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# The first crossing-time density for Brownian motion with a perturbed linear boundary

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An expansion is derived for the density of the first time a Brownian path crosses a perturbed linear boundary $\alpha + \varepsilon f(t)$. When the perturbation $f(t)$ is a finite mixture of negative exponentials of either sign the expansion is shown to converge for all values of the perturbation parameter $\varepsilon$. Numerical examples suggest that the technique works well for a wider choice of $f(t)$, including cases where $f(t)$ is periodic.

*Keywords:* Brownian motion; first crossing-time density; perturbed linear boundary

## 1. Introduction

Apart from the classical case of a linear boundary, the density of the first crossing time of a Brownian path is known exactly for only a few boundaries such as the square root and the parabolic, and each of these involves considerable computation. Recently a straightforward approximate method applicable to any boundary has become available which in most cases approximates the density with high accuracy (Daniels 1996; Lo *et al.* in preparation), so that from a practical point of view the problem can be regarded as essentially solved. However, it is still of interest to discover new boundaries for which a theoretical solution for the density is easily available. One approach is to consider perturbations of a linear boundary. A recent interesting paper by Hobson *et al.* (1999) addresses this problem for the distribution function of the first crossing time. In the present paper we adopt an entirely different approach to derive expansions for the density function which works well in practice even for periodic perturbations.

In general such expansions can be regarded as asymptotic, though the range of validity of the perturbation parameter $\varepsilon$ is usually difficult to establish. In the special case where the perturbation is a finite mixture of negative exponentials with positive or negative weights, we show that the expansion converges for all values of $\varepsilon$.

## 2. The perturbation expansion

Suppose there is standard Brownian motion $W(t)$ with $W(0) = 0$ and a one-sided boundary

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$\xi(t) > 0$ such that $W(t)$ is certain to cross it eventually. Then the first crossing-time density $g(t)$ satisfies the Wald identity

$$\int_0^\infty g(t) e^{\theta\xi(t) - \frac{1}{2}\theta^2 t}\, dt = 1, \qquad \theta \geqslant 0. \tag{2.1}$$

With $s = \frac{1}{2}\theta^2$ it becomes

$$\int_0^\infty g(t) e^{\sqrt{2s}\xi(t) - st}\, dt = 1, \qquad s \geqslant 0. \tag{2.2}$$

We consider boundaries of the form

$$\xi(t) = \alpha + \varepsilon f(t) \tag{2.3}$$

which are perturbations of a horizontal boundary of height $\alpha > 0$. The solution can be easily modified to allow for a gradient by the Cameron–Martin–Girsanov device. For such a boundary (2.2) can be rearranged as

$$\int_0^\infty g(t) e^{\varepsilon\sqrt{2s} f(t) - st}\, dt = e^{-\alpha\sqrt{2s}}. \tag{2.4}$$

It is assumed that $f(t)$ and its derivatives are continuous, $f'(0)$ is finite and $|f(t)|$ is bounded.

Following the usual perturbation routine, $g(t)$ is taken to be expandable in the form

$$g(t) = g_0(t) + \varepsilon g_1(t) + \varepsilon^2 g_2(t) + \cdots.$$

When $t$ is small the boundary is approximately $\xi(t) \sim a + bt$, where $a = \alpha + \varepsilon f(0)$, $b = \varepsilon f'(0)$, so that $g(t) \sim a \exp\{-(a+bt)^2/2t\}/(\sqrt{2\pi} t^{3/2})$. It follows that $g(0) = 0$, $g'(0) = 0$, ... and hence $g_r(0) = 0$, $g'_r(0) = 0 \ldots$ for all $r$. From (2.4),

$$\int_0^\infty (g_0(t) + \varepsilon g_1(t) + \varepsilon^2 g_2(t) + \cdots)\{1 + \varepsilon\sqrt{2s} f(t) + \tfrac{1}{2}\varepsilon^2(\sqrt{2s})^2 f^2(t) + \cdots\} e^{-st}\, dt = e^{-\alpha\sqrt{2s}}. \tag{2.5}$$

The terms $g_0(t)$, $g_1(t)$, ... are found sequentially by equating coefficients of powers of $\varepsilon$ on both sides and solving the resulting equations for the Laplace transforms $g_0^*(s)$, $g_1^*(s)$, ... which are then inverted to give the terms of $g(t)$. We shall use the notation $g^*(s) \doteqdot g(t)$ to denote the Laplace transform relation.

