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# Mixed fractional Brownian motion

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We show that the sum of a Brownian motion and a non-trivial multiple of an independent fractional Brownian motion with Hurst parameter $H \in (0, 1]$ is not a semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, that it is equivalent to a multiple of Brownian motion if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$. As an application we discuss the price of a European call option on an asset driven by a linear combination of a Brownian motion and an independent fractional Brownian motion.

*Keywords:* equivalent measures; mixed fractional Brownian motion; semimartingale; weak semimartingale

## 1. Introduction

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space.

***Definition 1.1.*** *A fractional Brownian motion* $(B_t^H)_{t\in\mathbb{R}}$ *with Hurst parameter* $H \in (0, 1]$ *is an almost surely continuous, centred Gaussian process with*

$$\operatorname{cov}(B_t^H, B_s^H) = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}. \tag{1.1}$$

These processes were first studied by Kolmogorov (1940) within a Hilbert space framework. For $H \in (0, 1)$, Mandelbrot and Van Ness (1968) defined fractional Brownian motion more constructively as

$$B_t^H = c_H \int_{\mathbb{R}} [1_{\{s\leq t\}}(t-s)^{H-1/2} - 1_{\{s\leq 0\}}(-s)^{H-1/2}]\mathrm{d}W_s, \qquad t \in \mathbb{R}, \tag{1.2}$$

where $(W_s)_{s\in\mathbb{R}}$ is a two-sided Brownian motion and $c_H$ a normalizing constant. For $H = 1$, fractional Brownian motion can be constructed by setting

$$B_t^1 = t\xi, \qquad t \in \mathbb{R}, \tag{1.3}$$

where $\xi$ is a standard normal random variable. It can be deduced from (1.1) that fractional Brownian motions divide into three different families. $B^{1/2}$ is a two-sided Brownian motion. For $H \in (\frac{1}{2}, 1]$ the covariance between two increments over non-overlapping time intervals is positive, and for $H \in (0, \frac{1}{2})$ it is negative. From the representations (1.2) and (1.3) it can be seen that fractional Brownian motion has stationary increments. Furthermore, it can easily be checked that, for all $a > 0$,

$$\left(a^H B^H_{t/a}\right)_{t\in\mathbb{R}} \quad \text{has the same distribution as} \quad \left(B^H_t\right)_{t\in\mathbb{R}}.$$

This property is called self-similarity.

By 'mixed fractional Brownian motion' we mean a linear combination of different fractional Brownian motions. In this paper we examine whether a mixed fractional Brownian motion is a semimartingale when it is of the special form

$$M^{H,\alpha} := B + \alpha B^H,$$

where $B$ is a Brownian motion, $B^H$ an independent fractional Brownian motion and $\alpha \in \mathbb{R}\backslash\{0\}$.

To avoid localization arguments we consider $(M^{H,\alpha}_t)_{t\in[0,T]}$ for $T < \infty$. It follows from self-similarity of fractional Brownian motion that the process

$$\left(B_t + \alpha B^H_t\right)_{t\in[0,T]}$$

has the same distribution as

$$\left(T^{1/2} B_{t/T} + \alpha T^H B^H_{t/T}\right)_{t\in[0,T]} = T^{1/2}\left(B_{t/T} + \alpha T^{H-1/2} B^H_{t/T}\right)_{t\in[0,T]}.$$

This shows that there is no loss of generality in assuming $T = 1$.

