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On some inequalities of local times for Azéma martingales

TSUNG-MING CHAO and CHING-SUNG CHOU

Institute of Mathematics, National Central University, Chung-Li, Taiwan, ROC

Let $(u_t, \mathscr{G}_t)_{t\geqslant 0}$ be an Azéma martingale and its filtration, and let $(\lambda_t^x;\ x \in \mathbb{R},\ t \geqslant 0)$ be the local times of the Azéma martingale defined by the following Tanaka formula:

$$u_t 1_{\{u_t > x\}} = \int_0^t 1_{\{u_{s-} > x\}}\, \mathrm{d}u_s + \tfrac{1}{2}\lambda_t^x.$$

Then, for every $(\mathscr{G}_t)_{t\geqslant 0}$ stopping time T and every $p > 0$, there exist two universal constants c_p, $C_p > 0$ depending only on p, such that

$$c_p\|T^{1/2}\|_p \leqslant \|\lambda_T^*\|_p \leqslant C_p\|T^{1/2}\|_p, \tag{1}$$

where $\lambda_t^* = \sup_{x\in\mathbb{R}}\lambda_t^x$.

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1. Introduction

In this paper, we deal only with the Azéma martingale and its local times. The maximal local time inequality for a continuous martingale M was first obtained by Barlow and Yor (1981). They established the existence of universal constants c_p, $C_p > 0$ such that, for all continuous martingales M, with $M_0 = 0$,

$$c_p\|\langle M\rangle_\infty^{1/2}\|_p \leqslant \|\sup_a L_\infty^a(M)\|_p \leqslant C_p\|\langle M\rangle_\infty^{1/2}\|_p.$$

for all $p > 0$.

Barlow and Yor (1982) also extended the right-hand-side inequality to any continuous semimartingale. Hence, it is natural to consider possible extensions of these inequalities to right-continuous martingales.

2. Basic notation and some results

Let $(\Omega, F, (F_t)_{t\geqslant 0}, (B_t)_{t\geqslant 0}, P)$ be a standard Brownian motion, $B_0 = 0$, with the filtration $(F_t)_{t\geqslant 0}$ completed. We define

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$$\mathrm{sign}(x) = \begin{cases} 1, & \textit{for } x > 0, \\ -1, & \textit{for } x \leqslant 0. \end{cases}$$

Set $\mathscr{G}^0_t = \sigma\{\mathrm{sign}(B_s);\ s \leqslant t\}$, and let $(\mathscr{G}_t)_{t\geqslant 0}$ denote the filtration $(\mathscr{G}^0_t)_{t\geqslant 0}$ completed. It is then right continuous as a consequence of the strong Markov property of Brownian motion.

Let $(M_t)_{t\geqslant 0}$ denote the optional projection of B onto the filtration $\mathscr{G}_t$. That is to say, $M_t = \mathrm{E}(B_t|\mathscr{G}_t)$. Then $(M_t)_{t\geqslant 0}$ is a cadlag martingale with the explicit form $\mathrm{sign}(B_t)(\pi/2)^{1/2}(t - g_t)^{1/2}$, where $g_t = \sup\{s \leqslant t\colon B_s = 0\}$. If we define $u_t = \mathrm{sign}(B_t)(t - g_t)^{1/2}$, then $[u, u]_t = g_t$ and $\langle u, u\rangle_t = t/2$ (see Azéma and Yor 1989).

If X, Y are two random variables defined on some probability space, we define $\|X\|_p = (\mathrm{E}(|X|^p))^{1/p}$ for any $p > 0$. And $\|X\|_p \sim \|Y\|_p$ means that there exist two constants a_p, b_p, depending only on p, such that $a_p\|Y\|_p \leqslant \|X\|_p \leqslant b_p\|Y\|_p$. An increasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be moderate if there exists an $\alpha > 1$ such that $\sup_{x>0} F(\alpha x)/F(x) < \infty$.