The constant term immediately gives $g_0^*(s) = \exp(-\alpha\sqrt{2s})$, which transforms to the density $g_0(t) = \alpha \exp(-\alpha^2/2t)/(\sqrt{2\pi} t^{3/2})$ for the horizontal boundary. Writing

$$g_{n,k}^*(s) \doteqdot g_{n,k}(t) = g_n(t) f^k(t), \tag{2.6}$$

with $g_{n,0}^*(s) = g_n^*(s)$, we obtain the following equations:

$$g_0^*(s) = \mathrm{e}^{-\alpha\sqrt{2s}},$$

$$g_1^*(s) + \sqrt{2s}\,g_{0,1}^*(s) = 0,$$

$$g_2^*(s) + \sqrt{2s}\,g_{1,1}^*(s) + \frac{(\sqrt{2s})^2}{2!}\,g_{0,2}^*(s) = 0,$$

$$\vdots$$

$$g_n^*(s) + \sqrt{2s}\,g_{n-1,1}^*(s) + \frac{(\sqrt{2s})^2}{2!}\,g_{n-2,2}^*(s) + \cdots + \frac{(\sqrt{2s})^n}{n!}\,g_{0,n}^*(s) = 0. \tag{2.7}$$

The inversion of terms like $s^{\frac{1}{2}k} g_{r,k}^*(s)$ differs according to whether $k$ is even or odd. For even $k = 2j$,

$$s^j g_{r,2j}^*(s) = s^{j-1} g_{r,2j}(0) + s^{j-2} g'_{r,2j}(0) + \cdots + g_{r,2j}^{(j-1)}(0) + \int_0^\infty g_{r,2j}^{(j)}(t)\mathrm{e}^{-st}\,\mathrm{d}t,$$

that is,

$$s^j g_{r,2j}^*(s) \doteqdot \frac{\mathrm{d}^j}{\mathrm{d}t^j}\{g_r(t) f^{2j}(t)\}, \tag{2.8}$$

because $g_r(t)$ and all its derivatives vanish at $t = 0$. For odd $k = 2j - 1$,

$$s^{j-1/2} g_{r,2j-1}^*(s) = s^{-3/2} s^{j+1} g_{r,2j-1}^*(s),$$

and since $s^{-3/2} \doteqdot 2(t/\pi)^{1/2}$,

$$s^{j-1/2} g_{r,2j-1}^*(s) \doteqdot \frac{2}{\sqrt{\pi}} \int_0^t \frac{\mathrm{d}^{j+1}}{\mathrm{d}u^{j+1}} \{g_r(u) f^{2j-1}(u)\} \sqrt{t-u}\,\mathrm{d}u. \tag{2.9}$$

The terms $g_n(t)$ can then be found sequentially by straightforward numerical differentiation and integration.

The Cameron–Martin–Girsanov factor enables the boundary $\xi(t)$ to be replaced by $\xi(t) + \beta t$. The density then becomes

$$g(t|\beta) = g(t)\exp\{-\beta\xi(t) - \tfrac{1}{2}\beta^2 t\}.$$

This automatically allows for the possibility that a Brownian path may no longer be certain to cross the new boundary.