***Definition 1.2.*** *A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ is said to fulfil the usual assumptions if it is right-continuous, $\mathcal{F}_1$ is complete and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_1$. For an arbitrary filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ we denote by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in[0,1]}$ the smallest filtration that contains $\mathbb{F}$ and satisfies the usual assumptions.*

The classical notion of a semimartingale stands at the end of a chain of generalizations of Brownian motion, each of which extended the class of stochastic processes that can play the role of the integrator in stochastic integration in the Itô sense – see Itô (1944) for Itô's construction of the stochastic integral. It reached its final form in Doléans-Dade and Meyer (1970). In their paper a stochastic process $(X_t)$ that is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)$ satisfying the usual assumptions is called an $\mathbb{F}$-semimartingale if it admits a decomposition of the form

$$X_t = X_0 + M_t + A_t, \tag{1.4}$$

where $X_0$ is an $\mathcal{F}_0$-measurable random variable, $M_0 = A_0 = 0$, $M$ is an a.s. right-continuous local martingale with respect to $\mathbb{F}$ and $A$ an a.s. right-continuous, $\mathbb{F}$-adapted finite-variation process. Later it was found that if a filtration $\mathbb{F} = (\mathcal{F})_{t\in[0,1]}$ satisfies the usual assumptions, an a.s. right-continuous, $\mathbb{F}$-adapted stochastic process $(X_t)_{t\in[0,1]}$ is of the form (1.4) if and only if $X$ fulfils the following condition:

$$I_X(\beta(\mathbb{F})) \text{ is bounded in } L^0, \tag{1.5}$$

where

$$\beta(\mathbb{F}) = \left\{ \sum_{j=0}^{n-1} f_j 1_{(t_j, t_{j+1}]} | n \in \mathbb{N},\ 0 \leqslant t_0 < \ldots < t_n \leqslant 1, \right.$$

$$\left. \forall j,\ f_j \text{ is } \mathcal{F}_{t_j}\text{-measurable and } |f_j| \leqslant 1 \text{ a.s.} \right\} \tag{1.6}$$

and

$$I_X(\vartheta) = \sum_{j=0}^{n-1} f_j (X_{t_{j+1}} - X_{t_j}) \qquad \text{for } \vartheta = \sum_{j=0}^{n-1} f_j 1_{(t_j, t_{j+1}]} \in \beta(\mathbb{F}).$$

This result is usually referred to as the Bichteler–Dellacherie theorem – see, for example, Section VIII.4 of Dellacherie and Meyer (1980) for a proof. For our purposes it is more convenient to work with condition (1.5) than with the decomposition property (1.4). If one does not require the process to be a.s. right-continuous and the filtration to satisfy the usual assumptions, one obtains a weaker form of the semimartingale property than the classical one.

***Definition 1.3.*** *A stochastic process* $(X_t)_{t\in[0,1]}$ *is a weak semimartingale with respect to a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ *if* $X$ *is* $\mathbb{F}$*-adapted and satisfies* (1.5).

Let $(X_t)_{t\in[0,1]}$ be a stochastic process. If $\mathbb{F}^1 = (\mathcal{F}^1_t)_{t\in[0,1]}$ and $\mathbb{F}^2 = (\mathcal{F}^2_t)_{t\in[0,1]}$ are two filtrations with $\mathcal{F}^1_t \subset \mathcal{F}^2_t$ for all $t \in [0, 1]$, then $\beta(\mathbb{F}^1) \subset \beta(\mathbb{F}^2)$. Hence, $L^0$-boundedness of $I_X(\beta(\mathbb{F}^2))$ implies $L^0$-boundedness of $I_X(\beta(\mathbb{F}^1))$. This shows that if $X$ is not a weak semimartingale with respect to the filtration generated by $X$, then it is not a weak semimartingale with respect to any other filtration. Therefore it is natural to introduce the following definition.

***Definition 1.4.*** *Let* $(X_t)_{t\in[0,1]}$ *be a stochastic process. We define the filtration* $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\in[0,1]}$ *by*

$$\mathcal{F}^X_t = \sigma((X_s)_{0\leqslant s\leqslant t}), \qquad t \in [0, 1].$$

*We call* $X$ *a weak semimartingale if it is a weak semimartingale with respect to* $\mathbb{F}^X$*. We call* $X$ *a semimartingale if it is a semimartingale with respect to* $\overline{\mathbb{F}}^X$.

