Approximation and support theorem for a wave equation in two space dimensions

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We prove a characterization of the support of the law of the solution for a stochastic wave equation with two-dimensional space variable, driven by a noise white in time and correlated in space. The result is a consequence of an approximation theorem, in the convergence of probability, for equations obtained by smoothing the random noise. For some particular classes of coefficients, approximation in the L^p-norm for $p \geqslant 1$ is also proved.

Keywords: approximations; stochastic partial differential equations; support theorem

1. Introduction and preliminaries

In this paper we characterize the topological support of the law of the solution to the stochastic wave equation with two-dimensional spatial variable

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t, x) = \sigma(u(t, x))F(\mathrm{d}t, \mathrm{d}x) + b(u(t, x)), \qquad u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x) \tag{1.1}$$

where $(t, x) \in [0, \infty[\times\mathbb{R}^2$. In this equation $F(t, x)$ is a generalized Gaussian field with covariance

$$\mathrm{E}(F(t, x)F(s, y)) = \delta(t - s)f(|x - y|), \tag{1.2}$$

where δ denotes the Dirac delta function and f is a continuous function on $]0, \infty[$ satisfying an integrability condition made precise in assumption (C1) below. In addition, the functional $J : \mathscr{D}(\mathbb{R}^3) \times \mathscr{D}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$J(\varphi, \psi) = \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^2} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y \varphi(t, x)f(|x - y|)\psi(t, y)$$

is positive definite. With these hypotheses the process $\{F(t, x), (t, x) \in [0, \infty[\times\mathbb{R}^2\}$ exists.
   We consider the mild formulation of (1.1). Let

$$S(t, x) = \frac{1}{2\pi}(t^2 - |x|^2)^{-1/2}1_{\{|x| < t\}};$$

then a solution of (1.1) is a stochastic process $\{u(t, x), (t, x) \in [0, \infty[\times\mathbb{R}^2\}$ satisfying

$$u(t, x) = \int_{\mathbb{R}^2} S(t, x - y)v_0(y)\,\mathrm{d}y + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t, x - y)u_0(y)\mathrm{d}y\right)$$

$$+ \int_0^t\int_{\mathbb{R}^2} S(t - s, x - y)[\sigma(u(s, y))F(\mathrm{d}s, \mathrm{d}y) + b(u(s, y))\mathrm{d}s\,\mathrm{d}y]. \qquad (1.3)$$

Consider the following set of assumptions on the elements defining (1.3):

(C1) There exist $\beta \in (0, 1)$, $r_0 > 0$, such that, for $0 < t < r_0$,

$$\int_0^t rf(r)\ln\left(1 + \frac{t}{r}\right)\mathrm{d}r \leq Ct^\beta.$$

(C2) $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathscr{C}^1$ and bounded with $\beta/2(1 + \beta)$-Hölder continuous partial derivatives, and $v_0 : \mathbb{R}^2 \to \mathbb{R}$ is such that $|v_0| + |\nabla u_0| \in L^{q_0}$ for some $q_0 \in]4, \infty]$.

(C3) $\sigma, b : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz functions.

Equation (1.3) was solved by Dalang and Frangos (1998) up to some small time $t_0 > 0$. In a previous paper (Millet and Sanz-Solé 1999), we proved that, assuming (C1), (C2) and (C3), equation (1.3) has a unique solution on $[0, T] \times \mathbb{R}^2$ for any $T > 0$. Moreover, the trajectories of u are γ-Hölder continuous in $(t, x) \in [0, T] \times K$ for every $\gamma \in (0, \beta/2(1 + \beta))$ and every compact subset K of $\mathbb{R}^2$.

Let $\mathscr{E}$ denote the inner product space of measurable functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} \mathrm{d}x\int_{\mathbb{R}^2} \mathrm{d}y|\varphi(x)|f(|x - y|)|\varphi(y)| < +\infty,$$

endowed with the scalar product

$$\langle\varphi, \psi\rangle_{\mathscr{E}} := \int_{\mathbb{R}^2} \mathrm{d}x\int_{\mathbb{R}^2} \mathrm{d}y\varphi(x)f(|x - y|)\psi(y),$$

and let $\tilde{H}$ denote the completion of $\mathscr{E}$. Set $H = L^2([0, T], \tilde{H})$; notice that H and $\tilde{H}$ need not be spaces of functions. For $h \in H$, let $\{\Phi^h(t, x), (t, x) \in [0, \infty) \times \mathbb{R}^2\}$ be the solution of

$$\Phi^h(t, x) = \int_{\mathbb{R}^2} S(t, x - y)v_0(y)\mathrm{d}y + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t, x - y)u_0(y)\mathrm{d}y\right)$$

$$+ \langle S(t - \cdot, x - *)\sigma(\Phi^h(\cdot, *)), h\rangle_H + \int_0^t\int_{\mathbb{R}^2} S(t - s, x - y)b(\Phi^h(s, y))\mathrm{d}s\,\mathrm{d}y. \quad (1.4)$$

Set

$$\|\varphi\|_{\gamma, K} = \sup_{\substack{t\in[0,T]\\x\in K}} |\varphi(t, x)| + \sup_{\substack{t,t'\in[0,T]\\x,x'\in K\\t\neq t',x\neq x'}} \frac{|\varphi(t, x) - \varphi(t', x')|}{(|t - t'| + |x - x'|)^\gamma}. \qquad (1.5)$$

We prove in Section 2 that the support of the law of $\{u(t, x), (t, x) \in [0, T] \times K\}$ is the closure with respect to the norm $\|\cdot\|_{\gamma, K}$ of the set of functions $\{\Phi^h, h \in H\}$, where $\{\Phi^h(t, x), (t, x) \in [0, T] \times K\}$ is the solution of (1.4). The proof is based on an approximation result for equations more general than (1.3) constructed by smoothing the random noise $F(t, x)$. We refer the reader to Millet and Sanz-Solé (1994a; 1994b) and Bally *et al.* (1995) for a presentation of the method and applications to stochastic differential and stochastic partial differential equations.

In the framework of stochastic partial differential equations, the regularization of the noise raises technical difficulties connected with the explosion of the corresponding integral (see, for instance, Bally *et al.* 1995). This problem does not appear here for the following reasons. The noise F is smoother than space-time white noise. On the other hand, the integrability condition (C1) and Lemma A.1 in Millet and Sanz-Solé (1999) yield

$$\mu(t) \leqslant C t^{\beta+1}$$

(see (A.2) and (A.11) below). This fact prevents explosions, as is made explicit in the proofs.

We now introduce some preliminaries and notation used throughout this paper. Let $\{e_j, j \in \mathbb{N}\}$ be a sequence of functions of $\mathscr{E}$ which is a complete orthonormal system of $\tilde{H}$ and is taken to be fixed. Define

$$W_j(t) = \int_0^t \int_{\mathbb{R}^2} e_j(x) F(\mathrm{d}s, \mathrm{d}x), \ j \in \mathbb{N}, \ t \in [0, T]. \tag{1.6}$$

Clearly $\{W_j, j \in \mathbb{N}\}$ is a sequence of independent Brownian motions.

Let $\mathscr{H}$ be the separable Hilbert space of functions $k : [0, T] \to \mathbb{R}^{\mathbb{N}}$ such that $\int_0^T \sum_{j=1}^{\infty} |k_j(s)|^2 \, \mathrm{d}s < \infty$ and endowed with the inner product

$$\langle k, \bar{k} \rangle_{\mathscr{H}} = \int_0^T \sum_{j=1}^{\infty} k_j(s) \bar{k}_j(s) \mathrm{d}s.$$

Notice that the mapping

$$\mathscr{J} : H \to \mathscr{H}$$

$$\varphi \mapsto (\langle \varphi(s, *), e_j(*) \rangle_{\bar{H}})_{j \in \mathbb{N}} \tag{1.7}$$

provides an isometry between H and $\mathscr{H}$.

Let $\mathscr{F}_t = \sigma(F([0, s] \times A); \ 0 \leqslant s \leqslant t, \ A \in \mathscr{B}(\mathbb{R}^2)), \ t \geqslant 0$. For any $\mathscr{F}_t$-predictable process $\varphi \in L^2(\Omega; H)$ we have

$$\int_0^t \int_{\mathbb{R}^2} \varphi(s, x) F(\mathrm{d}s, \mathrm{d}x) = \sum_{j=1}^{\infty} \int_0^t \langle \varphi(s, *), e_j(*) \rangle_{\bar{H}} W_j(\mathrm{d}s), \tag{1.8}$$

$t \in [0, T]$, so that the stochastic integral with respect to the martingale measure F can also be viewed as a stochastic integral with respect to the infinite-dimensional Gaussian process $\{W_j(t), t \in [0, T], j \in \mathbb{N}\}$. We introduce smooth approximations of F constructed as follows. Fix $n \in \mathbb{N}$ and let $\mathscr{P}_n$ be the partition of $[0, T]$ determined by $iT/2^n, \ i = 0, 1, \ldots, 2^n$.

Denote by Δ_i the interval $[iT/2^n, (i+1)T/2^n)$ and by $|\Delta_i|$ its length. We write $W_j(\Delta_i)$ for the increment $W_j((i+1)T/2^n) - W_j(iT/2^n)$, $i = 0, \ldots, 2^n - 1$. Define

$$W^n = \left(W_j^n = \int_0^{\cdot} \dot{W}_j^n(s) \mathrm{d}s, \, j \in \mathbb{N} \right), \tag{1.9}$$

where, for $j > n$,

$$\dot{W}_j^n = 0,$$

and, for $1 \leqslant j \leqslant n$,

$$\dot{W}_j^n(t) = \begin{cases} \sum_{i=1}^{2^n} 2^n T^{-1} W_j(\Delta_{i-1}) 1_{\Delta_i}(t) & \text{if } t \in [2^{-n}T, T], \\ 0 & \text{if } t \in [0, 2^{-n}T). \end{cases} \tag{1.10}$$

Set

$$\omega^n(t, x) = \sum_{j \in \mathbb{N}} \dot{W}_j^n(t) e_j(x). \tag{1.11}$$

It is easy to check that almost surely, $\omega^n \in H$ and, more precisely,

$$\|\omega^n\|_{L^p(\Omega;H)} \leqslant Cn^{1/2} 2^{n/2}, \quad \forall p \in [1, \infty). \tag{1.12}$$

Indeed, fix $p \in [2, \infty)$; then

$$\|\omega^n\|_{L^p(\Omega;H)}^p = \mathrm{E} \left(\left| \sum_{j=1}^{n} \sum_{i=1}^{2^n} 2^{2n} T^{-2} |\Delta_i| W_j(\Delta_{i-1})^2 \right|^{p/2} \right)$$

$$\leqslant Cn^{p/2} 2^{np/2}.$$

Moreover, for any $0 \leqslant t_1 \leqslant t_2 \leqslant T$, similar computations imply

$$\|\omega^n 1_{[t_1, t_2]}\|_{L^p(\Omega;H)} \leqslant Cn^{1/2} 2^{n/2} |t_2 - t_1|^{1/2}. \tag{1.13}$$

Let $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ be the canonical space associated with a standard Brownian motion. Denote by $(\Omega, \mathscr{F}, P)$ the product space $(\bar{\Omega}^{\mathbb{N}}, \bar{\mathscr{F}}^{\otimes \mathbb{N}}, \bar{P}^{\otimes \mathbb{N}})$, which will be our reference probability space.