For any $a \in \mathbb{R}$, let L^a_t, $\mathscr{L}^a_t$, λ^a_t be defined by the following formulae (see Meyer 1976; Azéma and Yor 1989):

$$(u_t - a)^+ = (u_0 - a)^+ + \int_0^t 1_{\{u_{s-} > a\}}\, \mathrm{d}u_s + \sum_{0<s\leqslant t} 1_{\{u_{s-} > a\}}(u_s - a)^- + 1_{\{u_{s-} \leqslant a\}}(u_s - a)^+ + \tfrac{1}{2}L^a_t, \tag{2}$$

$$(u_t - a)^+ = (u_0 - a)^+ + \int_0^t 1_{\{u_{s-} > a\}}\, \mathrm{d}u_s + \tfrac{1}{2}\mathscr{L}^a_t, \tag{3}$$

$$u_t 1_{\{u_t > a\}} = \int_0^t 1_{\{u_{s-} > a\}}\, \mathrm{d}u_s + \tfrac{1}{2}\lambda^a_t. \tag{4}$$

Then, it is well known that $L^a_t = 0$, $t \geqslant 0$, if $a \neq 0$ and $L^0_t = (2/\pi)^{1/2} l^0_t$, where l^0_t denotes the local time at 0 of Brownian motion B. Extending Barlow and Yor's local time inequality to cadlag martingales gives us a counterexample: the inequality (1) does not hold for all $p \geqslant 1$ as we consider L^*_t instead of λ^*_t (see Yor 1979). If we denote by Λ^a_t the dual predictable projection of $\mathscr{L}^a_t$, then Λ^a_t satisfies the 'occupation time density' formula (see Meyer 1976):

$$\langle u, u\rangle_t = \int_{-\infty}^{\infty} \mathrm{d}x \Lambda^x_t.$$

Let $f : \mathbb{R} \to \mathbb{R}_+$ be a Borel function; then Azéma and Yor (1989) showed that λ^a_t also satisfies the similar formula:

$$\int_0^t \mathrm{d}s f(u_s) = \int_{-\infty}^{\infty} \mathrm{d}x \lambda^x_t f(x).$$

This confirms that some time-dependent properties of u will be transmitted to the local times of u, in some analogous fashion, relative to the space parameter.

Barlow and Yor (1982) deduced the following useful lemma by applying the Garsia–Rodemich–Rumsey lemma.

Lemma 1 (Barlow and Yor (1982). *Let $\{U(a)\}$ be a family of random variables indexed by $a \in \mathbb{R}$ and taking real values. If the classical Kolmogorov's criterion*

$$\mathrm{E}(|U(a) - U(b)|^r) \leqslant H|a-b|^{r/2}$$

holds for some $H \geqslant 0$, $r > 2$ and for all $a, b \in \mathbb{R}$, then for r and any $\varepsilon \in (0, \frac{1}{2}]$ such that $r\varepsilon > 1$,

$$\left\| \sup_{a,b \in I, a \neq b} \frac{|U(a) - U(b)|}{|a-b|^{1/2-\varepsilon}} \right\|_r \leqslant C_{r,\varepsilon} H^{1/r} |I|^{\varepsilon},$$

where I is a finite interval of $\mathbb{R}$.

Another important lemma comes from the regularity of sharp bracket of u.

Lemma 2 (Chao and Chou 1998a). *Let X be a cadlag martingale with $\langle X, X \rangle_t = t$ and H be a predictable process; then there exist c_p, $C_p > 0$, depending only on p, such that*

$$c_p \left\| \left(\int_0^\infty H_s^2 \, \mathrm{d}s \right)^{1/2} \right\|_p \leqslant \left\| \sup_t \left| \int_0^t H_s \, \mathrm{d}X_s \right| \right\|_p \leqslant C_p \left\| \left(\int_0^\infty H_s^2 \, \mathrm{d}s \right)^{1/2} \right\|_p,$$

where the right-hand-side inequality holds for all $p > 0$ and the left-hand-side inequality holds for all $p \geqslant 2$.