***Example 1.5.*** It is easy to see that the deterministic process

$$X_t = \begin{cases} 0 & \text{for } t \in [0, \frac{1}{2}], \\ 1 & \text{for } t \in (\frac{1}{2}, 1], \end{cases}$$

is a weak semimartingale. But it is not a semimartingale because it is not a.s. right-continuous.

However, it follows from Lemma 2.4 below that for every filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$, an a.s right-continuous $\mathbb{F}$-weak semimartingale is also an $\overline{\mathbb{F}}$-semimartingale.

The problem of determining whether $M^{H,\alpha}$ is a semimartingale is easiest when $H \in \{\frac{1}{2}, 1\}$. It is clear that

$$\frac{1}{\sqrt{1+\alpha^2}} M^{1/2,\alpha}$$

is a Brownian motion. In particular, it is an $\overline{\mathbb{F}}^{M^{1/2,\alpha}}$-semimartingale. Hence, $M^{1/2,\alpha}$ is a semimartingale. $M^{1,\alpha}$ can be represented as

$$M_t^{1,\alpha} = B_t + \alpha t \xi, \qquad t \in [0, 1],$$

where $B$ is a Brownian motion and $\xi$ an independent standard normal random variable. This shows that $M^{1,\alpha}$ is a semimartingale with respect to $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in[0,1]}$, where

$$\mathcal{F}_t = \sigma(\xi, (B_s)_{0\leqslant s\leqslant t}), \qquad t \in [0, 1].$$

With the help of Girsanov's theorem we can show even more. Unlike $M^{1/2,\alpha}$, $M^{1,\alpha}$ is not a multiple of a Brownian motion under the measure $P$. But it is a Brownian motion under an equivalent measure $Q$. It can be deduced from Fubini's theorem that

$$\mathrm{E}[\exp(-\alpha\xi B_1 - \tfrac{1}{2}(\alpha\xi)^2)] = 1.$$

Therefore,

$$Q = \exp(-\alpha\xi B_1 - \tfrac{1}{2}(\alpha\xi)^2) \cdot P$$

is a probability measure that is equivalent to $P$ and it follows from Girsanov's theorem that $M^{1,\alpha}$ is a Brownian motion under $Q$. Hence, $M^{1,\alpha}$ is equivalent to Brownian motion in the sense of the following definition.

***Definition 1.6.*** *Let $(C[0, 1], \mathcal{B})$ be the space of continuous functions with the $\sigma$-algebra generated by the cylinder sets. If $(X_t)_{t\in[0,1]}$ is an a.s. continuous stochastic process, we denote by $P_X$ the measure induced by $X$ on $(C[0, 1], \mathcal{B})$. We call two a.s. continuous stochastic processes $(X_t)_{t\in[0,1]}$ and $(Y_t)_{t\in[0,1]}$ equivalent if $P_X$ and $P_Y$ are equivalent.*

It can be seen from Definition 1.3 that the weak semimartingale property is invariant under a change of the probability measure within the same equivalence class. The same is true for the semimartingale property. Hence, all processes that are equivalent to Brownian motion are semimartingales.

We express the main results of this paper in the following theorem.

**Theorem 1.7.** *$(M^{H,\alpha})_{t\in[0,1]}$ is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, it is equivalent to $\sqrt{1+\alpha^2}$ times Brownian motion if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$.*

For $H \in \{\frac{1}{2}, 1\}$, we have already proved Theorem 1.7. For $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ it has been shown by several authors (for example, Lipster and Shiryaev, 1989; Lin, 1995; Rogers,