## 3. An example

The density $g(t)$ was computed for the boundary $\xi(t) = \alpha + \beta t + \varepsilon \sin t$ with $\alpha = 2.0$, $\beta = 0.1$, $\varepsilon = 0.25$. (This example was simulated by Roberts and Shortland (1995) for comparison with their hazard rate tangent approximation.) Figure 1 shows the results of

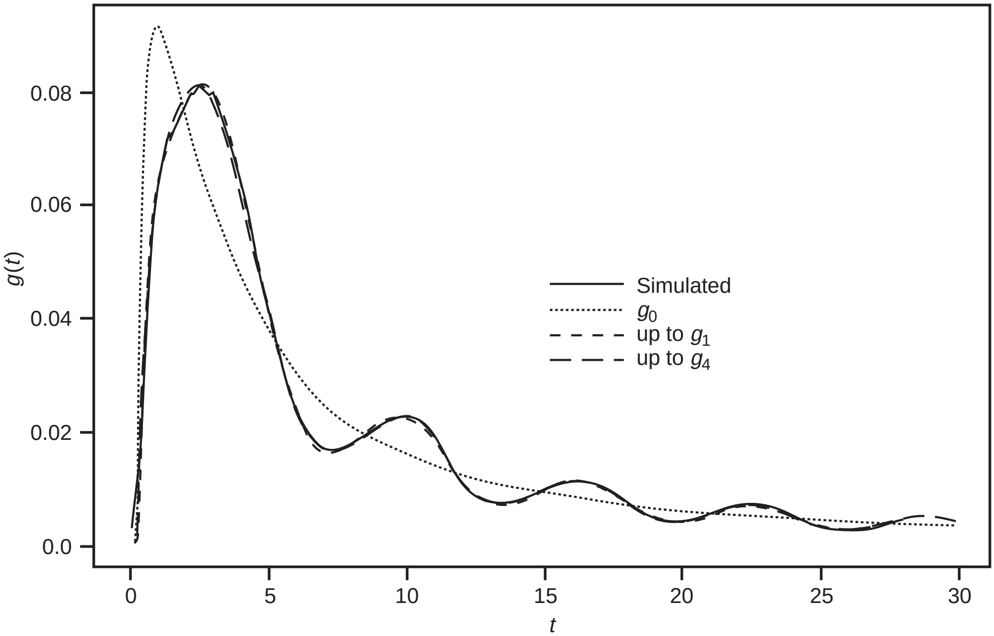


**Figure 1.** Approximations to the density of first crossing time of Brownian motion for the boundary $\xi(t) = \alpha + \beta t + \varepsilon \sin t$, with $\alpha = 2.0$, $\beta = 0.1$, $\varepsilon = 0.25$. The density for $\varepsilon = 0$ is shown for comparison.

stopping at $g_1$ and $g_4$ respectively, together with the result obtained from $2 \times 10^6$ simulations. The density $g_0(t)$ for the unperturbed boundary $\alpha + \beta t$ is also shown for comparison. The computation was most expeditiously done by alternately using trapezoidal integration and Lagrange differentiation on a sufficiently fine lattice.

# 4. Convergence

We now discuss the class of boundaries where $f(t)$ is a mixture of $m$ negative exponentials each with positive or negative weight. When $m$ is finite we show that the perturbation expansion converges for all $\varepsilon$. When $m$ is infinite a weaker result is obtained.

In the simplest case where $f(t)$ is a constant $c$, it is obvious that the expansion converges for all $\varepsilon$ since $g^*(s) = \exp - (\alpha + \varepsilon c)\sqrt{2s}$. The next simplest boundary is $f(t) = c\exp(-\lambda t)$, $\lambda > 0$, but since $c$ can be absorbed into $\varepsilon$ we need only consider $f(t) = \exp(-\lambda t)$. In that case $g^*_{n,k}(s) = g^*_n(s + k\lambda)$ and equations (2.7) become

$$
\begin{aligned}
&g_0^*(s) = \mathrm{e}^{-\alpha\sqrt{2s}}\\
&g_1^*(s) + \sqrt{2s}\,g_0^*(s+\lambda) = 0\\
&g_2^*(s) + \sqrt{2s}\,g_1^*(s+\lambda) + \frac{(\sqrt{2s})^2}{2!}\,g_0^*(s+2\lambda) = 0\\
&\qquad\qquad\vdots\\
&g_n^*(s) + \sqrt{2s}\,g_{n-1}^*(s+\lambda) + \frac{(\sqrt{2s})^2}{2!}\,g_{n-2}^*(s+2\lambda) + \cdots + \frac{(\sqrt{2s})^n}{n!}\,g_0^*(s+n\lambda) = 0. \qquad (4.1)
\end{aligned}
$$