Set $\bar{k}(t) = \int_0^t k(s) \mathrm{d}s$ for $k \in \mathscr{H}$. For any integer $n \geqslant 1$, let T_n^k denote the transformation of Ω defined by

$$T_n^k(\omega) = \omega + \bar{k} - W^n(\omega). \tag{1.14}$$

Notice that $T_n^k(\omega) = \omega + \int_0^{\cdot} \varphi_n(s, \omega) \mathrm{d}s$, where $\{\varphi_n(t, \omega), \, t \in [0, T]\}$ is an $\mathscr{H}$-valued process adapted to the filtration generated by $\{W_j(t), \, t \in [0, 1], \, j \in \mathbb{N}\}$. Therefore, by Girsanov's theorem, $P_{\circ}(T_n^k)^{-1} \ll P$. This fact will be used in the proof of Theorem 2.1.

The paper is organized as follows. In Section 2 we prove the characterization of the support by means of an approximation in probability. In Section 3 we prove approximations

in L^p-norm under stronger hypotheses on the coefficients. As usual, all constants are denoted by C, regardless of their values.

2. Approximation in probability and support theorem

The purpose of this section is to prove the following result:

Theorem 2.1. *Assume* (C1)–(C3)*, fix a compact set $K \subset \mathbb{R}^2$ and let $\{u(t, x), t \in [0, T], x \in K\}$ be the solution of (1.3). Then for any $\gamma \in (0, \beta/2(1 + \beta))$ the topological support of the law of u in the space $\mathscr{C}^\gamma([0, T] \times K)$ of γ-Hölder continuous functions in (t, x) is given by the closure in $\mathscr{C}^\gamma([0, T] \times K)$ of the set of functions $\{\Phi^h, h \in H\}$, where $\{\Phi^h(t, x), t \in [0, T], x \in K\}$ is the solution of (1.4).*

The proof of Theorem 2.1 is a consequence of an approximation result, concerning convergence in probability, for an equation more general than (1.3).

More precisely, let us introduce the following hypothesis:

(C3′) The coefficients A, B, D, $b \colon \mathbb{R} \to \mathbb{R}$ are globally Lipschitz functions.

Then we consider the evolution equations

$$X_n(t, x) = X^0(t, x) + \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)A(X_n(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)B(X_n(\cdot, *)), \omega^n\rangle_H + \langle S(t - \cdot, x - *)D(X_n(\cdot, *)), h\rangle_H$$

$$+ \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)b(X_n(s, y))\mathrm{d}s\mathrm{d}y, \tag{2.1}$$

$$X(t, x) = X^0(t, x) + \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)(A + B)(X(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)D(X(\cdot, *)), h\rangle_H + \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)b(X(s, y))\mathrm{d}s\,\mathrm{d}y, \tag{2.2}$$

where $n \geqslant 1$, A, B, D, b satisfy (C3′), $h \in H$, ω^n is defined in (1.11) and

$$X^0(t, x) = \int_{\mathbb{R}^2} S(t, x - y)v_0(y)\mathrm{d}y + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t, x - y)u_0(y)\mathrm{d}y\right). \tag{2.3}$$

Our aim is to prove the following:

Proposition 2.1. *Assume* (C1)*,* (C2) *and* (C3′)*. For any $\gamma \in (0, \beta/2(1 + \beta))$, $\eta > 0$ and every compact set $K \subset \mathbb{R}^2$,*

$$\lim_{n\to\infty} P(\|X_n - X\|_{\gamma,K} > \eta) = 0, \tag{2.4}$$

where $\|\cdot\|_{\gamma,K}$ has been defined in (1.5).

We first show that Theorem 2.1 is an easy consequence of this proposition.

Proof of Theorem 2.1. Assume that Proposition 2.1 has been proved. For $n \geq 1$, set

$$u_n(t, x) = X^0(t, x) + \langle S(t - \cdot, x - *)\sigma(u_n(\cdot, *)), \omega^n \rangle_H$$

$$+ \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)b(u_n(s, y))\mathrm{d}s\,\mathrm{d}y, \tag{2.5}$$

$$v_n(t, x) = X^0(t, x) + \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)\sigma(v_n(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)\sigma(v_n(\cdot, *)), h - \omega^n \rangle_H$$

$$+ \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)b(v_n(s, y))\mathrm{d}s\,\mathrm{d}y. \tag{2.6}$$

Clearly, equations (2.5) and (2.6) are particular cases of (2.1), while equations (1.3) and (1.4) are particular cases of (2.2), obtained by choosing $A = D = 0$, $B = \sigma$ and $A = D = \sigma$, $B = -\sigma$, respectively.

Moreover, $u_n = \Phi^{\omega^n}$. Given $h \in H$, set $k = \mathscr{J}(h)$, where $\mathscr{J}$ is the isometry defined in (1.7). Then, by (1.8), equation (2.6) can be rewritten as follows:

$$v_n(t, x) = X^0(t, x) + \sum_{j=1}^{\infty} \int_0^t \langle S(t - s, x - *)\sigma(v_n(s, *)), e_j \rangle_{\tilde{H}} W_j(\mathrm{d}s)$$

$$+ \sum_{j=1}^{\infty} \int_0^t \langle S(t - s, x - *)\sigma(v_n(s, *)), e_j \rangle_{\tilde{H}}(k_j(s) - \dot{W}_j^n(s))\mathrm{d}s$$

$$+ \int_0^t\!\!\int_{\mathbb{R}^2} S(t - s, x - y)b(v_n(s, y))\mathrm{d}s\,\mathrm{d}y,$$

with $\dot{W}_j^n$ defined in (1.10). Therefore, $v^n(\omega) = u \circ T_n^k(\omega)$, where T_n^k is the absolutely continuous transformation on Ω defined by (1.14).

The convergence (2.4) implies, for any $\eta > 0$,

$$\lim_{n\to\infty} P(\|\Phi^{\omega^n} - u\|_{\gamma,K} > \eta) = 0,$$

$$\lim_{n\to\infty} P(\|u(T_n^h) - \Phi^h\|_{\gamma,K} > \eta) = 0.$$

These two convergences yield the characterization of the support stated in Theorem 2.1 (see, for instance, Millet and Sanz-Solé (1994a; or 1994b). Indeed, since $\omega_n \in H$, the first convergence implies that the support of u in $\mathscr{C}^{\gamma}([0, T] \times K)$ is included in the closure of

$\{\Phi^h : h \in H\}$. Since ω^n is adapted, $P \circ (T_n^h)^{-1} \ll P$; the second convergence yields the converse inclusion. □

The question of the existence and uniqueness of a solution to equations (2.1) and (2.2) is solved in Lemma A.1, which is a slight extension of Theorem 1.2 in Millet and Sanz-Solé (1999). We remark that the existence of a solution X_n to (2.1) as well as upper estimates of X_n require some localization, due to the term involving ω^n which has an unbounded H-norm. For this reason we localize ω^n as follows: for any positive integer n, $M \in \mathbb{N}_+$ and $t \in [0, T]$, set

$$A_{n,M}(t) = \left\{ \sup_{1 \leq j \leq n} \sup_{0 \leq i \leq ([2^n t T^{-1}]-1)^+} 2^n |W_j(\Delta_i)| \leq M \right\} \tag{2.7}$$

and

$$\omega^{n,M}(t, x) = \omega^n(t, x) 1_{A_{n,M}(t)}.$$

Notice that

$$\sup_{0 \leq t \leq T} \|\omega^{n,M}(t, *)\|_{\tilde{H}} \leq M \sqrt{n}. \tag{2.8}$$

Fix $\alpha > (2 \ln 2)^{1/2}$ and, for every $n > 0$, set

$$M(n) = \alpha 2^{n/2} n^{1/2} \tag{2.9}$$

and

$$A_n(t) = A_{n,M(n)}(t). \tag{2.10}$$

Lemma 2.1. *The following convergence holds:*

$$\lim_{n \to \infty} P(A_n(T)^c) = 0.$$

Proof. Let Z denote an $N(0, 1)$ random variable. Then

$$P(A_n(T)^c) \leq n 2^n P(|Z| > 2^{-n/2} M(n))$$

$$\leq C n 2^n \frac{2^{n/2}}{M(n)} \exp\left(-\frac{2^{-n} M(n)^2}{2} \right)$$

$$= C \sqrt{n} \exp\left(-n \left(\frac{\alpha^2}{2} - \ln 2 \right) \right) \underset{n \to \infty}{\to} 0.$$

□

Remark 2.1. Due to (2.8), on the set $A_n(T)$ we have

$$\|\omega^n\|_H \leq C n 2^{n/2} \tag{2.11}$$

and, for any $0 \leq t \leq t' \leq T$, on $A_n(t')$ we have

$$\|\omega^n 1_{[t,t']}\|_H \leqslant Cn2^{n/2}|t' - t|^{1/2}. \tag{2.12}$$

In particular, if $[t, t'] \subset \Delta_i$ for some $i = 0, \ldots, 2^n - 1$, on $A_n(t')$ we have

$$\|\omega^n 1_{[t,t']}\|_H \leqslant Cn. \tag{2.13}$$

Lemma A.1 yields the existence and uniqueness of the solution X_n to (2.1) and Remark 2.2 shows that the trajectories of X_n almost surely have γ-Hölder continuous trajectories for $\gamma < \beta/2(1 + \beta)$; since X is a particular case of X_n, it also has γ-Hölder continuous trajectories.

Our next purpose is to check that the sequence of processes $Y_n(t, x) := X_n(t, x) - X(t, x)$, $n \geqslant 1$, satisfies the requirements of Lemma A.2. To this end, we introduce some notation and prove several lemmas. For any $n \geqslant 1$, $t \in [0, T]$, set

$$\underline{t}_n = \max\{k2^{-n}T; \; k = 1, \ldots, 2^n - 1 : k2^{-n}T \leqslant t\},$$

$$t_n = \max\{\underline{t}_n - 2^{-n}T, 0\}, \tag{2.14}$$

$$X_n^-(t, x) = X^0(t, x) + \int_0^{t_n}\int_{\mathbb{R}^2} S(t - s, x - y)A(X_n(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)B(X_n(\cdot, *))1_{[0,t_n]}(\cdot), \omega^n\rangle_H$$

$$+ \langle S(t - \cdot, x - *)D(X_n(\cdot, *))1_{[0,t_n]}(\cdot), h\rangle_H$$

$$+ \int_0^{t_n}\int_{\mathbb{R}^2} S(t - s, x - y)b(X_n(s, y))\mathrm{d}s\,\mathrm{d}y, \tag{2.15}$$

$$X^-(t, x) = X^0(t, x) + \int_0^{t_n}\int_{\mathbb{R}^2} S(t - s, x - y)(A + B)(X(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)D(X(\cdot, *))1_{[0,t_n]}(\cdot), h\rangle_H$$

$$+ \int_0^{t_n}\int_{\mathbb{R}^2} S(t - s, x - y)b(X(s, y))\mathrm{d}s\,\mathrm{d}y. \tag{2.16}$$

For convenience' sake, we do not write explicitly the fact that the process X^- depends on n. In what follows, $\|\,\|_p$ denotes the $L^p(\Omega)$-norm.