1997) that fractional Brownian motion $B^H$ cannot be a semimartingale. Since $\mathbb{F}^{B^H}$ does not satisfy the usual assumptions, the statement that $B^H$ is not a weak semimartingale is slightly stronger. We will prove this statement in Section 2. Nothing in this proof is essentially new. We give it to make clear which parts of it can also be used to deal with $M^{H,\alpha}$ and when new methods are needed. For $H \in \left(0, \frac{1}{2}\right)$ the proof is based on the fact that the quadratic variation of $B^H$ is infinite. The same argument can be used to show that $M^{H,\alpha}$ is not a weak semimartingale for $H \in \left(0, \frac{1}{2}\right)$ because, as we will show in Section 3, in this case $M^{H,\alpha}$ has also infinite quadratic variation. For $H \in \left(\frac{1}{2}, 1\right)$, $B^H$ is not a weak semimartingale because it is a stochastic process with vanishing quadratic variation and a.s. paths of infinite variation. This reasoning cannot be applied to treat $M^{H,\alpha}$ for $H \in \left(\frac{1}{2}, 1\right)$ because then $M^{H,\alpha}$ has the same quadratic variation as Brownian motion. In this case we need more refined methods to see whether $M^{H,\alpha}$ is a semimartingale. Surprisingly, $M^{H,\alpha}$ is not a weak semimartingale if $H \in (\frac{1}{2}, \frac{3}{4}]$ and it is equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$. In Section 4 we prove Theorem 1.7 for $H \in (\frac{1}{2}, \frac{3}{4}]$. The proof depends on a theorem of Stricker (1984) on Gaussian processes. In Section 5 we prove Theorem 1.7 for $H \in (\frac{3}{4}, 1]$. In this case we use the concept of relative entropy and the fact that two Gaussian measures are either equivalent or singular. In Section 6 we discuss the price of a European call option on a stock that is modelled as an exponential mixed fractional Brownian motion with drift.

## 2. $B^H$ is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$

From now on we use the following notation. For a stochastic process $(X_t)_{t\in[0,1]}$ and $n \in \mathbb{N}$, we set, for $j = 1, \ldots, n$, $\Delta_j^n X = X_{j/n} - X_{(j-1)/n}$.

That $B^H$ is not a weak semimartingale for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ can be derived from the fact that in this case $B^H$ does not have the 'right' variation. The following facts about the $p$-variation of fractional Brownian motion are well known.

**Lemma 2.1.** *Let* $p, q > 0$. *Then*

(*a*) $n^{pH-1}\sum_{j=1}^n |\Delta_j^n B^H|^p \overset{(n\to\infty)}{\longrightarrow} \mathrm{E}[|B_1^H|^p]$ in $L^1$;
(*b*) $n^{pH-1-q}\sum_{j=1}^n |\Delta_j^n B^H|^p \overset{(n\to\infty)}{\longrightarrow} 0$ in $L^1$;
(*c*) $n^{pH-1+q}\sum_{j=1}^n |\Delta_j^n B^H|^p \overset{(n\to\infty)}{\longrightarrow} \infty$ in probability,
*i.e. for all* $L > 0$ *there exists an* $n_0$ *such that*
$P[n^{pH-1+q}\sum_{j=1}^n |\Delta_j^n B^H|^p < L] < 1/L$ *for all* $n \geqslant n_0$.

***Proof.*** To show (a) we recall that the sequence

$$(B_j^H - B_{j-1}^H)_{j=1}^\infty$$

is stationary. Since it is Gaussian and

$$\mathrm{cov}(B_1^H - B_0^H, B_j^H - B_{j-1}^H) \overset{(n\to\infty)}{\longrightarrow} 0,$$

it is also mixing. Hence, the ergodic theorem implies that

$$\frac{1}{n}\sum_{j=1}^{n}\left|B_j^H - B_{j-1}^H\right|^p \overset{(n\to\infty)}{\longrightarrow} \mathrm{E}[|B_1^H|^p] \text{ in } L^1. \tag{2.1}$$

On the other hand, it follows from the self-similarity of $B^H$ that, for all $n \in \mathbb{N}$,

$$n^{pH-1}\sum_{j=1}^{n}|\Delta_j^n B^H|^p = \frac{1}{n}\sum_{j=1}^{n}|B_j^H - B_{j-1}^H|^p \text{ in law.}$$

This, together with (2.1), proves (a).

Now (b) follows from (a).