In the $r$th equation replace $s$ by $s+(n-r)\lambda$, $r = 0, 1, 2, \ldots, n$, so that the argument of $g_0^*$ in each equation becomes $s + n\lambda$. On rearranging the terms the equations can then be put in the form

$$
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0\\
\sqrt{2(s+(n-1)\lambda)} & 1 & 0 & \ldots & 0\\
\frac{[\sqrt{2(s+(n-2)\lambda}]^2}{2!} & \sqrt{2(s+(n-2)\lambda)} & 1 & \ldots & 0\\
 & \vdots & & & \\
\frac{(\sqrt{2s})^n}{n!} & \frac{(\sqrt{2s})^{n-1}}{(n-1)!} & \frac{(\sqrt{2s})^{n-2}}{(n-2)!} & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
g_0^*(s+n\lambda)\\
g_1^*(s+(n-1)\lambda)\\
g_2^*(s+(n-2)\lambda)\\
\vdots\\
g_n^*(s)
\end{bmatrix}
=
\begin{bmatrix}
\mathrm{e}^{-\alpha\sqrt{2(s+n\lambda)}}\\
0\\
0\\
\vdots\\
0
\end{bmatrix}, \qquad (4.12)
$$

leading to the solution

$$
g_n^*(s) = (-1)^n
\begin{vmatrix}
\sqrt{2(s+(n-1)\lambda} & 1 & 0 & \ldots & 0\\
\frac{[\sqrt{2(s+(n-2)\lambda}]^2}{2!} & \sqrt{2(s+(n-2)\lambda} & 1 & \ldots & 0\\
 & \vdots & & & \\
\frac{(\sqrt{2s})^n}{n!} & \frac{(\sqrt{2s})^{n-1}}{(n-1)!} & \frac{(\sqrt{2s})^{n-2}}{(n-2)!} & \ldots & \sqrt{2s}
\end{vmatrix}
\mathrm{e}^{-\alpha\sqrt{2(s+n\lambda)}}. \qquad (4.3)
$$

Determinants of the kind appearing in (4.3) have been discussed in quite a different context by Daniels (1945; 1963), where their asymptotic properties were investigated. We recall some of the results which are applicable here. Consider

$$B_n = n!\begin{vmatrix} b_n & \dfrac{b_n^2}{2!} & \dfrac{b_n^3}{3!} & \dots & \dfrac{b_n^{n-1}}{(n-1)!} & \dfrac{b_n^n}{n!} \\ 1 & b_{n-1} & \dfrac{b_{n-1}^2}{2!} & \dots & \dfrac{b_{n-1}^{n-2}}{(n-2)!} & \dfrac{b_{n-1}^{n-1}}{(n-1)!} \\ 0 & 1 & b_{n-2} & \dots & \dfrac{b_{n-2}^{n-3}}{(n-3)!} & \dfrac{b_{n-2}^{n-2}}{(n-2)!} \\ & \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 & b_1 \end{vmatrix} \tag{4.4}$$

where $b_n$ is a descending positive sequence with $b_0 = 1$. It can also be expressed in the form

$$B_n = n!\int_0^{b_n} dx_{n-1}\int_{x_{n-1}}^{b_{n-1}} dx_{n-2}\dots\int_{x_1}^{b_1} dx_0. \tag{4.5}$$

This follows from the fact that the associated Gontcharoff polynomial (Gontcharoff 1937)

$$B_0(x) = 1, \qquad B_n(x) = n!\int_x^{b_n} dx_{n-1}\int_{x_{n-1}}^{b_{n-1}} dx_{n-2}\dots\int_{x_1}^{b_1} dx_0, \quad n \geqslant 1 \tag{4.6}$$

satisfies the equation

$$\frac{dB_n(x)}{dx} = -nB_{n-1}(x),$$

with $B_n(0) = B_n$, $B_n(b_n) = 0$. Expanding $B_r(b_r)$ in powers of $b_r$ and solving the resulting equations for $r = 1, 2, \dots, n$ leads to (4.4).