Lemma 2.2. *Suppose that conditions* (C1), (C2) *and* (C3′) *hold. Then, for any* $p \in [1, \infty)$ *and every integer* $n \geqslant 1$,

$$\sup_{(s,x)\in[0,T]\times\mathbb{R}^2} \|X(s, x) - X^-(s, x)\|_p \leqslant C2^{-n(\beta+1)/2} \tag{2.17}$$

and

$$\sup_{n \geqslant 1} \sup_{(s,x) \in [0,T] \times \mathbb{R}^2} \|X^-(s, x)\|_p \leqslant \infty. \tag{2.18}$$

Proof. Set $\|X(t, x) - X^-(t, x)\|_p^p \leqslant C\sum_{i=1}^3 V_i(t, x)$, with

$$V_1(t, x) = \mathrm{E}\left(\left|\int_{t_n}^t \int_{\mathbb{R}^2} S(t - s, x - y)(A + B)(X(s, y))F(\mathrm{d}s, \mathrm{d}y)\right|^p\right),$$

$$V_2(t, x) = \mathrm{E}(\langle S(t - \cdot, x - *)D(X(\cdot, *))1_{(t_n, t]}(\cdot), h\rangle_H|^p),$$

$$V_3(t, x) = \mathrm{E}\left(\left|\int_{t_n}^t \int_{\mathbb{R}^2} S(t - s, x - y)b(X(s, y))\mathrm{d}s\,\mathrm{d}y\right|^p\right).$$

Burkholder's and Hölder's inequalities, along with (A.15) and (A.11), yield

$$V_1(t, x) \leqslant C\mu(t - t_n)^{p/2}\left(1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \mathrm{E}(|X(t, x)|^p)\right) \leqslant C2^{-n(\beta+1)p/2},$$

with $\mu(t - t_n)$ given by (A.2).

The Cauchy-Schwarz and Hölder inequalities imply

$$V_2(t, x) \leqslant C\|h\|_H^p \mu(t - t_n)^{p/2}\left(1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \mathrm{E}(|X(t, x)|^p)\right) \leqslant C2^{-n(\beta+1)p/2}.$$

Finally, Hölder's inequality implies, for $\nu(t)$ defined by (A.3):

$$V_3(t, x) \leqslant C\left(\int_{t_n}^t \int_{\mathbb{R}^2} S(t - s, x - y)\mathrm{d}s\,\mathrm{d}y\right)^p\left(1 + \sup_{\substack{0 \leqslant t \leqslant T \\ x \in \mathbb{R}^2}} \mathrm{E}(|X(t, x)|^p)\right)$$

$$\leqslant C\nu(t - t_n)^p \leqslant C2^{-2np},$$

which completes the proof of (2.17). Finally, (2.18) is a straightforward consequence of (A.15) and (2.17). $\qquad\square$

Consider the Picard iteration scheme associated with (2.1):

$$X_n^0(t, x) = X^0(t, x),$$

$$X_n^{k+1}(t, x) = X^0(t, x) + \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)A(X_n^k(s, y))F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *)B(X_n^k(\cdot, *)), \omega^n\rangle_H + \langle S(t - \cdot, x - *)D(X_n^k(\cdot, *)), h\rangle_H$$

$$+ \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)b(X_n^k(s, y))\mathrm{d}s\,\mathrm{d}y, \qquad k \geqslant 0. \tag{2.19}$$

For any $0 \leqslant r \leqslant t \leqslant T$ and every integer $k \geqslant 0$, set

$$X_n^0(t, r; x) = X^0(t, x),$$

$$X_n^{k+1}(t, r; x) = X^0(t, x) + \int_0^r \int_{\mathbb{R}^2} S(t - s, x - y) A(X_n^k(s, y)) F(\mathrm{d}s, \mathrm{d}y)$$

$$+ \langle S(t - \cdot, x - *) B(X_n^k(\cdot, *)) 1_{[0,r]}(\cdot), \omega^n \rangle_H$$

$$+ \langle S(t - \cdot, x - *) D(X_n^k(\cdot, *)) 1_{[0,r]}(\cdot), h \rangle_H$$

$$+ \int_0^r \int_{\mathbb{R}^2} S(t - s, x - y) b(X_n^k(s, y)) \mathrm{d}s \, \mathrm{d}y, \tag{2.20}$$

$$\bar{X}_n^{k+1}(t, x) = X_n^{k+1}(t, t_n; x). \tag{2.21}$$

Notice that $X_n^k(t, t; x) = X_n^k(t, x)$.

Lemma 2.3. *Assume* (C1), (C2) *and* (C3′). *Then, for every* $p \in [1, \infty)$, $t \in [0, T]$, $k \geq 1$, $n \geq 1$,

$$\sup_{(s,y) \in [0,t] \times \mathbb{R}^2} \mathrm{E}(|X_n^k(s, y) - \bar{X}_n^k(s, y)|^p 1_{A_n(s)}) \leq C n^p 2^{-n(1+\beta)p/2}$$

$$\times \left[1 + \sup_{(s,y) \in [0,t] \times \mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)}) \right] \tag{2.22}$$

and

$$\sup_{(s,y) \in [0,t] \times \mathbb{R}^2} \mathrm{E}(|X_n(s, y) - X_n^-(s, y)|^p 1_{A_n(s)}) \leq C n^p 2^{-n(1+\beta)p/2}$$

$$\times \left[1 + \sup_{(s,y) \in [0,t] \times \mathbb{R}^2} \mathrm{E}(|X_n(s, y)|^p 1_{A_n(s)}) \right]. \tag{2.23}$$

Proof. Consider the decomposition

$$\mathrm{E}(|X_n^k(t, x) - \bar{X}_n^k(t, x)|^p 1_{A_n(t)}) \leq C \sum_{i=1}^4 T_n^{k,i}(t, x), \tag{2.24}$$

with

$$T_n^{k,1}(t, x) = \mathrm{E}\left(\left| \int_{t_n}^t \int_{\mathbb{R}^2} S(t - s, x - y) A(X_n^{k-1}(s, y)) F(\mathrm{d}s, \mathrm{d}y) \right|^p 1_{A_n(t)} \right),$$

$$T_n^{k,2}(t, x) = \mathrm{E}\left(\left| \langle S(t - \cdot, x - *) B(X_n^{k-1}(\cdot, *)) 1_{(t_n,t]}(\cdot), \omega^n \rangle_H \right|^p 1_{A_n(t)} \right),$$

$$T_n^{k,3}(t, x) = \mathrm{E}\left(\left| \langle S(t - \cdot, x - *) D(X_n^{k-1}(\cdot, *)) 1_{(t_n,t]}(\cdot), h \rangle_H \right|^p 1_{A_n(t)} \right),$$

$$T_n^{k,4}(t, x) = \mathrm{E}\left(\left| \int_{t_n}^t \int_{\mathbb{R}^2} S(t - s, x - y) b(X_n^{k-1}(s, y)) \mathrm{d}s \, \mathrm{d}y \right|^p 1_{A_n(t)} \right).$$

The local property of stochastic integrals, the inclusion $A_n(s) \supset A_n(t)$ for $s \leq t$, Burkholder's and Hölder's inequalities and (A.11) yield

$$T_n^{k,1}(t, x) \leq C\mu(t - t_n)^{p/2}\left[1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)})\right]$$

$$\leq C2^{-n(\beta+1)p/2}\left[1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)})\right]. \qquad (2.25)$$

The Cauchy-Schwarz and Hölder inequalities, along with (2.13) and (A.11), imply

$$T_n^{k,2}(t, x) \leq \mathrm{E}[\|\omega^n 1_{[t_n,t]} 1_{A_n(t)}\|_H^p \|S(t - \cdot, x - *)B(X_n^{k-1}(\cdot, *))1_{[t_n,t]}(\cdot)1_{A_n(t)}\|_H^p]$$

$$\leq Cn^p 2^{-n(1+\beta)p/2}\left[1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)})\right]. \qquad (2.26)$$

Similarly, using (A.3) for the last inequality, we have

$$T_n^{k,3}(t, x) \leq C\|h\|_H^p 2^{-n(1+\beta)p/2}\left[1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)})\right], \qquad (2.27)$$

$$T_n^{k,4}(t, x) \leq C2^{-2np}\left[1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^p 1_{A_n(s)})\right]. \qquad (2.28)$$

Thus (2.24)–(2.28) conclude the proof of (2.22).

Similar computations can be carried out to prove upper estimates of the p moments of the increments $|X_n^{k+1}(s, x) - X_n^k(s, x)|1_{A_n(s)}$ and $|\bar{X}_n^{k+1}(s, x) - \bar{X}_n^k(s, x)|1_{A_n(s)}$; they yield

$$\lim_{k\to\infty} \sup_{(s,x)\in[0,t]\times\mathbb{R}^2} \mathrm{E}[(|X_n^k(s, x) - X_n(s, x)|^p + |\bar{X}_n^k(s, x) - X_n^-(s, x)|^p)1_{A_n(s)}] = 0. \qquad (2.29)$$

Therefore, (2.22) and (2.29) yield (2.23). $\qquad\qquad\square$

We now prove the convergence of $X_n^-(s, y)$ to $X_n(s, y)$.

Lemma 2.4. *Assume* (C1)*,* (C2) *and* (C3′)*. Then, for any* $p \in [1, +\infty)$*,*

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \mathrm{E}[1_{A_n(t)}(|X_n(t, x)|^p + |X_n^-(t, x)|^p)] < \infty \qquad (2.30)$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \|(X_n(t, x) - X_n^-(t, x))1_{A_n(t)}\|_p \leq Cn2^{-n(1+\beta)/2}. \qquad (2.31)$$

Proof. We want to show that, for any $p \in [1, \infty)$,

$$\sup_{n\geq 1} \sup_{k\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \mathrm{E}[1_{A_n(t)}(|X_n^k(t, x)|^p + |\bar{X}_n^k(t, x)|^p)] < +\infty. \qquad (2.32)$$

Indeed, (2.30) is a consequence of (2.32) and (2.29). For $r \leq t$, consider the decomposition

$$\mathrm{E}(|X_n^{k+1}(t,\,r;\,x)|^p 1_{A_n(t)}) \leqslant C \sum_{i=1}^{6} T_n^{k+1,i}(t,\,r;\,x), \tag{2.33}$$

where

$$T_n^{k+1,1}(t,\,r;\,x) = |X^0(t,\,x)|^p,$$

$$T_n^{k+1,2}(t,\,r;\,x) = \mathrm{E}\left(\left|\int_0^r\!\!\int_{\mathbb{R}^2} S(t-s,\,x-y)A(X_n^k(s,\,y))F(\mathrm{d}s,\,\mathrm{d}y)\right|^p 1_{A_n(t)}\right),$$

$$T_n^{k+1,3}(t,\,r;\,x) = \mathrm{E}(|\langle S(t-\cdot,\,x-*)B(\bar{X}_n^k(\cdot,\,*))1_{[0,r]}(\cdot),\,\omega^n\rangle_H|^p 1_{A_n(t)}),$$

$$T_n^{k+1,4}(t,\,r;\,x) = \mathrm{E}(|\langle S(t-\cdot,\,x-*)[B(X_n^k)-B(\bar{X}_n^k)](\cdot,\,*)1_{[0,r]}(\cdot),\,\omega^n\rangle_H|^p 1_{A_n(t)}),$$

$$T_n^{k+1,5}(t,\,r;\,x) = \mathrm{E}(|\langle S(t-\cdot,\,x-*)D(X_n^k)(\cdot,\,*)1_{[0,r]}(\cdot),\,h\rangle_H|^p 1_{A_n(t)}),$$

$$T_n^{k+1,6}(t,\,r;\,x) = \mathrm{E}\left(\left|\int_0^r\!\!\int_{\mathbb{R}^2} S(t-s,\,x-y)b(X_n^k(s,\,y))\mathrm{d}s\,\mathrm{d}y\right|^p 1_{A_n(t)}\right).$$