To prove (c) we choose $L > 0$. It follows from (a) that

$$n^{pH-1}\sum_{j=1}^{n}|\Delta_j^n B^H|^p \overset{(n\to\infty)}{\longrightarrow} \mathrm{E}[|B_1^H|^p] \text{ in probability.}$$

In particular, there exists an $n_1 \in \mathbb{N}$ such that

$$P\left[\left|\mathrm{E}[|B_1^H|^p] - n^{pH-1}\sum_{j=1}^{n}|\Delta_j^n B^H|^p\right| > \frac{1}{2}\mathrm{E}[|B_1^H|^p]\right] < \frac{1}{L},$$

for all $n \geqslant n_1$. This implies that, for all $n \geqslant n_1$,

$$P\left[n^{pH-1}\sum_{j=1}^{n}|\Delta_j^n B^H|^p < \frac{1}{2}\mathrm{E}[|B_1^H|^p]\right] < \frac{1}{L}$$

or, equivalently,

$$P\left[n^{pH-1+q}\sum_{j=1}^{n}|\Delta_j^n B^H|^p < n^q\frac{1}{2}\mathrm{E}[|B_1^H|^p]\right] < \frac{1}{L}.$$

This shows that there exists an $n_0 \in \mathbb{N}$, such that

$$P\left[n^{pH-1+q}\sum_{j=1}^{n}|\Delta_j^n B^H|^p < L\right] < \frac{1}{L} \qquad \text{for all } n \geqslant n_0,$$

and (c) is proved. □

It follows from Lemma 2.1(c) that, for $H \in (0, \frac{1}{2})$, $B^H$ has infinite quadratic variation. The next proposition shows that this implies that $B^H$ cannot be a weak semimartingale if $H \in (0, \frac{1}{2})$.

**Proposition 2.2.** *Let $(X_t)_{t\in[0,1]}$ be an a.s. cadlag process and denote by $\tau$ the set of all finite partitions*

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

*of* [0, 1]. *If*

$$\left\{\sum_{j=0}^{n-1}(X_{t_{j+1}} - X_{t_j})^2 | (t_0, t_1, \ldots, t_n) \in \tau\right\}$$

*is unbounded in* $L^0$, *then* $X$ *is not a weak semimartingale.*

***Proof.*** To simplify calculations we define $Y_t = X_t - X_0$, $t \in [0, 1]$. Then $(Y_t)_{t\in[0,1]}$ is an $\mathbb{F}^X$-adapted, a.s. cadlag process with $Y_0 = 0$. It is clear that $I_Y = I_X$ and

$$\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 = \sum_{j=0}^{n-1}\left(X_{t_{j+1}} - X_{t_j}\right)^2$$

for all partitions $(t_0, t_1, \ldots, t_n) \in \tau$.

To prove the lemma, we must show that $I_Y(\beta(\mathbb{F}^X))$ is unbounded in $L^0$. The key ingredient in our derivation of this from the $L^0$-unboundedness of

$$\left\{\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 | (t_0, t_1, \ldots, t_n) \in \tau\right\}$$

is the equality

$$\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 = Y_1^2 - 2\sum_{j=1}^{n-1} Y_{t_j}\left(Y_{t_{j+1}} - Y_{t_j}\right), \tag{2.2}$$

which holds for all partitions $(t_0, t_1, \ldots, t_n) \in \tau$.

That

$$\left\{\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 | (t_0, t_1, \ldots, t_n) \in \tau\right\}$$

is unbounded in $L^0$ means that

$$c := \lim_{L\to\infty} \sup_{\tau} P\left[\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 > L\right] > 0. \tag{2.3}$$

We will deduce from this that

$$\lim_{L\to\infty} \sup_{\vartheta\in\beta(\mathbb{F}^X)} P[|I_X(\vartheta)| > L] \geqslant \frac{c}{4}, \tag{2.4}$$

which implies $L^0$-unboundedness of $I_Y(\beta(\mathbb{F}^X))$. To do this we choose $L > 0$. Since $Y$ is a.s. cadlag, $\sup_{t\in[0,1]} |Y_t| < \infty$ a.s. Therefore there exists an $N > 0$ such that

$$P\left[\sup_{t\in[0,1]} |Y_t| > N\right] < \frac{c}{4}. \tag{2.5}$$

It follows from (2.3) that there exists a partition $(t_0, t_1, \ldots, t_n) \in \tau$ with

$$P\left[\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 > 2LN + N^2\right] > \frac{c}{2}. \tag{2.6}$$