An obvious upper bound for $B_n$ is obtained when each $b_r$ is replaced by 1, giving $B_n < 1$, which is adequate for our purpose. With a trivial transposition the determinant in (4.3) has the same form as that in (4.4) and can be expressed as

$$\int_0^{\sqrt{2[s+(n-1)\lambda]}} dx_{n-1}\int_{x_{n-1}}^{\sqrt{2[s+(n-2)\lambda]}} dx_{n-2}\dots\int_{x_1}^{\sqrt{2(s+\lambda)}} dx_0 = \frac{(2(s+n\lambda))^{n/2}}{n!}B_n,$$

where $b_r = \sqrt{1 - r\lambda/(s+n\lambda)}$.

We can now examine the convergence of the perturbation expansion when $f(t) = \exp(-\lambda t)$. Since

$$|g_n^*(s)| < \frac{[2(s+n\lambda)]^{n/2}}{n!}\,e^{-\alpha\sqrt{2(s+n\lambda)}} = h_n,$$

the expansion is dominated by $\sum_n |\varepsilon|^n h_n$, and when $n$ is large the ratio of the $(n+1)$th to the $n$th term is approximately

$$|\varepsilon| h_{n+1}/h_n \sim \sqrt{\frac{2\lambda\,\mathrm{e}}{n}}|\varepsilon|.$$

The expansion therefore converges for all $\varepsilon$.

The rate of convergence established here is slower than when $\lambda = 0$. An asymptotic expression for $B_n$ when $n$ is large is available and might be expected to improve the upper bound. In Daniels (1963, Section 4), $b_n$ is regarded as the value taken by a continuous differentiable concave function $\beta(\omega)$ at $\omega = r/n$ with $\beta(0) = 1$, $\beta(\xi) = 0$. In our case $\beta(\omega) = \sqrt{1 - n\lambda\omega/(s + n\lambda)}$, $\xi = 1 + s/n\lambda$, so that $\beta'(\xi) = -\infty$ and the approximation reduces to $B_n \sim 1 - |\beta'(0)| = 1 - \frac{1}{2}n\lambda/(s + n\lambda)$. But this does not improve the rate of convergence by much.

# 5. Mixture of exponentials

The same approach can be used when $f(t)$ is a finite mixture of $m$ negative exponentials, but the details are more complicated. For simplicity we first consider the case $f(t) = c_1\,\mathrm{e}^{-\lambda_1 t} + c_2\,\mathrm{e}^{-\lambda_2 t}$, $0 < \lambda_1 < \lambda_2$. Writing $\varepsilon_1 = c_1\varepsilon$, $\varepsilon_2 = c_2\varepsilon$, (2.4) becomes

$$\int_0^\infty g(t)\mathrm{e}^{\sqrt{2s}(\varepsilon_1 \mathrm{e}^{-\lambda_1 t} + \varepsilon_2 \mathrm{e}^{-\lambda_2 t}) - st}\,\mathrm{d}t = \mathrm{e}^{-\alpha\sqrt{2s}}. \tag{5.1}$$

It turns out that the right way to proceed is to expand in powers of $\varepsilon_1$ and $\varepsilon_2$ *preserving the order of the products*, that is, distinguishing between $\varepsilon_1\varepsilon_2$ and $\varepsilon_2\varepsilon_1$ and so on. We therefore expand $g$ in the form

$$\begin{aligned} g = {} & g_0 + (\varepsilon_1 g_1 + \varepsilon_2 g_2) + (\varepsilon_1\varepsilon_1 g_{11} + \varepsilon_1\varepsilon_2 g_{12} + \varepsilon_2\varepsilon_1 g_{21} + \varepsilon_2\varepsilon_2 g_{22}) \\ & + (\varepsilon_1\varepsilon_1\varepsilon_1 g_{111} + \varepsilon_1\varepsilon_1\varepsilon_2 g_{112} + \varepsilon_1\varepsilon_2\varepsilon_1 g_{121} + \varepsilon_2\varepsilon_1\varepsilon_1 g_{211} \\ & + \varepsilon_1\varepsilon_2\varepsilon_2 g_{122} + \varepsilon_2\varepsilon_1\varepsilon_2 g_{212} + \varepsilon_2\varepsilon_2\varepsilon_1 g_{221} + \varepsilon_2\varepsilon_2\varepsilon_2 g_{222}) + \cdots. \end{aligned} \tag{5.2}$$