Under hypotheses weaker than (C2), we proved in Millet and Sanz-Solé (1999) that

$$|X^0(t,\,x)| \leqslant C(\|v_0\|_{q_0} + \|\nabla u_0\|_{q_0}). \tag{2.34}$$

Burkholder's and Hölder's inequalities yield

$$T_n^{k+1,2}(t,\,r;\,x) \leqslant C\int_0^r J(t-s)\left[1 + \sup_{(u,y)\in[0,s]\times\mathbb{R}^2} \mathrm{E}(|X_n^k(u,\,y)|^p 1_{A_n(u)})\right]\mathrm{d}s. \tag{2.35}$$

Let τ_n be the operator defined on real-valued functions by

$$\tau_n(\rho)(s,\,x) = \rho((s+2^{-n})\wedge T,\,x).$$

Let $\mathscr{E}_n$ be the closed subspace of H generated by the orthonormal system

$$2^n T^{-1} 1_{\Delta_i}(\cdot)\otimes e_j(*), \qquad i = 0,\,\ldots,\,2^n-1,\,j = 1,\,\ldots,\,n,$$

and denote by π_n the orthogonal projection operator on $\mathscr{E}_n$. Then since $\bar{X}_n^k(s,\,y)$ is $\mathscr{F}_{s_n}$-measurable, the definition of ω_n in (1.11) implies

$$T_n^{k+1,3}(t,\,r;\,x) = \mathrm{E}\left(\left|\int_0^1\!\!\int_{\mathbb{R}^2} (\pi_n\circ\tau_n)[S(t-\cdot,\,x-*)B(\bar{X}_n^k(\cdot,\,*))1_{[0,r]}(\cdot)1_{A_n(\cdot)}](s,\,y)F(\mathrm{d}s,\,\mathrm{d}y)\right|^p\right)$$

and, by Burkholder's and Hölder's inequalities, if J is defined by (A.1),

$$T_n^{k+1,3}(t,\,r;\,x) \leqslant C\,\mathrm{E}\|(\pi_n\circ\tau_n)(S(t-\cdot,\,x-*)B(\bar{X}_n^k(\cdot,\,*)1_{[0,r]}(\cdot)1_{A_n(\cdot)})\|_H^p$$

$$\leqslant C\int_0^r J(t-s)\left[1 + \sup_{(u,y)\in[0,s]\times\mathbb{R}^2} \mathrm{E}(|\bar{X}_n^k(u,\,y)|^p 1_{A_n(u)})\right]\mathrm{d}s. \tag{2.36}$$

The Cauchy-Schwarz and Hölder inequalities, along with (2.11) and (2.22), imply

$$T_n^{k+1,4}(t, r; x) \leqslant \mathrm{E}(\|\omega^n 1_{A_n(t)}\|_H^p \|S(t - \cdot, x - *)[B(X_n^k) - B(\bar{X}_n^k)](\cdot, *) 1_{[0,r]}(\cdot) 1_{A_n(t)}\|_H^p)$$

$$\leqslant Cn^p 2^{np/2} \int_0^r J(t - s) \sup_{y \in \mathbb{R}^2} \mathrm{E}(|X_n^k(s, y) - \bar{X}_n^k(s, y)|^p 1_{A_n(t)}) \mathrm{d}s$$

$$\leqslant Cn^{2p} 2^{-n\beta np/2} \int_0^r J(t - s) \left[1 + \sup_{(u,y) \in [0,s] \times \mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(u, y)|^p 1_{A_n(u)}) \right] \mathrm{d}s. \tag{2.37}$$

Analogously,

$$T_n^{k+1,5}(t, r; x) \leqslant C\|h\|_H^p \int_0^r J(t - s) \left[1 + \sup_{(u,y) \in [0,s] \times \mathbb{R}^2} \mathrm{E}(|X_n^k(u, y)|^p 1_{A_n(u)}) \right] \mathrm{d}s, \tag{2.38}$$

$$T_n^{k+1,6}(t, r; x) \leqslant C\int_0^r (t - s) \left[1 + \sup_{(u,y) \in [0,s] \times \mathbb{R}^2} \mathrm{E}(|X_n^k(u, y)|^p 1_{A_n(u)}) \right] \mathrm{d}s. \tag{2.39}$$

Therefore, (2.33)–(2.39) yield

$$\mathrm{E}(|X_n^{k+1}(t, r; x)|^p 1_{A_n(t)}) \leqslant \int_0^r \left[1 + \sup_{(u,y) \in [0,s] \times \mathbb{R}^2} \{ \mathrm{E}([|X_n^k(u, y)|^p \right.$$

$$\left. + |X_n^{k-1}(u, y)|^p + |\bar{X}_n^k(u, y)|^p] 1_{A_n(u)}) \} \right] \mathrm{d}s. \tag{2.40}$$

Set, for any $k \geqslant 0$, $t \in [0, T]$,

$$\varphi_n^k(t) = \sup_{(s,y) \in [0,t] \times \mathbb{R}^2} \mathrm{E}((|X_n^k(s, y)|^p + |\bar{X}_n^k(s, y)|^p) 1_{A_n(s)}).$$

Then, using (2.40) with $r = t$ and $r = t_n$ and adding term by term the corresponding inequalities, we obtain

$$\varphi_n^{k+1}(t) \leqslant C\int_0^t [1 + \varphi_n^k(s) + \varphi_n^{k-1}(s)] \mathrm{d}s, \tag{2.41}$$

with the convention $\varphi_n^{-1}(\cdot) \equiv 0$. Since, by (2.34),

$$\varphi_n^0(t) \leqslant 2 \sup_{x \in \mathbb{R}^2} |X^0(t, x)|^p \leqslant C,$$

(2.41) yields

$$\sup_{n \geqslant 1} \sup_{k \geqslant 0} \sup_{t \in [0,T]} \varphi_n^k(t) \leqslant C,$$

which establishes (2.32). Finally, inequalities (2.23) and (2.30) inply (2.31), which completes the proof of the lemma. □

In what follows, K denotes an arbitrary compact subset of $\mathbb{R}^2$. For any s, t, $\bar{t} \in [0, T]$, $x, \bar{x} \in K$, $y \in \mathbb{R}^2$, set

$$\gamma(t, \bar{t}, x, \bar{x}; s, y) = S(t - s, x - y) - S(\bar{t} - s, \bar{x} - y)$$

and

$$\Gamma(t, \bar{t}, x, \bar{x}; s, y) = |\gamma(t, \bar{t}, x, \bar{x}; s, y)|.$$

Lemma 2.5. *Assume that the function f satisfies condition* (C1). *For any* $0 < \gamma < \beta/(2(1 + \beta))$, t, $\bar{t} \in [0, T]$, $x, \bar{x} \in K$,

$$\|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H \leq C(|t - \bar{t}|^\gamma + |x - \bar{x}|^\gamma), \tag{2.42}$$

$$\int_0^T \int_{\mathbb{R}^2} \Gamma(t, \bar{t}, x, \bar{x}; s, y) \mathrm{d}s \, \mathrm{d}y \leq C(|t - \bar{t}|^{1/2} + |x - \bar{x}|^{1/2}). \tag{2.43}$$

Proof. Assume $0 \leq t \leq \bar{t} \leq T$ and set

$$\Gamma_1(t, \bar{t}, x, \bar{x}; s, y) = (S(t - s, x - y) - S(\bar{t} - s, x - y)) 1_{[0,t]}(s),$$

$$\Gamma_2(t, \bar{t}, x, \bar{x}; s, y) = |(S(\bar{t} - s, x - y) - S(\bar{t} - s, \bar{x} - y)) 1_{[0,t]}(s)|,$$

$$\Gamma_3(t, \bar{t}, x, \bar{x}; s, y) = S(\bar{t} - s, \bar{x} - y) 1_{[t,\bar{t}]}(s);$$

then

$$\|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^2 \leq C \sum_{i=1}^3 \|\Gamma_i(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^2.$$

For $i = 1, 2$, it is easy to check that

$$\|\Gamma_1(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^2 \leq \mu_{t,\bar{t}-t} + \tilde{\mu}_{t,\bar{t}-t} + 2(\mu_{t,\bar{t}-t} \tilde{\mu}_{t,\bar{t}-t})^{1/2},$$

$$\|\Gamma_2(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^2 \leq M_{t,\bar{x}-x} + N_{t,\bar{x}-x} + 2(M_{t,\bar{x}-x} N_{t,\bar{x}-x})^{1/2},$$

where $\mu_{t,\bar{t}-t}$, $\tilde{\mu}_{t,\bar{t}-t}$, $M_{t,\bar{x}-x}$, $N_{t,\bar{x}-x}$ are defined in (A.6)–(A.9), respectively. Finally, for $\mu(t)$ defined by (A.2),

$$\|\Gamma_3(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^2 = \mu(\bar{t} - t).$$

Thus the estimates (A.11)–(A.13) show (2.42).

In order to prove (2.43), set $\Gamma(t, \bar{t}, x, \bar{x}; s, y) \leq \sum_{i=1}^2 \bar{\Gamma}_i(t, \bar{t}, x, \bar{x}; s, y)$, with

$$\bar{\Gamma}_1(t, \bar{t}, x, \bar{x}; s, y) = |S(t-s, x-y) - S(\bar{t}-s, x-y)|,$$

$$\bar{\Gamma}_2(t, \bar{t}, x, \bar{x}; s, y) = |S(\bar{t}-s, x-y) - S(\bar{t}-s, \bar{x}-y)|.$$

Assume $0 \leqslant t \leqslant \bar{t} \leqslant T$; then

$$\int_0^T ds \int_{\mathbb{R}^2} dy \bar{\Gamma}_1(t, \bar{t}, x, \bar{x}; s, y) \leqslant C(\nu_{t,\bar{t}-t} + \tilde{\nu}_{t,\bar{t}-t} + \nu(\bar{t}-t)),$$

with $\nu(\bar{t}-t)$, $\nu_{t,\bar{t}-t}$, $\tilde{\nu}_{t,\bar{t}-t}$, defined in (A.3)–(A.5). Hence, inequalities (A.3) and (A.10) imply

$$\int_0^T ds \int_{\mathbb{R}^2} dy \bar{\Gamma}_1(t, \bar{t}, x, \bar{x}; s, y) \leqslant C(\bar{t}-t)^{1/2}. \qquad (2.44)$$