These two lemmas play a crucial role on the proof of our main result.

A slight modification from Lemma 2 for the Azéma martingale in Chao and Chou (1998b) proved that for any $\mathscr{G}_t$ stopping time T and every $p > 0$,

$$\|u_T^*\|_p \sim \|T^{1/2}\|_p. \tag{5}$$

Combining (5) with the classical Burkholder–Davis–Gundy inequality for cadlag martingales, we obtain

$$\|T^{1/2}\|_p \sim \|[u, u]_T^{1/2}\|_p \tag{6}$$

for all $p \geqslant 1$.

The Burkholder–Davis–Gundy inequality for cadlag martingales does not hold for the case $0 < p < 1$. For further interesting results, see Monat (1994).

3. Some Azéma martingale inequalities

In the following, let $\hat{u}_t^a = \int_0^t 1_{\{u_{s-} > a\}} \, \mathrm{d}u_s$ denote the stochastic integral of the predictable process $1_{\{u_{s-} > a\}}$ with respect to u and $\mathscr{O}$ denote the optional σ-field. Yor (1978), investigating the regularity of the parameter about local time for cadlag semimartingales, established the following important sufficient condition: Let X be a semimartingale and $\mathscr{L}_t^a(X)$ be its local time defined as formula (3); if X satisfies

$$\sum_{0<s\leqslant t} |\Delta X_s| < \infty \text{ a.s.,} \qquad \text{for each } t > 0,$$

then there exists a $\mathscr{B}(\mathbb{R}) \otimes \mathscr{O}$-measurable version of $(a, t, \omega) \to \mathscr{L}_t^a(\omega)(X)$ which is everywhere jointly right continuous in a and t. Moreover the limits $\mathscr{L}_t^{a^-}(X) = \lim_{b\to a, b<a} \mathscr{L}_t^b(X)$ almost surely exist.

As is well known, the Azéma martingale does not satisfy this condition, but the next proposition can be used to derive regularity properties for $(\lambda^a)_{a\in\mathbb{R}}$ and $(\mathscr{L}^a)_{a\in\mathbb{R}}$.

Proposition 1. *Let u be an Azéma martingale; there exists a universal constant $c_p > 0$ such that, for any finite stopping time T and $a, b \in \mathbb{R}$,*

$$\mathrm{E}\left(\sup_{s\leqslant T} |\hat{u}_s^a - \hat{u}_s^b|^p\right) \leqslant c_p \mathrm{E}(T^{p/4})|b-a|^{p/2}$$

holds for any $p \geqslant 4$.

Proof. As usual, the values of the universal constant will vary from one line to another. Take $a < b$ to simplify the second expectation. By the right-hand-side inequality of Lemma 2, we have

$$\begin{aligned}
\mathrm{E}(\sup_{s\leqslant T} |\hat{u}_s^a - \hat{u}_s^b|^p) &\leqslant c_p \mathrm{E}\left(\left(\int_0^T 1_{\{a<u_{s-}<b\}}\,\mathrm{d}s\right)^{p/2}\right) \\
&= c_p \mathrm{E}\left(\left|\int_a^b \lambda_T^x\,\mathrm{d}x\right|^{p/2}\right) \\
&= c_p |b-a|^{p/2} \mathrm{E}\left(\left|\frac{1}{b-a}\int_a^b \lambda_T^x\,\mathrm{d}x\right|^{p/2}\right) \\
&\leqslant c_p |b-a|^{p/2} \mathrm{E}\left(\left|\frac{1}{b-a}\int_a^b (\lambda_T^x)^{p/2}\,\mathrm{d}x\right|\right) \\
&\leqslant c_p |b-a|^{p/2} \sup_x \mathrm{E}((\lambda_T^x)^{p/2}).
\end{aligned}$$