Inequalities (2.5) and (2.6) show that

$$P\left[\left\{\sup_{t\in[0,1]}|Y_t| > N\right\} \cup \left\{\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 \leqslant 2LN + N^2\right\}\right]$$

$$\leqslant P\left[\sup_{t\in[0,1]}|Y_t| > N\right] + P\left[\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 \leqslant 2LN + N^2\right] < 1 - \frac{c}{4}.$$

Hence,

$$P\left[\left\{\sup_{t\in[0,1]}|Y_t| \leqslant N\right\} \cap \left\{\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 > 2LN + N^2\right\}\right] > \frac{c}{4}. \tag{2.7}$$

It is clear that

$$\vartheta = \sum_{j=1}^{n-1} -1_{\{|Y_{t_j}|\leqslant N\}}\frac{Y_{t_j}}{N}1_{(t_j,t_{j+1}]}$$

is in $\beta(\mathbb{F}^X)$, and it can be seen from (2.2) that on the event

$$\left\{\sup_{t\in[0,1]}|Y_t| \leqslant N\right\} \cap \left\{\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 > 2LN + N^2\right\}$$

we have

$$I_Y(\vartheta) = \frac{1}{2N}\left(\sum_{j=0}^{n-1}(Y_{t_{j+1}} - Y_{t_j})^2 - Y_1^2\right)$$

$$> \frac{1}{2N}(2LN + N^2 - N^2) = L.$$

Together with (2.7), this implies that

$$P[I_Y(\vartheta) > L] > \frac{c}{4}.$$

Since $L$ was chosen arbitrarily, this shows (2.4) and the proposition is proved. □

**Corollary 2.3.** $(B_t^H)_{t\in[0,1]}$ *is not a weak semimartingale if* $H \in (0, \frac{1}{2})$.

***Proof.*** It follows from Lemma 2.1(c) that

$$\sum_{j=1}^{n}\left(\Delta_j^n B^H\right)^2 \overset{(n\to\infty)}{\longrightarrow} \infty \text{ in probability.}$$

This implies that

$$\left\{ \sum_{j=1}^{n} (\Delta_j^n B^H)^2 | n \in \mathbb{N} \right\}$$

is unbounded in $L^0$. Since $B^H$ is a.s. continuous, the corollary follows from Proposition 2.2. □

For $H \in (\frac{1}{2}, 1)$ a direct proof of the fact that $B^H$ is not a weak semimartingale appears to be difficult. Our roundabout method permits us to use already existing results on semimartingales.

**Lemma 2.4.** *Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$ be a filtration. Then every stochastically right-continuous $\mathbb{F}$-weak semimartingale $(X_t)_{t \in [0,1]}$ is also an $\overline{\mathbb{F}}$-weak semimartingale. In particular, if $X$ is a.s. right-continuous, it is an $\overline{\mathbb{F}}$-semimartingale.*

***Proof.*** Define $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t \in [0,1]}$ as follows. Let $\mathcal{F}_1^0$ be the completion of $\mathcal{F}_1$, $\mathcal{N}$ the null sets of $\mathcal{F}_1^0$ and set

$$\mathcal{F}_t^0 = \sigma(\mathcal{F}_t \cup \mathcal{N}), \qquad t \in [0, 1].$$

Let $t \in [0, 1]$ and $f \in L^0(\mathcal{F}_t^0)$ such that $|f| \leqslant 1$ a.s. We set

$$A = \{f > \mathrm{E}[f|\mathcal{F}_t]\}, \text{ and } B = \{f < \mathrm{E}[f|\mathcal{F}_t]\}.$$