Moreover,

$$\int_0^T ds \int_{\mathbb{R}^2} dy \bar{\Gamma}_2(t, \bar{t}, x, \bar{x}; s, y) \leqslant \bar{\Gamma}_{2,1}(t, \bar{t}, x, \bar{x}) + 2\bar{\Gamma}_{2,2}(t, \bar{t}, x, \bar{x}),$$

with

$$\bar{\Gamma}_{2,1}(t, \bar{t}, x, \bar{x}) = \int_0^{\bar{t}-\frac{|x-\bar{x}|}{2}} ds \int_{\substack{|x-y|<\bar{t}-s \\ |\bar{x}-y|<\bar{t}-s}} dy |S(\bar{t}-s, x-y) - S(\bar{t}-s, \bar{x}-y)|,$$

$$\bar{\Gamma}_{2,2}(t, \bar{t}, x, \bar{x}) = \int_0^{\bar{t}} ds \int_{\substack{|\bar{x}-y|<\bar{t}-s \\ |x-y|\geqslant\bar{t}-s}} dy S(\bar{t}-s, \bar{x}-y).$$

Using (A.14), we obtain $\bar{\Gamma}_{2,1}(t, \bar{t}, x, \bar{x}) \leqslant C|x-\bar{x}|^{1/2}$. Finally, an explicit computation implies $\bar{\Gamma}_{2,2}(t, \bar{t}, x, \bar{x}) \leqslant C|x-\bar{x}|^{1/2}$. Thus

$$\int_0^T ds \int_{\mathbb{R}^2} dy \bar{\Gamma}_2(t, \bar{t}, x, \bar{x}; s, y) \leqslant C|x-\bar{x}|^{1/2}. \qquad (2.45)$$

Inequalities (2.44) and (2.45) show (2.43) and conclude the proof of the lemma. $\qquad \square$

In the next proposition, we show that the sequence of processes $\{X_n(t, x), \ n \geqslant 1\}$ satisfies assumption (P1) of Lemma A.2. It proves estimates similar to those in Millet and Sanz-Solé (1999, Proposition 1.4) which are uniform in n.

Proposition 2.2. *Assume* (C1)*,* (C2)*,* (C3$'$)*. For any* $p \in [1, \infty)$*,* $0 \leqslant t \leqslant \bar{t} \leqslant T$*,* $x, \bar{x} \in K$*,* $\gamma \in]0, \beta/(2(1+\beta))[$*,*

$$\sup_n \|(X_n(t, x) - X_n(\bar{t}, \bar{x}))1_{A_n(\bar{t})}\|_p \leqslant C(|t-\bar{r}|^\gamma + |x-\bar{x}|^\gamma).$$

Proof. Consider the decomposition

$$\mathrm{E}(|X_n(t, x) - X_n(\bar{t}, \bar{x})|^p 1_{A_n(\bar{t})}) \leq C \sum_{i=1}^{6} R_n^i(t, \bar{t}; x, \bar{x}),$$

where

$$R_n^1(t, \bar{t}; x, \bar{x}) = |X_0(t, x) - X_0(\bar{t}, \bar{x})|^p,$$

$$R_n^2(t, \bar{t}; x, \bar{x}) = \mathrm{E}\left(\left|\int_0^T \int_{\mathbb{R}^2} \gamma(t, \bar{t}, x, \bar{x}; s, y) A(X_n(s, y)) F(\mathrm{d}s, \mathrm{d}y)\right|^p 1_{A_n(\bar{t})}\right),$$

$$R_n^3(t, \bar{t}; x, \bar{x}) = \mathrm{E}(|\langle \gamma(t, \bar{t}, x, \bar{x}; \cdot, *) B(X_n^-(\cdot, *)), \omega^n \rangle|_H^p 1_{A_n(\bar{t})}),$$

$$R_n^4(t, \bar{t}; x, \bar{x}) = \mathrm{E}(|\langle \gamma(t, \bar{t}, x, \bar{x}; \cdot, *) (B(X_n) - B(X_n^-))(\cdot, *), \omega^n \rangle|_H^p 1_{A_n(\bar{t})}),$$

$$R_n^5(t, \bar{t}; x, \bar{x}) = \mathrm{E}(|\langle \gamma(t, \bar{t}, x, \bar{x}; \cdot, *) D(X_n(\cdot, *)), h \rangle|_H^p 1_{A_n(\bar{t})}),$$

$$R_n^6(t, \bar{t}; x, \bar{x}) = \mathrm{E}\left(\left|\int_0^T \int_{\mathbb{R}^2} \gamma(t, \bar{t}, x, \bar{x}; s, y) b(X_n(s, y)) \mathrm{d}s\, \mathrm{d}y\right|^p 1_{A_n(\bar{t})}\right).$$

In the proof of Proposition 1.4 in Millet and Sanz-Solé (1999) we checked that (C2) implies that

$$R_n^1(t, \bar{t}; x, \bar{x}) \leq C(|t - \bar{t}|^\delta + |x - \bar{x}|^\delta) \tag{2.46}$$

for $\delta = \beta/(2(1 + \beta))$. Burkholder's and Hölder's inequalities yield

$$R_n^2(t, \bar{t}; x, \bar{x}) \leq C\|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^p \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X_n(s, y)|^p 1_{A_n(t)})\right]. \tag{2.47}$$

Using the operators τ_n and π_n introduced in the proof of Lemma 2.4, we may rewrite the scalar product appearing in R_n^3 as a stochastic integral; thus Burkholder's inequality implies

$$R_n^3(t, \bar{t}; x, \bar{x}) = \mathrm{E}\left(\left|\int_0^T \int_{\mathbb{R}^2} (\pi_n \circ \tau_n)(\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *) B(X_n^-(\cdot, *)) 1_{A_n(\bar{t})})(s, y) F(\mathrm{d}s, \mathrm{d}y)\right|^p\right)$$

$$\leq C\|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^p \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X_n^-(s, y)|^p 1_{A_n(s)})\right]. \tag{2.48}$$

The Cauchy-Schwarz and Hölder inequalities, along with (2.11) and (2.31), imply

$$R_n^4(t, \bar{t}; x, \bar{x}) \leqslant \{ \mathrm{E}(\|\omega_n\|_H^{2p} 1_{A_n(T)}) \mathrm{E}(\|\gamma(t, \bar{t}, x, \bar{x}; \cdot, *)(B(X_n) - B(X_n^-))(\cdot, *)\|_H^{2p} 1_{A_n(\bar{t})}) \}^{1/2}$$

$$\leqslant C n^p 2^{np/2} \|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^p \left\{ \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X_n(s, y) - X_n^-(s, y)|^{2p} 1_{A_n(s)}) \right\}^{1/2}$$

$$\leqslant C n^{2p} 2^{-n\beta p/2} \|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^p. \tag{2.49}$$

Finally

$$R_n^5(t, \bar{t}; x, \bar{x}) \leqslant C \|h\|_H^p \|\Gamma(t, \bar{t}, x, \bar{x}; \cdot, *)\|_H^p \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X_n(s, y)|^p 1_{A_n(s)}) \right], \tag{2.50}$$

$$R_n^6(t, \bar{t}; x, \bar{x}) \leqslant C \left(\int_0^T \int_{\mathbb{R}^2} \Gamma(t, \bar{t}, x, \bar{x}; s, y) \mathrm{d}s\, \mathrm{d}y \right)^p \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X_n(s, y)|^p 1_{A_n(s)}) \right].$$

$$\tag{2.51}$$

Hence, (2.46)–(2.51), (2.30) and Lemma 2.5 yield the proposition. □

Remark 2.2. Proposition 2.2 establishes the γ-Hölder continuity for the trajectories of X_n on $A_n(T)$, because the sets $A_n(t)$, $t \in [0, T]$, are decreasing. In Lemma 2.1 we have shown $\lim_{\alpha\to\infty} P(A_n(T)) = 1$, so that the trajectories of X_n are almost surely γ-Hölder continuous on $[0, T] \times K$ for any $\gamma < \beta/2(1 + \beta)$ and any compact subset K of $\mathbb{R}^2$.

We now prove that the processes $\{X_n(t, x), n \geqslant 1\}$ satisfy condition (P2) of Lemma A2.

Proposition 2.3. *Suppose that conditions* (C1), (C2) *and* (C3′) *are satisfied. Then, for any* $p \in [1, \infty)$, $(t, x) \in [0, T] \times K$,

$$\lim_{n\to\infty} \|(X_n(t, x) - X(t, x)) 1_{A_n(t)}\|_p = 0. \tag{2.52}$$

Proof. Set

$$X_n(t, x) - X(t, x) = \sum_{i=1}^8 U_n^i(t, x),$$

where

$$U_n^1(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)[(A + B)(X_n(s, y)) - (A + B)(X(s, y))]F(ds, dy),$$

$$U_n^2(t, x) = \langle S(t - \cdot, x - *)[D(X_n(\cdot, *)) - D(X(\cdot, *))], h \rangle_H,$$

$$U_n^3(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)[b(X_n(s, y)) - b(X(s, y))]ds\, dy,$$

$$U_n^4(t, x) = \langle S(t - \cdot, x - *)[B(X_n(\cdot, *)) - B(X_n^-(\cdot, *))], \omega^n \rangle_H,$$

$$U_n^5(t, x) = \langle S(t - \cdot, x - *)[B(X_n^-(\cdot, *)) - B(X^-(\cdot, *))], \omega^n \rangle_H,$$

$$U_n^6(t, x) = \langle S(t - \cdot, x - *)B(X^-(\cdot, *)), \omega^n \rangle_H - \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)B(X^-(s, y))F(ds, dy),$$

$$U_n^7(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)[B(X^-(s, y)) - B(X_n^-(s, y))]F(ds\, dy),$$

$$U_n^8(t, x) = \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)[B(X_n^-(s, y)) - B(X_n(s, y))]F(ds\, dy),$$

with X_n^- and X^- defined in (2.15) and (2.16), respectively.

For $t \in [0, T]$, let $K(t) = \{x \in \mathbb{R}^2 : d(x, K) \leq T - t\}$; fix $t \in [0, T]$ and $x \in K(t)$. Burkholder's and Hölder's inequalities imply

$$\|U_n^1(t, x)1_{A_n(t)}\|_p^p \leq C \int_0^t J(t - s) \sup_{y \in K(s)} \|(X_n(s, y) - X(s, y))1_{A_n(s)}\|_p^p\, ds.$$

The Cauchy-Schwarz and Hölder inequalities yield

$$\|U_n^2(t, x)1_{A_n(t)}\|_p^p \leq C \|h\|_H^p \int_0^t J(t - s) \sup_{y \in K(s)} \|(X_n(s, y) - X(s, y))1_{A_n(s)}\|_p^p\, ds.$$

Analogously,

$$\|U_n^3(t, x)1_{A_n(t)}\|_p^p \leq C \int_0^t (t - s) \sup_{y \in K(s)} \|(X_n(s, y) - X(s, y))1_{A_n(s)}\|_p^p\, ds.$$

Since

$$U_n^5(t, x) = \int_0^t \int_{\mathbb{R}^2} (\pi_n \circ \tau_n)[S(t - \cdot, x - *)[B(X_n^-) - B(X^-)](\cdot, *)1_{A_n(\cdot)}](s, y)F(ds, dy),$$

Burkholder's and Hölder's inequalities easily yield

$$\|(U_n^5(t, x) + U_n^7(t, x))1_{A_n(t)}\|_p^p \leq C \int_0^t J(t - s) \sup_{y \in K(s)} \|(X_n^-(s, y) - X^-(s, y))1_{A_n(s)}\|_p^p\, ds.$$

Thus, (2.31) and (2.17) ensure that

$$\|(U_n^5(t, x) + U_n^7(t, x))1_{A_n(t)}\|_p^p$$

$$\leq Cn^p 2^{-n(1+\beta)p/2} + C\int_0^t J(t-s) \sup_{y\in K(s)} \|(X_n(s, y) - X(s, y))1_{A_n(s)}\|^p \, ds.$$

Thus, by Gronwall's lemma, if suffices to check

$$\sup_{0\leq t\leq T} \sup_{x\in K(t)} \|U_n^i(t, x)1_{A_n(t)}\|_p \underset{n\to\infty}{\to} 0, \qquad i = 4, 6, 8. \tag{2.53}$$

The Cauchy-Schwarz and Hölder inequalities, along with (2.11) and (2.31), imply that

$$\|U_n^4(t, x)1_{A_n(t)}\|_p^p$$

$$\leq C\{E(\|\omega^n 1_{A_n(t)}\|_H^{2p})\}^{1/2} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} \{E(|X_n(t, x) - X_n^-(t, x)|^{2p}1_{A_n(t)})\}^{1/2}$$

$$\leq Cn^{2p}2^{-n\beta p/2}.$$

Burkholder's and Hölder's inequalities and (2.31) imply

$$\|U_n^8(t, x)1_{A_n(t)}\|_p^p \leq C\int_0^t J(t-s) \sup_{x\in\mathbb{R}^2}(\|(X_n^-(s, x) - X_n(s, x))1_{A_n(s)}\|_p^p)ds$$

$$\leq Cn^p 2^{-n(1+\beta)p/2};$$

thus (2.53) holds for $i = 4, 8$.