Expanding the exponential in the same way, (4.1) becomes

$$\begin{aligned} & \int_0^\infty \left\{ g_0 + \sum_{i=1}^2 \varepsilon_i g_i + \sum_{i=1}^2\sum_{j=1}^2 \varepsilon_i\varepsilon_j g_{ij} + \sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2 \varepsilon_i\varepsilon_j\varepsilon_k g_{ijk} + \cdots \right\} \\ & \left\{ 1 + \sqrt{2s}\sum_{i=1}^2 \varepsilon_i\,\mathrm{e}^{-\lambda_i t} + \frac{(\sqrt{2s})^2}{2!}\sum_{i=1}^2\sum_{j=1}^2 \varepsilon_i\varepsilon_j\,\mathrm{e}^{-(\lambda_i+\lambda_j)t} \right. \\ & \left. + \frac{(\sqrt{2s})^3}{3!}\sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2 \varepsilon_i\varepsilon_j\varepsilon_k\,\mathrm{e}^{-(\lambda_i+\lambda_j+\lambda_k)t} + \cdots \right\} \mathrm{e}^{-st}\,\mathrm{d}t = \mathrm{e}^{-\alpha\sqrt{2s}} \end{aligned} \tag{5.3}$$

To illustrate the way the coefficients are determined, consider $g_{ijk}$. The equations arising from the constant term and the coefficients of $\varepsilon_i$, $\varepsilon_i\varepsilon_j$, $\varepsilon_i\varepsilon_j\varepsilon_k$ are

$$g_0^*(s) = \mathrm{e}^{-\alpha\sqrt{2s}},$$

$$g_i^*(s) + \sqrt{2s}\, g_0^*(s+\lambda_i) = 0,$$

$$g_{ij}^*(s) + \sqrt{2s}\, g_i^*(s+\lambda_j) + \frac{(\sqrt{2s})^2}{2!}\, g_0^*(s+\lambda_i+\lambda_j) = 0,$$

$$g_{ijk}^*(s) + \sqrt{2s}\, g_{ij}^*(s+\lambda_k) + \frac{(\sqrt{2s})^2}{2!}\, g_i^*(s+\lambda_j+\lambda_k) + \frac{(\sqrt{2s})^3}{3!}\, g_0^*(s+\lambda_i+\lambda_j+\lambda_k) = 0. \quad (5.4)$$

In each equation add the required quantity to $s$ to make the $g_0^*$ term have argument $s+\lambda_i+\lambda_j+\lambda_k$. Rearranging the terms as before, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \sqrt{2(s+\lambda_j+\lambda_k)} & 1 & 0 & 0 \\ \frac{(\sqrt{2(s+\lambda_k)})^2}{2!} & \sqrt{2(s+\lambda_k)} & 1 & 0 \\ \frac{(\sqrt{2s})^3}{3!} & \frac{(\sqrt{2s})^2}{2!} & \sqrt{2s} & 1 \end{bmatrix} \begin{bmatrix} g_0^*(s+\lambda_i+\lambda_j+\lambda_k) \\ g_i^*(s+\lambda_j+\lambda_k) \\ g_{ij}^*(s+\lambda_k) \\ g_{ijk}^*(s) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\alpha\sqrt{2(s+\lambda_i+\lambda_j+\lambda_k)}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (5.5)$$

and hence

$$g_{ijk}^*(s) = -\begin{vmatrix} \sqrt{2(s+\lambda_j+\lambda_k)} & 1 & 0 \\ \frac{(\sqrt{2(s+\lambda_k)})^2}{2!} & \sqrt{2(s+\lambda_k)} & 1 \\ \frac{(\sqrt{2s})^3}{3!} & \frac{(\sqrt{2s})^2}{2!} & \sqrt{2s} \end{vmatrix} \mathrm{e}^{-\alpha\sqrt{2(s+\lambda_i+\lambda_j+\lambda_k)}}. \quad (5.6)$$