The first equality comes from the occupation time density formula for λ. It is easy to deduce from Tanaka's formula (4) that

$$\mathrm{E}((\lambda_T^x)^{p/2}) \leqslant c_p \mathrm{E}\left(|u_T|^{p/2} + \left| \int_0^T 1_{\{u_{s-} > x\}}\, \mathrm{d}u_s \right|^{p/2} \right)$$

$$\leqslant c_p \mathrm{E}(T^{p/4}).$$

Since the right-hand side is independent of x, this completes the proof. □

Corollary 1. *There exists a version of* $(\hat{u})_t^a$ *such that* $(a, t, \omega) \to (\hat{u})_t^a(\omega)$ *is* $\mathscr{B}(\mathbb{R}) \otimes \mathscr{O}$-*measurable, and everywhere continuous in* a *and right continuous in* t.

Proof. The result follows from Kolmogorov's continuity criterion. □

Corollary 2. *There exists a* $\mathscr{B}(\mathbb{R}) \otimes \mathscr{O}$-*measurable version of* $(a, t, \omega) \to \lambda_t^a(\omega)$ *which is everywhere jointly right continuous in* t *and right continuous in* a. *Besides,* $\lambda_t^{a^-} \equiv \lim_{b < a, b \to a} \lambda_t^b$ *exists, and* $\lambda_t^a - \lambda_t^{a^-} = -2u_t 1_{\{u_t = a\}}$.

Proof. By Corollary 1, $(\hat{u}_t^a)$ has a continuous version in a, hence formula (4) implies that λ_t^a has a measurable version which is right continuous in a and

$$\lambda_t^{a^-} = 2u_t 1_{\{u_t \geqslant a\}} - 2(\hat{u}_t^a).$$

In consequence, the result comes directly from formula (4). □

Corollary 3. *There exists a* $\mathscr{B}(\mathbb{R}) \otimes \mathscr{O}$-*measurable version of* $(a, t, \omega) \to \mathscr{L}_t^a(\omega)$ *which is jointly continuous in* a *and right continuous in* t.

Proof. This result is derived immediately from Tanaka's formula (3) and Corollary 1. □

By applying Lemmas 1 and 2, one can deduce the following proposition which is a fundamental part of our main result.

Proposition 2. *Let* $\varepsilon \in (0, \frac{1}{2}]$. *Then, for any finite* $\mathscr{G}_t$ *stopping time* T, *and any moderate function* Φ, *one has*

$$\mathrm{E}\left(\Phi\left(\sup_{s \leqslant T; a \neq b} \frac{|\hat{u}_s^a - \hat{u}_s^b|}{|a-b|^{1/2-\varepsilon}} \right) \right) \leqslant C_{\Phi,\varepsilon} \mathrm{E}(\Phi(T^{1/2(\varepsilon+1/2)})).$$

In particular,

$$\left\| \sup_{s \leqslant T; a \neq b} \frac{|\hat{u}_s^a - \hat{u}_s^b|}{|a-b|^{1/2-\varepsilon}} \right\|_r \leqslant C_{r,\varepsilon} \| T^{(\varepsilon+1/2)/2} \|_r$$

holds for all $r \geqslant 1$.

Proof. Let S, T be two $\mathscr{G}_t$ stopping times with $S \leqslant T$. Define

$$U_\varepsilon(u;\ T) \equiv \sup_{s\leqslant T; a\neq b} \frac{|\hat{u}^a_s - \hat{u}^b_s|}{|a-b|^{1/2-\varepsilon}}, \qquad \hat{Z}^a_t = \hat{u}^a_{S+t\wedge T} 1_{\{S<T\}}.$$

Then, $\hat{Z}^a_t$ is a $(\mathscr{G}_{S+t})_{t\geqslant 0}$ martingale with sharp bracket

$$\langle \hat{Z}^a, \hat{Z}^a\rangle_t = \int_S^{S+t\wedge T} 1_{\{u_{s-}>a\}}\,\mathrm{d}s 1_{\{S<T\}}.$$

