mathscr{S}'$ (see Walsh 1986, Corollary 4.2). So, we can consider the process $(\Pi_t^C;\ t \geqslant 0)$ as a $\mathscr{S}'$-valued process. Obviously, the process $(\Pi_t;\ t \geqslant 0)$ can also be considered as a $\mathscr{S}'$-valued process.

As in the previous section, we can express the semigroup Q_t^C of the process Π_t^C by

$$\begin{aligned} Q_t^C(\Phi_f)(\mu) &= \hat{\mathrm{E}}\left(\exp(-\langle \mu, f(\hat{B}_t + \cdot)\rangle)\exp\left(-\int_0^t f(\hat{B}_t - \hat{B}_u)\,\mathrm{d}\hat{C}_u\right)\right) \\ &= \mathrm{E}\left(\exp(-\langle \mu, f(B_t + \cdot)\rangle)\exp\left(-\int_0^t f^2(B_u)\,\mathrm{d}u\right)\right) \end{aligned}$$

for $f \in \mathscr{S}$ and $\mu \in \mathscr{S}'$.

Proposition 4.1. *The semigroups Q_t and Q_t^C enjoy the intertwining relationship*

$$Q_t \mathscr{M} = \mathscr{M} Q_t^C,$$

where $\mathscr{M}$ is a Markov kernel from $\mathscr{S}'$ to $\mathscr{S}'$ defined on the functions Φ_f $(f \in \mathscr{S})$ by

$$\mathscr{M}(\Phi_f)(\mu) = \mathrm{E}(\exp(-\mu(f^2)^{1/2}N) = \exp(\tfrac{1}{2}\mu(f^2))$$

in which N denotes a standard Gaussian variable. In other words, $\mathscr{M}(\mu, \mathrm{d}\nu)$ is a centred Gaussian measure over $\mathscr{S}'$ with intensity μ.

Sketch of proof. We define $\mathscr{G}_t = \sigma\{B_u, C_u;\ u \leqslant t\}$. We compute the expression

$$A = \mathrm{E}(\Phi_f(\Pi_{t+s}^C)|\mathscr{F}_t)$$

first by conditioning with respect to $\mathscr{F}_{t+s}$. Now, conditionally to $\mathscr{F}_{t+s}$,

$$\Pi_{t+s}^C(f) \overset{(\text{law})}{=} (\Pi_{t+s}(f^2))^{1/2} N,$$

where N is a standard Gaussian variable, independent of B. Then, we obtain

$$A = Q_s(\mathscr{M}\Phi_f)(\Pi_t).$$

On the other hand, by conditioning first with respect to $\mathscr{G}_t$, we find

$$A = \mathscr{M}(Q_f^C(\Phi_s))(\Pi_t).$$

□

5. The process $\int_0^t f(B_t - B_s)\,\mathrm{d}B_s$

It seems natural to extend the definition of the process Π_t^A defined by (2.3) to the case where $A = B$. Since, for t fixed, the process $(B_t - B_s;\ s < t)$ is not $\mathscr{F}_s$-adapted, we must make precise the meaning of the stochastic integral $\int_0^t f(B_t - B_s)\,\mathrm{d}B_s$. $(B_t - B_s;\ s \leqslant t)$ is $\mathscr{F}^s := \sigma\{B_u - B_t;\ s \leqslant u \leqslant t\}$ adapted; therefore, we can define this integral as a backward Itô integral and we denote it by

$$\int_0^t f(B_t - B_s)\,\mathrm{d}_- B_s.$$

We recall briefly the definition of the backward integral: for an $\mathscr{F}^s$-measurable process H_s,

$$\int_0^t \mathrm{d}_- B_s H_s \stackrel{\text{def}}{=} -\int_0^t \mathrm{d}\hat{B}_s^{(t)} H_{t-s}$$

where $\hat{B}_s^{(t)} = B_t - B_{t-s}$, and on the right-hand side, the integral is a forward integral with respect to the Brownian motion $\hat{B}^{(t)}$.

Note that this integral coincides with the Skorohod integral (see Nualart and Pardoux 1988).

Proposition 5.1. *The $\mathscr{S}'$-valued process $(\Pi_t^B;\ t \geqslant 0)$ defined by*

$$\Pi_t^B(f) = \int_0^t f(B_t - B_s) d_- B_s$$

satisfies, for every f in C_b^2,

$$\Pi_t^B(f) = B_t f(0) + \int_0^t \mathrm{d}B_s \Pi_s^B(f') + \tfrac{1}{2}\int_0^t \mathrm{d}s \Pi_s^B(f''). \tag{5.1}$$

Proof. We apply Itô's formula to $f(B_t - B_s)$ and we use the following Fubini-type identity (see Rosen and Yor 1991, (2.2) and (2.3)):

$$\int_0^t \mathrm{d}_- B_s \int_s^t \mathrm{d}B_u \varphi(B_u - B_s) = \int_0^t \mathrm{d}B_u \int_0^u \mathrm{d}_- B_s \varphi(B_u - B_s). \tag{5.2}$$

□

Remark. We can also prove (5.1) without using (5.2). Take f of the form

$$f(x) = \int g(\xi)\exp(\mathrm{i}x\xi)\,\mathrm{d}\xi.$$

Then,

$$X_t := \int_0^t \exp(\mathrm{i}\xi(B_t - B_u))\,\mathrm{d}_- B_u$$

$$= \exp(\mathrm{i}\xi B_t)\int_0^t \exp(-\mathrm{i}\xi B_u)\,\mathrm{d}B_u - i\xi\exp(\mathrm{i}\xi B_t)\int_0^t \exp(-\mathrm{i}\xi B_u)\,\mathrm{d}u,$$

using the well-known property for Skorohod integrals (see Nualart and Pardoux 1988):

$$\delta(Fu_\cdot) = F\delta(u_\cdot) - \int_0^t D_t F u_t\,\mathrm{d}t.$$

We now apply Itô's formula to X_t. Integrating then with respect to $g(\xi)\,\mathrm{d}\xi$ (and using a classical Fubini theorem) yields the result.

6. A measure-valued process related to Pitman's theorem

It is shown in Matsumoto and Yor (1998) that for $\lambda \in \mathbb{R}$, the process

$$\exp(-\lambda B_t)\int_0^t \mathrm{d}s\,\exp(2\lambda B_s)$$

is a Markov process with respect to its own filtration, a result from which one recovers asymptotically Pitman's celebrated theorem (see Pitman 1975).

By analogy with our present work, this prompted us to define a measure-valued process $(\tilde{\Pi}_t)$ by

$$\tilde{\Pi}_t(f) = \int_0^t \mathrm{d}s\, f(2B_s - B_t),$$

which satisfies the equation

$$\tilde{\Pi}_t(f) = \int_0^t \mathrm{d}s f(B_s) + \int_0^t \mathrm{d}B_s \tilde{\Pi}_s(f') + \tfrac{1}{2}\int_0^t \mathrm{d}s\tilde{\Pi}_s(f''). \tag{6.1}$$

However, the analogy with (Π_t) cannot be pushed much further, as discussed in Matsumoto and Yor (1998), to which we refer the reader: in particular, $(\tilde{\Pi}_t)_t$ is not a Markov process. On the other hand, note how similar the equation (6.1) is to equation (2.1), the only change being that 'the given data' $t\delta_0$ has been changed in (6.1) into the occupation measure $\int_0^t \mathrm{d}s\delta_{B_s}$.

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