Set $\|U_n^6(t, x)1_{A_n(t)}\|_p^p \leq C(U_n^{6,1}(t, x) + U_n^{6,2}(t, x))$, with

$$U_n^{6,1}(t, x) = \left\|\int_0^t\int_{\mathbb{R}^2} [\pi_n(\{\tau_n(S(t-\cdot, x-*)B(X^-(\cdot, *))))\right.$$

$$\left. - S(t-\cdot, x-*)B(X^-(\cdot, *))\}1_{A_n(.)})](s, y)F(ds, dy)\right\|_p^p,$$

$$U_n^{6,2}(t, x) = \left\|\int_0^t\int_{\mathbb{R}^2} [\pi_n(S(t-\cdot, x-*)B(X^-(\cdot, *))1_{A_n(\cdot)})(s, y)\right.$$

$$\left. - S(t-s, x-y)B(X^-(s, y))1_{A_n(s)}]F(ds, dy)\right\|_p^p.$$

By Burkholder's inequality and the fact that π_n is a contraction of H, we deduce that

$$U_n^{6,1}(t, x) \leq C E(\|[\tau_n(S(t-\cdot, x-*)B(X^-(\cdot, *))) - S(t-\cdot, x-*)B(X^-(\cdot, *))]1_{A_n(\cdot)}\|_H^p).$$

Define

$$U_n^{6,1,1}(t, x) = \mathrm{E}(\|[\tau_n(S(t - \cdot, x - *)) - S(t - \cdot, x - *)]\tau_n(B(X^-(\cdot, *)))1_{A_n(\cdot)}\|_H^p),$$

$$U_n^{6,1,2}(t, x) = \mathrm{E}(\|S(t - \cdot, x - *)1_{A_n(\cdot)}[\tau_n(B(X^-(\cdot, *))) - B(X^-(\cdot, *))]\|_H^p).$$

Expressions (2.18) and (A.12) together yield

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} U_n^{6,1,1}(t, x) \leq C2^{-n\gamma p} \tag{2.54}$$

for any $\gamma < \beta/2(1 + \beta)$. Moreover, since X is a particular case of X_n, by (2.17) and Proposition 2.2 we have

$$\sup_{(s,x)\in[0,T]\times\mathbb{R}^2} \|(X^-(s + 2^{-n}, x) - X^-(s, x))1_{A_n(s+2^{-n})}\|_p \leq C2^{-n\gamma} \tag{2.55}$$

for $0 < \gamma < \beta/2(1 + \beta)$. Furthermore, (2.18) yields that if $1/p + 1/q = 1$, then

$$\sup_{(s,x)\in[0,T]\times\mathbb{R}^2} \left\|(|X^-(s + 2^{-n}, x)| + |X^-(s, x)|)1_{A_n(s)\setminus A_n(s+2^{-n})}\right\|_p \leq C[P(A_n(T))^c]^{1/q},$$

so that, using Lemma 2.1 and (2.55), we conclude that

$$\lim_n \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} U_n^{6,1,2}(t, x) = 0. \tag{2.56}$$

Inequalities (2.54) and (2.56) imply

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} U_n^{6,1}(t, x) = 0. \tag{2.57}$$

Let I_H denote the identity operator on the Hilbert space H. Burkholder's inequality yields $U_n^{6,2}(t, x) \leq CZ_n(t, x)$, with, for $(t, x) \in [0, T] \times \mathbb{R}^2$,

$$Z_n(t, x) = \mathrm{E}(\|(\pi_n - I_H)(S(t - \cdot, x - *)B(X^-(\cdot, *))1_{A_n(\cdot)}\|_H^p).$$

The sequence $(\|(\pi_n - I_H)(S(t - \cdot, x - *)B(X^-(\cdot, *)) \times 1_{A_n(\cdot)}\|_H)$, $n \geq 1$, decreases to 0 as $n \to \infty$. It is bounded by $\sup_n 2\|S(t - \cdot, x - *)B(X^-(\cdot, *))\|_H$; we prove that this last random variable belongs to $L^p(\Omega)$. Indeed, The Cauchy-Schwarz inequality implies

$$\mathrm{E}\left(\sup_n\|S(t - \cdot, x - *)B(X^-(\cdot, *))\|_H^p\right) \leq C\sum_{i=1}^4 T_i,$$

with

$$T_1 = \mathrm{E}(\|S(t - \cdot, x - *)(1 + |X^0(\cdot, *))\||_H^p),$$

$$T_2 = \mathrm{E}\left(\sup_n \left|\int_0^t \mathrm{d}s \iint \mathrm{d}y \, \mathrm{d}z \, S(t - s, x - y) f(|y - z|) S(t - s, x - z)\right.\right.$$

$$\times \left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, y - \eta)(A + B)(X(r, \eta)) F(\mathrm{d}r, \, \mathrm{d}\eta)\right|$$

$$\times \left.\left.\left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, z - \zeta)(A + B)(X(r, \zeta)) F(\mathrm{d}r, \mathrm{d}\zeta)\right|\right|^{p/2}\right),$$

$$T_3 = \mathrm{E}\left(\sup_n \left|\int_0^t \mathrm{d}s \iint \mathrm{d}y \, \mathrm{d}z \, S(t - s, x - y) f(|y - z|) S(t - s, x - z).\right.\right.$$

$$\times |\langle 1_{[0,s_n]}(\cdot) S(s - \cdot, y - *) D(X(\cdot, *)), \, h\rangle_H|$$

$$\times \left.\left.|\langle 1_{[0,s_n]}(\cdot) S(s - \cdot, z - *) D(X(\cdot, *)), \, h\rangle_H|\right|^{p/2}\right),$$

$$T_4 = \mathrm{E}\left(\sup_n \left|\int_0^t \mathrm{d}s \iint \mathrm{d}y \, \mathrm{d}z S(t - s, x - y) f(|y - z|) S(t - s, x - z)\right.\right.$$

$$\times \left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, y - \eta) b(X(r, \eta)) \mathrm{d}r \, \mathrm{d}\eta\right|$$

$$\times \left.\left.\left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, z - \zeta) b(X(r, \zeta)) \mathrm{d}r \, \mathrm{d}\zeta\right|\right|^{p/2}\right).$$

Hölder's inequality implies that

$$T_2 \leqslant C \int_0^t \mathrm{d}s \iint \mathrm{d}y \mathrm{d}z \, S(t - s, x - y) f(|y - z|) S(t - s, x - z)$$

$$\times \mathrm{E}\left(\sup_n \left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, y - \eta)(A + B)(X(r, \eta)) F(\mathrm{d}r, \mathrm{d}\eta)\right|^p\right)^{1/2}$$

$$\times \mathrm{E}\left(\sup_n \left|\int_0^{s_n} \int_{\mathbb{R}^2} S(s - r, z - \zeta)(A + B)(X(r, \zeta)) F(\mathrm{d}r, \mathrm{d}\zeta)\right|^p\right)^{1/2}.$$

Doob's maximal inequality applied to the martingale

$$\left(\int_0^\tau \int_{\mathbb{R}^2} S(s - r, y - \eta)(A + B)(X(r, \eta)) F(\mathrm{d}r, \mathrm{d}\eta), \, \mathscr{F}_\tau\right)$$

and (A.11) imply that

$$T_2 \leq C \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X(s,\ y)|^p) \right].$$

A similar, easier computation using The Cauchy-Schwarz's and Hölder's inequality yields

$$T_3 + T_4 \leq C \left[1 + \sup_{(s,y)\in[0,T]\times\mathbb{R}^2} \mathrm{E}(|X(s,\ y)|^p) \right].$$

Therefore, these estimations and (A.15) imply, for $p \in [1, \infty[$,

$$\mathrm{E}\left(\sup_n \| S(t - \cdot,\ x - *)B(X^-(\cdot,\ *)) \|_H^p \right) < \infty.$$

Thus, by dominated convergence, the sequence $(Z_n(t, x))_{n\geq 1}$ decreases to 0. Moreover, $Z_n(t, x)$ is jointly continuous in (t, x); consequently, by Dini's theorem,

$$\sup_{(t,x)\in[0,T]\times K} Z_n(t,\ x) \downarrow 0 \text{ as } n \to \infty.$$

Thus,

$$\sup_{(t,x)\in[0,T]\times K} U_n^{6,2}(t,\ x) \underset{n\to\infty}{\longrightarrow} 0. \tag{2.58}$$

The convergences (2.57) and (2.58) complete the proof of (2.53) for $i = 6$, and hence that of the proposition. $\qquad\square$

Proof of Proposition 2.1. Since equation (2.2) defining the process $\{X(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$ is a particular case of (2.1) which defines $\{X_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$, Propositions 2.2 and 2.3 ensure the validity of conditions (P1) and (P2) of Lemma A.2 for the sequence of processes $Y_n(t, x) := X_n(t, x) - X(t, x)$ and the sequence of adapted sets $B_n(t) := A_n(t)$ defined in (2.10). Therefore, given any $0 < \gamma < \beta/2(1 + \beta)$, $p \in [1, \infty)$,

$$\lim_{n\to\infty} \mathrm{E}(\|X_n - X\|_{\gamma,K}^p 1_{A_n(T)}) = 0, \tag{2.59}$$

where $\|\cdot\|_{\gamma,K}$ is given by (1.5).