If we can show that

$$\|\sup_t |\hat{Z}^a_t - \hat{Z}^b_t|\|_r \leqslant C_r |b-a|^{1/2} \|T^{1/2}\|^{1/2}_\infty P(S<T)$$

holds for some $r\geqslant 2$, then, since $|a|, |b| \leqslant u^*_T \leqslant T^{1/2}$, this, together with Lemma 1, implies that

$$\begin{aligned}
\|U_\varepsilon(u;\ T^-) - U_\varepsilon(u;\ S^-)\|_r &\leqslant \left\| \sup_{t; a\neq b} \frac{|\hat{Z}^a_t - \hat{Z}^b_t|}{|a-b|^{1/2-\varepsilon}} \right\|_r \\
&\leqslant C_{r,\varepsilon} |b-a|^\varepsilon \|T^{1/2}\|^{1/2}_\infty P(S<T) \\
&\leqslant C_{r,\varepsilon} \|u^*_T\|^\varepsilon_\infty \|T^{1/2}\|^{1/2}_\infty P(S<T) \\
&\leqslant C_{r,\varepsilon} \|T^{1/2}\|^\varepsilon_\infty \|T^{1/2}\|^{1/2}_\infty P(S<T) \\
&= C_{r,\varepsilon} \|T^{1/2}\|^{1/2+\varepsilon}_\infty P(S<T).
\end{aligned}$$

Since we are considering an Azéma martingale u stopped by a $\mathscr{G}_t$ stopping time T, our result is a straightforward consequence of Lemma 7 and 8 in Jacka and Yor (1993).

Take $a<b$ to simplify the second expectation. For any $r\geqslant 4$, Lemma 2 implies that

$$\begin{aligned}
\|\sup_t |Z^a_t - Z^b_t|\|_r &\leqslant c_r \left\| \left(\int_S^T 1_{\{a<u_{s-}<b\}}\,\mathrm{d}s \right)^{1/2} 1_{\{S<T\}} \right\|_r \\
&= c_r \left\| \left(\int_a^b (\lambda^x_T - \lambda^x_S)\,\mathrm{d}x \right)^{1/2} 1_{\{S<T\}} \right\|_r \\
&\leqslant c_r |b-a|^{1/2} \left\| \left(\frac{1}{b-a} \int_a^b \lambda^x_T - \lambda^x_S\,\mathrm{d}x \right)^{1/2} 1_{\{S<T\}} \right\|_r \\
&\leqslant c_r |b-a|^{1/2} \sup_x \|(\lambda^x_T - \lambda^x_S)^{1/2} 1_{\{S<T\}}\|_r.
\end{aligned}$$

If we set $Y^x_t = \int_S^{S+t\wedge T} 1_{\{u_{s-}>x\}}\,\mathrm{d}u_s$, then $(Y^x_t)_{t\geqslant 0}$ is a zero-mean $(\mathscr{G}_{S+t})_{t\geqslant 0}$ martingale with sharp bracket $\langle Y^x, Y^x\rangle_\infty = \int_S^T 1_{\{u_{s-}>x\}}\,\mathrm{d}s$. Therefore, Tanaka's formula (4) implies that

$$
\begin{aligned}
&\|(\lambda_T^x - \lambda_S^x)^{1/2} 1_{\{S<T\}}\|_r \\
&\quad \leqslant c_r \left(\| |u_T - u_S|^{1/2} 1_{\{S<T\}} \|_r + \left\| \left| \int_0^T 1_{\{u_{s-}>x\}} \, \mathrm{d}u_s - \int_0^S 1_{\{u_{s-}>x\}} \, \mathrm{d}u_s \right|^{1/2} 1_{\{S<T\}} \right\|_r \right) \\
&\quad \leqslant c_r \| T^{1/4} 1_{\{S<T\}} \|_r + c_r \left\| \left(\int_S^T 1_{\{u_{s-}>x\}} \, \mathrm{d}s \right)^{1/4} 1_{\{S<T\}} \right\|_r \\
&\quad \leqslant c_r \| T^{1/4} 1_{\{S<T\}} \|_r \\
&\quad \leqslant c_r \| T^{1/2} \|_\infty^{1/2} P(S<T).
\end{aligned}
$$

Since the last term is independent of x, this finishes our proof. □

Before proceeding, we note the following elementary corollary.