Since $\lambda_1 \leqslant \lambda_i, \lambda_j, \lambda_k \leqslant \lambda_2$ the previous argument can now be used to give the upper bound

$$\begin{aligned} |g_{ijk}^*(s)| &< \frac{[2(s+\lambda_i+\lambda_j+\lambda_k)]^{3/2}}{3!}\, \mathrm{e}^{-\alpha\sqrt{2(s+\lambda_i+\lambda_j+\lambda_k)}} \\ &< \frac{[2(s+3\lambda_2)]^{3/2}}{3!} \mathrm{e}^{-\alpha\sqrt{2(s+3\lambda_1)}}, \end{aligned} \quad (5.7)$$

so that

$$\left|\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_i\varepsilon_j\varepsilon_k g^*_{ijk}\right| < \frac{[2(s+3\lambda_2)]^{3/2}}{3!}\,\mathrm{e}^{-\alpha\sqrt{2(s+3\lambda_1)}}(|\varepsilon_1|+|\varepsilon_2|)^3$$

$$= \frac{[2(s+3\lambda_2)]^{3/2}}{3!}\,\mathrm{e}^{-\alpha\sqrt{2(s+3\lambda_1)}}(|c_1|+|c_2|)^3|\varepsilon|^3. \qquad (5.8)$$

In the same way the $n$th term of the expansion of $g$ can be shown to be dominated by

$$\frac{[2(s+n\lambda_2)]^{n/2}}{n!}\mathrm{e}^{-\alpha\sqrt{2(s+n\lambda_1)}}(|c_1|+|c_2|)^n|\varepsilon|^n. \qquad (5.9)$$

The ratio of $(n+1)$th to $n$th term is approximately $(|c_1|+|c_2|)|\varepsilon|\sqrt{2\lambda_2\,\mathrm{e}/n}$ and the series converges for all $\varepsilon$.

The general case of a finite number $m$ of negative exponentials is susceptible to the same analysis. Let

$$f(t) = c_1\exp(-\lambda_1 t) + c_2\exp(-\lambda_2 t) + \cdots + c_m\exp(-\lambda_m t),$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_m$. Then in (5.9) $\lambda_2$ is replaced by $\lambda_m$, $(|c_1|+|c_2|)^n$ is replaced by $(|c_1|+|c_2|+\cdots+|c_m|)^n$ and the same conclusion follows. However, if $m$ is allowed to become infinite and $\lambda_m\to\infty$ the argument breaks down. For example, it cannot be applied to $f(t)=1/(1+c\exp(-\lambda t))=\sum_{n=0}^{\infty}(-c)^n\exp(-n\lambda t)$, $0<c<1$. In such cases one has to resort to the fact that $x^n\exp(-\alpha x)$ takes its maximum value $(n/\alpha)^n\mathrm{e}^{-n}$ at $x=n/\alpha$. This replaces the equivalent of (5.9) by a weaker upper bound

$$\left(\frac{n}{\alpha}\right)^n\frac{\mathrm{e}^{-n}}{n!}\left(\sum_{r=1}^{\infty}|c_r|\right)^n|\varepsilon|^n \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{\sum_{r=1}^{\infty}|c_r||\varepsilon|}{\alpha}\right)^n. \qquad (5.10)$$

The ratio of the $(n+1)$th to the $n$th term tends to $(\sum_1^\infty|c_r|)|\varepsilon|/\alpha$, and all one can say is that the expansion certainly converges if $|\varepsilon|<\alpha/\sum_1^\infty|c_n|$.

Periodic boundaries can be expressed as sums of exponentials with complex $\lambda_i$s. The coefficients of the perturbation expansion can be found as sums of terms like (5.6), but the complex $\lambda_i$s mean that there is no obvious upper bound which can be exploited.

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