Fix $\varepsilon > 0$; by Lemma 2.1 there exists $n_0 \in \mathbb{N}$ such that $P(A_n(T)^c) < \varepsilon$ for $n \geq n_0$. Then, for any $\eta > 0$,

$$P(\|X_n - X\|_{\gamma,K} > \eta) \leq \varepsilon + P(\|X_n - X\|_{\gamma,K} > \eta,\ A_n(T))$$

$$\leq \varepsilon + \eta^{-p}\mathrm{E}(\|X_n - X\|_{\gamma,K}^p 1_{A_n(T)}). \tag{2.60}$$

Since $\varepsilon > 0$ is arbitrary, (2.59) and (2.60) show (2.4). $\qquad\square$

3. Approximation in L^p

In the previous section, we proved an approximation theorem in probability, by showing the L^p convergence of the sequence X_n localized by $A_{n,M(n)}$. The aim of this section is to check that under a stronger growth assumption on the coefficients, a slight modification of the proof

yields the L^p-convergence of X^n to X without localization. Let us introduce the following growth condition:

(C4′) There exists $\delta \in (0, 1)$ and a constant $C > 0$ such that, for $x \in \mathbb{R}^2$,

$$|A(x)| + |B(x)| + |D(x)| + |b(x)| \leq C(1 + |x|^\delta).$$

Then we have the following proposition:

Proposition 3.1. *Assume* (C1)*,* (C2) *and* (C4′)*, and let X and X^n be defined by* (2.1) *and* (2.2)*, respectively. For any $\gamma \in (0, \beta/2(1 + \beta))$, every compact subset $K \subset \mathbb{R}^2$ and every $p \in [1, +\infty)$,*

$$\lim_n \| \|X_n - X\|_{\gamma, K} \|_p = 0. \tag{3.1}$$

The proof is very similar to that of Proposition 2.1, and will only be sketched. It depends on several technical lemmas, which are 'unlocalized' versions of Lemmas 2.3 and 2.4.

Lemma 3.1. *Suppose that conditions* (C1)*,* (C2) *and* (C4′) *hold. Then, for any $p \in [1, +\infty)$, $t \in [0, T]$, $\delta' \in]\delta, 1[$ and $n \geq 1$,*

$$\sup_{(s, y) \in [0, t] \times \mathbb{R}^2} \mathrm{E}(|X_n^k(s, y) - \bar{X}_n^k(s, y)|^p)$$

$$\leq Cn^{p/2} 2^{-n(1+\beta)p/2} \left[1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^2} \mathrm{E}(|X_n^{k-1}(s, y)|^{\delta' p})\right] \tag{3.2}$$

and

$$\sup_{(s, y) \in [0, t] \times \mathbb{R}^2} \mathrm{E}(|X_n(s, y) - X_n^-(s, y)|^p) \leq Cn^{p/2} 2^{-n(1+\beta)p/2} \left[1 + \sup_{(s, y) \in [0, t] \times \mathbb{R}^2} \mathrm{E}(|X_n(s, y)|^{\delta p})\right]. \tag{3.3}$$

Proof. Consider the decomposition

$$\mathrm{E}(|X_n^k(t, x) - \bar{X}_n^k(t, x)|^p) \leq C \sum_{i=1}^4 \tilde{T}^{k, i},$$

where each term $\tilde{T}_n^{k, i}$ is deduced from the corresponding term $T_n^{k, i}$ introduced in (2.24) by removing $1_{A_n(t)}$.

Let $\bar{p}$ and $\bar{q}$ be conjugate exponents such that $\delta \bar{p} = \delta'$. Then The Cauchy-Schwarz and Hölder inequalities, along with (1.13) and (A.11), imply

$$\tilde{T}^{k,2}(t,x) \leq \mathrm{E}(\|1_{(t_n,t]}\omega_n\|_H^{\bar{q}p})^{1/\bar{q}} \mathrm{E}(\|1_{[t_n,t]}(\cdot)S(t-\cdot, x-*)B(X_n^{k-1}(\cdot,*))\|_H^{\bar{p}p})^{1/\bar{p}}$$

$$\leq Cn^{p/2}2^{np/2}2^{-np/2}\left[\mu(t-t_n)^{p\bar{p}/2}\left(1+\sup_{(s,y)\in[0,t]\times\mathbb{R}^2}\mathrm{E}(|X_n^{k-1}(s,y)|^{\delta\bar{p}p})\right)\right]^{1/\bar{p}}$$

$$\leq Cn^{p/2}2^{-n((1+\beta)/2)p}\left[1+\sup_{(s,y)\in[0,t]\times\mathbb{R}^2}\mathrm{E}(|X_n^{k-1}(s,y)|^{\delta'p})\right].$$

The upper estimates of $\tilde{T}_n^{k,i}$, $i=1,3,4$ are obtained by means of a straightforward modification of those of $T_n^{k,i}$ in the proof of Lemma 2.3; this concludes the proof of (3.2).

Using the arguments in the proof of Millet and Sanz-Solé (1999, Theorem 1.2), we obtain the convergence of the Picard iteration scheme, that is, for $p \in [1, +\infty)$,

$$\lim_{k\to\infty}\sup_{(s,x)\in[0,T]\times\mathbb{R}^2}(\|X_n^k(s,x)-X_n(s,x)\|_p + \|\bar{X}_n^k(s,x)-X_n^-(s,x)\|_p)=0. \qquad (3.4)$$

Therefore, (3.2) and (3.4) imply (3.3). □

We now prove L^p-convergence of $X_n^-(s,y)$ to $X_n(s,y)$.

Lemma 3.2. *Assume* (C1), (C2) *and* (C4′); *then, for* $p \in [1, +\infty)$,

$$\sup_{n\geq 1}\sup_{(t,x)\in[0,T]\times\mathbb{R}^2}(\|X_n(t,x)\|_p + \|X_n^-(t,x)\|_p) < \infty \qquad (3.5)$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2}\|X_n(t,x)-X_n^-(t,x)\|_p \leq Cn^{1/2}2^{-n(1+\beta)/2}. \qquad (3.6)$$

Proof. The proof reduces to that of

$$\sup_{n\geq 1}\sup_{k\geq 0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^2}(\|X_n^k(t,x)\|_p + \|\bar{X}_n^k(t,x)\|_p) < \infty. \qquad (3.7)$$

Indeed, (3.4) and (3.7) imply (3.5), while (3.3) and (3.5) yield (3.6). For $r \leq t$, consider the decomposition

$$\mathrm{E}(|X_n^{k+1}(t,r;x)|^p) \leq C\sum_{i=1}^{6}\tilde{T}_n^{k+1,i}(t,r;x), \qquad (3.8)$$

where $\tilde{T}_n^{k+1,i}(t,r;x)$ is deduced from the term $T_n^{k+1,i}(t,r;x)$ in (2.33) by removing $1_{A_n(t)}$. The arguments used to obtain upper estimates of $\tilde{T}_n^{k+1,i}$ for $i \neq 4$ are similar to that in Lemma 2.4 and are omitted. They yield the analogues of (2.34)–(2.36), (2.38) and (2.39).

Let $\delta' \in]\delta, 1[$, $\bar{p}=(\delta')^{-1}$, $\bar{p}$ and $\bar{q}$ be conjugate exponents; then The Cauchy-Schwarz and Hölder inequalities, along with (A.11) and (1.12), yield

$$\tilde{T}_n^{k+1,4}(t, r; x) \leq \ \mathrm{E}(\|\omega^n\|_H^{\bar{q}p})^{1/\bar{q}} \ \mathrm{E}(\|S(t - \cdot, x - *)1_{[0,r]}(\cdot)[B(X_n^k) - B(\bar{X}_n^k)](\cdot, *)\|_H^{\bar{p}p})^{1/\bar{p}}$$

$$\leq Cn^{p/2}2^{\mathrm{n}p/2}\left[\sup_{(s,y)\in[0,r]\times\mathbb{R}^2}\mathrm{E}(|(X_n^k - \bar{X}_n^k)(s, y)|^{\bar{p}p})\right]^{1/\bar{p}}.$$

Hence (3.2) implies

$$\tilde{T}_n^{k+1,4}(t, r; x) \leq Cn^{p/2}2^{np/2}\left\{n^{\bar{p}p/2}2^{-n(1+\beta)\bar{p}p/2}\left[1 + \sup_{(s,y)\in[0,r]\times\mathbb{R}^2}\mathrm{E}(|X_n^{k-1}(s, y)|^{\delta'\bar{p}p})\right]\right\}^{1/\bar{p}}$$

$$\leq Cn^p2^{-n\beta p/2}\left[1 + \sup_{(s,y)\in[0,r]\times\mathbb{R}^2}\mathrm{E}(|X_n^{k-1}(s, y)|^p)\right]. \tag{3.9}$$

Set $\tilde{\varphi}_n^{-1} \equiv 0$ and, for every $k \geq 0$,

$$\tilde{\varphi}_n^k(t) = \sup_{0\leq s\leq t}\sup_{y\in\mathbb{R}^2}\mathrm{E}(|X_n^k(s, y)|^p + |\bar{X}_n^k(s, y)|^p).$$

Then, for every $k \geq 0$,

$$\tilde{\varphi}_n^{k+1}(t) \leq C\int_0^t[1 + \tilde{\varphi}_n^k(s) + \tilde{\varphi}_n^{k-1}(s)]\mathrm{d}s.$$

Since $\sup_{0\leq t\leq T}\tilde{\varphi}_n^0(t) = C < \infty$, this implies (3.7). $\quad\square$

Replacing (2.30) and (2.31) by (3.5) and (3.6) respectively, the arguments in the proofs of Propositions 2.2 and 2.3 yield the following result:

Proposition 3.2. *Assume* (C1)*,* (C2) *and* (C4′)*. Let K be a compact subset of $\mathbb{R}^2$ and $p \in [1, \infty)$; then we have the following:*

(i) *For $0 < \gamma < \beta/(2(1 + \beta))$, $0 \leq t \leq \bar{t} \leq T$, $x, \bar{x} \in K$,*

$$\sup_n\|X_n(t, x) - X_n(\bar{t}, \bar{x})\|_p + \|X(t, x) - X(\bar{t}, \bar{x})\|_p \leq C(|t - \bar{t}|^\gamma + |x - \bar{x}|^\gamma). \tag{3.10}$$

(ii) *For $(t, x) \in [0, T]xK$,*

$$\lim_n\|X_n(t, x) - X(t, x)\|_p = 0. \tag{3.11}$$

Proof of Proposition 3.1. To conclude the proof of this proposition, it suffices to apply Bally *et al.* (1995, Lemma A.1). Indeed, the results proved in the previous proposition ensure the validity of the hypothesis of that lemma. $\quad\square$

Appendix

This section quotes some notation introduced in our previous paper (Millet and Sanz-Solé 1999), which is extensively used along the paper. It also contains some technical results.