Corollary 4. *Let $\varepsilon \in (0, \frac{1}{2}]$. Then, for any moderate function Φ, and any finite $\mathscr{G}_t$ stopping time T, one has*

$$
\mathrm{E}\left(\Phi\left(\sup_{s \leqslant T; a \neq b} \frac{|\mathscr{L}_s^a - \mathscr{L}_s^b|}{|a-b|^{1/2-\varepsilon}} \right) \right) \leqslant C_{\Phi,\varepsilon} \mathrm{E}(\Phi(T^{1/2(\varepsilon+1/2)})).
$$

Moreover, for any $r \geqslant 1$,

$$
\left\| \sup_{s \leqslant T; a \neq b} \frac{|\mathscr{L}_s^a - \mathscr{L}_s^b|}{|a-b|^{1/2-\varepsilon}} \right\|_r \leqslant C_r \| T^{(1/2+\varepsilon)/2} \|_r.
$$

Proof. The proof is similar to that of Proposition 2; in addition, formula (3) gives us the elementary inequality

$$
|\mathscr{L}_t^a - \mathscr{L}_t^b| \leqslant 2|\hat{u}_t^a - \hat{u}_t^b| + 4|a-b|
$$

which helps us to get the transformation from $\hat{u}_t^a$ to $\mathscr{L}_t^a$. □

We now approach the core of the proof of the main result. As we apply the δ-variations of stochastic processes to the parameter of local time $\mathscr{L}$, a slight modification of the right-hand-side inequality of (1) can be obtained. For many references on δ-variations of stochastic processes, see Lépingle (1976).

Proposition 3. *Let $\delta > 2$, and let $V_\delta(\mathscr{L}) = \sup_\sigma V_\delta(\mathscr{L}; \sigma)$, where $V_\delta(\mathscr{L}; \sigma) = \sum_\sigma \sup_{s \leqslant T} |\mathscr{L}_s^{a_{i+1}} - \mathscr{L}_s^{a_i}|^\delta$, $\sigma = (0 < a_1 < a_2 \cdots < a_m < \infty)$ and T* is *$\mathscr{G}_t$ stopping time. If we define $W_\delta(\mathscr{L}) = V_\delta(\mathscr{L})^{1/\delta}$, then, for any $r > 0$,*

$$
\| W_\delta(\mathscr{L}) \|_r \leqslant C_r \| T^{1/2} \|_r.
$$

In particular, if we set $\mathscr{L}_t^* = \sup_a \mathscr{L}_t^a$, then

$$\|\mathscr{L}_T^*\|_r \leqslant C_r \|T^{1/2}\|_r. \tag{7}$$

Proof. Since $\delta > 2$, there exist $\varepsilon \in (0, \frac{1}{2})$ and $v \geqslant 1$ such that $v/\delta = \frac{1}{2} - \varepsilon$. Then the elementary inequality

$$V_\delta(\mathscr{L}) \leqslant \sup_{s \leqslant T; a \neq b} \frac{|\mathscr{L}_s^a - \mathscr{L}_s^b|^\delta}{|a-b|^v} (2T^{\frac{v}{2}})$$

implies, with the help of Hölder's inequality (with $1/q = \frac{1}{2} - \varepsilon$, and $1/p = \frac{1}{2} + \varepsilon$), that for any $r \geqslant 1$

$$\begin{aligned} \|V_\delta(\mathscr{L})^{1/\delta}\|_r &\leqslant 2^{v/\delta} \Big\| \sup_{s \leqslant T; a \neq b} \frac{|\mathscr{L}_s^a - \mathscr{L}_s^b|}{|a-b|^{v/\delta}} \Big\|_{rp} \|T^{v/(2\delta)}\|_{rq} \\ &\leqslant C_{r,\varepsilon} \|T^{(1/2+\varepsilon)/2}\|_{rp} \|T^{v/(2\delta)}\|_{rq} \\ &\leqslant C_{r,\varepsilon} \|T^{1/2}\|_r^{1/p} \|T^{1/2}\|_r^{1/q} \\ &= C_{r,\varepsilon} \|T^{1/2}\|_r. \end{aligned}$$