For any $t \in [0, T]$, $h \geqslant 0$, $\xi \in \mathbb{R}^2$, set

$$J(t) = \int_{|y|<|x|<t} \frac{1}{\sqrt{t^2 - |x|^2}} f(|x - y|) \frac{1}{\sqrt{t^2 - |y|^2}} \, dx \, dy, \tag{A.1}$$

$$\mu(t) = \int_0^t ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \, S(s, x) f(|x - y|) S(s, y) = \frac{1}{2\pi^2} \int_0^t J(s) ds, \tag{A.2}$$

$$\nu(t) = \frac{1}{2\pi} \int_0^t ds \int_{|x|<s} \frac{dx}{\sqrt{s^2 - |x|^2}} = \frac{t^2}{2}, \tag{A.3}$$

$$\nu_{t,h} = \int_0^t ds \int_{|y|<s} dy (S(s, y) - S(s + h, y)), \tag{A.4}$$

$$\tilde{\nu}_{t,h} = \int_0^t ds \int_{s \leqslant |y| < s+h} dy \, S(s + h, y), \tag{A.5}$$

$$\mu_{t,h} = \int_0^t ds \int_{|y|<s} dy \int_{|z|<s} dz [S(s, y) - S(s + h, y)] f(|y - z|) [S(s, z) - S(s + h, z)], \tag{A.6}$$

$$\tilde{\mu}_{t,h} = \int_0^t ds \int_{s \leqslant |y| < s+h} dy \int_{s \leqslant |z| < s+h} dz \, S(s + h, y) f(|y - z|) S(s + h, z), \tag{A.7}$$

$$M_{t,\xi} = \int_0^t \int_{\substack{|y|<s \\ |y-\xi| \geqslant s}} dy \int_{\substack{|z|<s \\ |z-\xi| \geqslant s}} dz \, S(s, y) f(|y - z|) S(s, z), \tag{A.8}$$

$$N_{t,\xi} = \int_{\frac{|\xi|}{2}}^t ds \int_{\substack{|y|<s \\ |y-\xi|<s}} dy \int_{\substack{|z|<s \\ |z-\xi|<s}} dz |S(s, y) - S(s, y - \xi)| f(|y - z|) |S(s, z) - S(s, z - \xi)|. \tag{A.9}$$

A direct computation shows that

$$\nu_{t,h} + \tilde{\nu}_{t,h} \leqslant Ch^{1/2}. \tag{A.10}$$

Assume that f satisfies assumption (C1); then Lemma A.1 in Millet and Sanz-Solé (1999) implies that

$$J(t) \leqslant Ct^\beta, \quad \mu(t) \leqslant Ct^{\beta+1}, \qquad t \in [0, T], \tag{A.11}$$

while, for $t \in [0, T]$, $h \vee |\xi| \leqslant \frac{1}{2}$ and $0 < \delta < \beta/(1 + \beta)$, Lemma A.5 in Millet and Sanz-Solé (1999) shows that

$$\mu_{t,h} + \tilde{\mu}_{t,h} \leqslant Ch^\delta, \tag{A.12}$$

$$M_{t,\xi} + N_{t,\xi} \leqslant C|\xi|^\delta. \tag{A.13}$$

Furthermore, (A.24) in Millet and Sanz-Solé (1999) yields that for $0 \leqslant s \leqslant T$, $x \in \mathbb{R}^2$,

$$\int_{|x|/2}^{x} dr \int_{|z-x|<|z|<s} dz(S(r, z) - S(r, z - x)) \leqslant C|x|^{1/2}. \qquad (A.14)$$

The following result is a slight generalization of Theorem 1.2 in Millet and Sanz-Solé (1999).

Lemma A.1. *Suppose* (C1)*,* (C2) *end* (C′3)*; for any* $T > 0$*, equation* (2.1) *(equation* (2.2)*) has a unique solution* X_n (X) *on* $[0, T] \times \mathbb{R}^2$*. Futhermore,*

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^2} \mathrm{E}(|X(t, x)|^p) < \infty, \quad p \in [1, \infty). \qquad (A.15)$$

Sketch of proof. Since (2.2) is a particular case of (2.1), we only prove the existence of the solution to (2.1). The argument for (2.2) is simpler, since no localization is required. Given $M > 0$, consider the evolution equation

$$X_{n,M}(t, x) = X^0(t, x) + \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)A(X_{n,M}(s, y))F(ds, dy)$$

$$+ \langle S(t - \cdot, x - *)B(X_{n,M}(\cdot, *)), \omega^{n,M} \rangle_H$$

$$+ \langle S(t - \cdot, x - *)D(X_{n,M}(\cdot, *)), h \rangle_H$$

$$+ \int_0^t \int_{\mathbb{R}^2} S(t - s, x - y)b(X_{n,M}(s, y))ds\, dy. \qquad (A.16)$$

A slight extension of the proof of Millet and Sanz-Solé (1999, Theorem 1.2) based on Picard's iteration scheme – using The Cauchy-Schwarz's inequality for the scalar products, Gronwall's lemma and (A.11) – provides the existence and uniqueness of the solution to (A.16). For any $(t, x) \in [0, 1] \times \mathbb{R}^2$, define

$$X_n(t, x) = X_{n,M}(t, x) \qquad \text{on } A_{n,M}(T).$$

For fixed n and M, the sets $(A_{n,M}(t))_{t \in [0,T]}$ are decreasing in t. Therefore, a standard argument based on the local property of stochastic integrals implies that this definition is consistent and, since $P(\cup_{M \geqslant 1} A_{n,M}(1)) = 1$ for every integer n, this provides the existence and uniqueness of solution to equation (2.1).

The proof of (A.15) is a straightforward extension of the corresponding inequality (1.8) in Millet and Sanz-Solé (1999). □

The following lemma is a localized version of Lemma A.1 in Bally *et al.* (1995). For the sake of completeness we give the main arguments of the proof.

Lemma A.2. *Let* $\{Y_n(t, x), (t, x) \in K_0\}$*,* $n \geqslant 1$*, be a sequence of processes indexed by* $K_0 = [0, T] \times K$*,* K *being a compact set of* $\mathbb{R}^2$*. Let* $\{B_n(t), t \in [0, T]\} \subset \mathscr{F}$ *be a sequence of adapted sets which, for every n, decreases in t. Make the following assumptions, for every* $p \in (1, \infty)$*:*

(P1) *There exists* $\delta > 0$ *such that, for any* $0 \leqslant t \leqslant \bar{t} \leqslant T$, $x, \bar{x} \in K$,

$$\sup_n \mathrm{E}(|Y_n(t, x) - Y_n(\bar{t}, \bar{x})|^p 1_{B_n(\bar{t})}) \leqslant C(|t - \bar{t}| + |x - \bar{x}|)^{3+\delta}.$$

(P2) *For every* $(t, x) \in [0, T] \times K$,

$$\lim_{n \to \infty} \mathrm{E}(|Y_n(t, x)|^p 1_{B_n(t)}) = 0.$$

Then, for any $\rho \in (0, \delta/p)$ *and any* $r \in [1, p)$,

$$\lim_{n \to \infty} \mathrm{E}(\|Y_n\|_{\rho, K}^r 1_{B_n(T)}) = 0.$$

Proof. Let $\zeta = 2d + \delta'$, $d = 3$, $0 < \delta' < \delta$; set $z = (t, x)$, $\bar{z} = (\bar{t}, \bar{x})$. Then, by (P1),

$$\int_{K_0} \int_{K_0} \mathrm{E}\left(\frac{|Y_n(z) - Y_n(\bar{z})|^p}{|z - \bar{z}|^\zeta} 1_{B_n(\bar{t})}\right) \mathrm{d}z \, \mathrm{d}\bar{z} \leqslant CB',$$

where

$$B' = \int_{K_0} \int_{K_0} |z - \bar{z}|^{-d+\delta-\delta'} \mathrm{d}z \, \mathrm{d}\bar{z} < +\infty.$$

Set

$$Z = \int_{K_0} \int_{K_0} \frac{|Y_n(z) - Y_n(\bar{z})|^p}{|z - \bar{z}|^\zeta} 1_{B_n(\bar{t})} \, \mathrm{d}z \, \mathrm{d}\bar{z}.$$

Clearly, by Fubini's theorem, $\mathrm{E}(Z) \leqslant CB'$, so that

$$\mathrm{P}(Z > \lambda^p) \leqslant C\lambda^{-p} B'.$$

The Garsia–Rodemich–Rumsey lemma yields

$$|Y_n(z) - Y_n(\bar{z})| 1_{B_n(\bar{t})} \leqslant \bar{C} Z^{1/p} |z - \bar{z}|^{\rho_0},$$

with $\rho_0 = \delta'/p$. Since $\{B_n(t), t \in [0, T]\}$ decreases in t, this yields, for any $\rho < \delta/p$,

$$P\left(\sup_{z \neq \bar{z}} \frac{|Y_n(z) - Y_n(\bar{z})|}{|z - \bar{z}|^\rho} > \lambda, B_n(T)\right) \leqslant \lambda^{-p} \mathrm{E}\left(\sup_{z \neq \bar{z}} \frac{|Y_n(z) - Y_n(\bar{z})|^p}{|z - \bar{z}|^{\rho p}} 1_{B_n(T)}\right)$$

$$\leqslant C\lambda^{-p} \mathrm{E}(Z) \leqslant C\lambda^{-p}.$$

On the set $B_n(T)$, we now proceed exactly as in Bally *et al.* (1995, Lemma A.1) and show that for any $\varepsilon > 0$, $r \in [1, p)$, there exists $N \in \mathbb{N}$ such that, for any $n \geqslant N$,

$$\mathrm{E}(\|Y_n\|_{\beta, K}^r 1_{B_n(T)}) \leqslant \varepsilon^r + C\varepsilon.$$

$\square$

Remark A.1. Under condition (C1), inequalities (A.12) and (A.13) have been improved in a recent paper of Millet and Morien (2000) as follows. For $0 < \delta < \beta < 1$,

$$\mu_{t,h} + \tilde{\mu}_{t,h} \leqslant Ch^{\delta}, \qquad M_{t,\xi} + N_{t,\xi} \leqslant C|\xi|^{\delta}. \tag{A.17}$$

Suppose that (C2) is replaced by the following condition:

(C2′) $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is of class $\mathscr{C}^1$ and bounded with β-Hölder continuous partial derivatives, and $v_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $|v_0| + |\nabla v_0| \in L^{q_0}$ for some $q_0 \in]4 \vee 2/(1 - \beta)[$.

The processes u and X have almost surely γ-Hölder continuous trajectories for $0 < \gamma < \beta$, and the same arguments show that Theorem 2.1 can be reinforced to characterize the topological support of u in $\mathscr{C}^{\gamma}([0, T] \times K)$ for $\gamma \in]0, \beta[$. All intermediary results can also be stated for $\gamma \in]0, \beta[$ instead of $\gamma \in]0, \beta/(2(1 + \beta))[$. $\qquad\qquad\square$

Acknowledgements

References

Bally, V., Millet, A. and Sanz-Solé, M. (1995) Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations. *Ann. Probab.*, **23**, 178–222.

Dalang, R. and Frangos, N. (1998) The stochastic wave equation in two spatial dimensions. *Annals of Probab.*, **26**, 187–212.

Millet, A. and Sanz-Solé, M. (1994a) The support of an hyperbolic stochastic partial differential equation. *Probab. Theory Related Fields*, **98**, 361–387.

Millet, A. and Sanz-Solé, M. (1994b) A simple proof of the support theorem for diffusion processes. In J. Azéma, P.-A. Meyer and M. Yor (eds), *Séminaire de Probabilités XXVIII*, Lecture Notes in Math. 1583, pp. 36–48. Berlin: Springer-Verlag.

Millet, A. and Sanz-Solé, M. (1999) A stochastic wave equation in two space dimensions: Smoothness of the law. *Ann. Probab.*, **27**, 803–844.

Millet, A. and Morien, P.-L. (2000) On a stochastic wave equation in two space dimensions: regularity of the solution and its density. *Stoch. Proc. Appl.*, **86**, 141–162.