The second inequality comes from Corollary 4. In the case of $0 < r < 1$, we use Lenglart's (1979) *relation de domination* to obtain our result. Actually, inequality (7) is also based on Corollary 4. □

From the above results, we can derive the main theorem.

Theorem 1. *For any $\mathscr{G}_t$ stopping time T, there exist c_p, $C_p > 0$ such that*

$$c_p \|T^{1/2}\|_p \leqslant \|\lambda_T^*\|_p \leqslant C_p \|T^{1/2}\|_p,$$

holds for all $p > 0$.

Proof. By the occupation time density formula, one has almost surely

$$T = \int_{-\infty}^{\infty} \lambda_T^x \, \mathrm{d}x \leqslant 2u_T^* \lambda_T^* \leqslant 2\lambda_T^* T^{1/2}.$$

Hence, the left-hand-side inequality is obvious. For the right-hand-side inequality, the comparison of formulae (3) and (4) yields:

$$\mathscr{L}_T^x - \lambda_T^x = 2((u_T - x)1_{\{u_T > x\}} - (-x)^+ - u_T 1_{\{u_T > x\}}).$$

Therefore,

$$\mathscr{L}_T^x - \lambda_T^x = \begin{cases} x1_{\{u_T \leqslant x\}}, & \textit{for } x < 0, \\ -x1_{\{u_T > x\}}, & \textit{for } x \geqslant 0, \end{cases}$$

implies

$$\lambda_T^* \leqslant \mathscr{L}_T^* + 2T^{1/2}.$$

For any $p \geqslant 1$, Minkowski's inequality and (7) imply

$$\begin{aligned}\|\lambda_T^*\|_p &\leqslant \|\mathscr{L}_T^*\|_p + 2\|T^{1/2}\|_p \\ &\leqslant C_p\|T^{1/2}\|_p.\end{aligned}$$

Again, Lenglart's *relation de domination* allows us to lower the exponent to the case $0 < p < 1$. □

The next theorem, which is analogous to Theorem 1, necessitates extending Barlow and Yor's inequality to cadlag martingales. For this purpose, we need an extra lemma.

Remark. However, the extension is not done in complete generality. More precisely, we are not sure whether it is true for normal martingales or not.

Lemma 3. *Let $N_t \equiv \int_{-\infty}^{\infty}(\mathscr{L}_t^x - \Lambda_t^x)\,\mathrm{d}x$; then (N_t) is a $(\mathscr{G}_t)$-martingale. In other words, $\int_{-\infty}^{\infty}\Lambda_t^x\,\mathrm{d}x$ is the dual predictable projection of $\int_{-\infty}^{\infty}\mathscr{L}_t^x\,\mathrm{d}x$.*

Furthermore, we have

$$[u, u]_t = \int_{-\infty}^{\infty}\mathscr{L}_t^x\,\mathrm{d}x.$$

Proof. It suffices to show that $\mathrm{E}(|N_t|) < \infty$.

$$\begin{aligned}\mathrm{E}(|N_t|) &\leqslant \mathrm{E}\left(\int_{-\infty}^{\infty}|\mathscr{L}_t^x - \Lambda_t^x|\,\mathrm{d}x\right) \\ &\leqslant \mathrm{E}\left(\int_{-\infty}^{\infty}(\mathscr{L}_t^x + \Lambda_t^x)\,\mathrm{d}x\right) \\ &= 2\int_{-\infty}^{\infty}\mathrm{E}(\Lambda_t^x)\,\mathrm{d}x \\ &= 2\mathrm{E}\left(\int_{-\infty}^{\infty}\Lambda_t^x\,\mathrm{d}x\right) \\ &= 2t < \infty.\end{aligned}$$

This completes the proof. □

Theorem 2. *For any finite $\mathscr{G}_t$ stopping time T, there exist $c_p, C_p > 0$ such that*

$$c_p\|T^{1/2}\|_p \leqslant \|\mathscr{L}_T^*\|_p \leqslant C_p\|T^{1/2}\|_p$$

for all $p > 0$.

Proof. It suffices to prove the left-hand-side inequality. Let $p \geqslant 1$. Lemma 3 implies that

$$[u, u]_T = \int_{-\infty}^{\infty} \mathscr{L}_T^a \, \mathrm{d}a \leqslant 2u_T^* \mathscr{L}_T^*.$$

Hence, using inequalities (6) and (5),

$$\begin{aligned} \|T^{1/2}\|_p &\leqslant C_p \|[u, u]_T^{1/2}\|_p \\ &\leqslant C_p \|\mathscr{L}_T^*\|_p^{1/2} \|u_T^*\|_p^{1/2} \\ &\leqslant C_p \|\mathscr{L}_T^*\|_p^{1/2} \|T^{1/2}\|_p^{1/2}. \end{aligned}$$

It remains to divide both sides by $\|T^{1/2}\|_p^{1/2}$, and the result holds.

Let $0 < p < 1$. Hence the jump of the Azéma martingale occurring at time t is

$$\Delta u_t = -u_{t^-} 1_{\{\Delta u_t \neq 0\}}.$$

By formulae (2) and (3), there is no problem in showing that

$$\mathscr{L}_t^0 = L_t^0 = \left(\frac{2}{\pi}\right)^{1/2} l_t^0. \tag{8}$$

Moreover, Lenglart *et al.* (1980) proved that if F is a slowly increasing function, there exists a universal constant C depending only on F such that

$$\mathrm{E}(F(T^{1/2})) \leqslant C\mathrm{E}(F(l_T^0)) \tag{9}$$

for any $\mathscr{F}_t$ stopping time T, where $\mathscr{F}_t$ denotes the Brownian filtration.

It is well known that the natural filtration $\mathscr{G}_t$ of an Azéma martingale is contained in the Brownian filtration $\mathscr{F}_t$; see Azéma and Yor (1989). Combining (8) with (9), if we consider $F(x) = x^p$, for $p \in (0, 1)$, then for any finite $\mathscr{G}_t$ stopping time T,

$$\mathrm{E}(T^{p/2}) \leqslant C_p \mathrm{E}((l_T^0)^p) = C_p \left(\frac{\pi}{2}\right)^{1/2} \mathrm{E}((\mathscr{L}_T^0)^p) \leqslant C_p \left(\frac{\pi}{2}\right)^{1/2} \mathrm{E}((\mathscr{L}_T^*)^p). \tag{10}$$

This completes the proof. □

The following corollary, which gives some norm-equivalent relations, is a direct consequence of Theorems 1 and 2 and inequality (5).

Corollary 5. *For any $\mathscr{G}_t$ stopping time T,*

$$\|u_T^*\|_p \sim \|T^{1/2}\|_p \sim \|\lambda_T^*\|_p \sim \|\mathscr{L}_T^*\|_p$$

for all $p > 0$.

Remark. The above inequalities are constructed for $\mathscr{G}_t$-stopping times. It is well known that an $\mathscr{F}_t$-stopping time is not necessarily a $\mathscr{G}_t$-stopping time, and (10) is still valid for $\mathscr{F}_t$-stopping times. This inspires us to consider some possible extensions of the local time inequalities for Azéma martingales to positive random times. For some inequalities of local time for continuous local martingales stopped at a random time, see Chou (1995).

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