Since

$$\mathcal{F}_t^0 = \{G \subset \Omega | \exists F \in \mathcal{F}_t \text{ such that } G \triangle F \in \mathcal{N}\},$$

there exist $\tilde{A}$, $\tilde{B} \in \mathcal{F}_t$ with $A \triangle \tilde{A}$, $B \triangle \tilde{B} \in \mathcal{N}$. The equalities

$$\int_A f - \mathrm{E}[f|\mathcal{F}_t] \mathrm{d}P = \int_{\tilde{A}} f - \mathrm{E}[f|\mathcal{F}_t] \mathrm{d}P = 0$$

and

$$\int_B f - \mathrm{E}[f|\mathcal{F}_t] \mathrm{d}P = \int_{\tilde{B}} f - \mathrm{E}[f|\mathcal{F}_t] \mathrm{d}P = 0$$

imply $P[A] = P[B] = 0$. Hence,

$$f = \mathrm{E}[f|\mathcal{F}_t] \text{ a.s.} \tag{2.8}$$

Let $(X_t)_{t \in [0,1]}$ be an $\mathbb{F}$-weak semimartingale. It follows from (2.8) that for every $\vartheta \in \beta(\mathbb{F}^0)$ there exists a $\tilde{\vartheta} \in \beta(\mathbb{F})$ with $I_X(\tilde{\vartheta}) = I_X(\vartheta)$ a.s. Therefore

$$I_X(\beta(\mathbb{F})) = I_X(\beta(\mathbb{F}^0)) \text{ in } L^0.$$

This shows that $X$ is also an $\mathbb{F}^0$-weak semimartingale.

Let

$$\psi = \sum_{j=0}^{n-1} f_j 1_{(t_j, t_{j+1}]} \in \beta(\overline{\mathbb{F}}).$$

For all $t \in [0, 1]$,

$$\bar{\mathcal{F}}_t = \bigcap_{s > t} \mathcal{F}^0_{s \wedge 1}.$$

Therefore,

$$\psi^\varepsilon = \sum_{j=0}^{n-1} f_j 1_{(t_j+\varepsilon, t_{j+1}]} \in \beta(\mathbb{F}^0), \tag{2.9}$$

for all $\varepsilon$ with $0 < \varepsilon < \min_j (t_{j+1} - t_j)$. If $(X_t)_{t\in[0,1]}$ is stochastically right-continuous, then

$$\lim_{\varepsilon \searrow 0} I_X(\psi^\varepsilon) = I_X(\psi) \text{ in probability.}$$

This, together with (2.9) and the fact that $I_X(\beta(\mathbb{F}^0))$ is bounded in $L^0$, implies that $I_X(\beta(\overline{\mathbb{F}}))$ also is bounded in $L^0$, and therefore $X$ is an $\overline{\mathbb{F}}$-weak semimartingale. □

**Proposition 2.5.** *Let* $(X_t)_{t\in[0,1]}$ *be an a.s. right-continuous process such that*

$$P[(X_t)_{t\in[0,1]} \text{ has finite variation}] < 1 \tag{2.10}$$

*and, for all* $\varepsilon > 0$, *there exists a partition*

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

*with*

$$\max_j (t_{j+1} - t_j) < \varepsilon \tag{2.11}$$

*and*

$$P\left[\sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2 > \varepsilon\right] < \varepsilon. \tag{2.12}$$

*Then* $X$ *is not a weak semimartingale.*

***Proof.*** Suppose $X$ is a weak semimartingale. By Lemma 2.4 $X$ is also an $\overline{\mathbb{F}}^X$-semimartingale. Hence, $X$ is of the form

$$X_t = X_0 + M_t + A_t,$$

where $X_0$ is an $\bar{\mathcal{F}}_0$-measurable random variable, $M_0 = A_0 = 0$, $M$ is an a.s. right-continuous local martingale with respect to $\overline{\mathbb{F}}$ and $A$ an a.s. right-continuous, $\overline{\mathbb{F}}$-adapted finite-variation process. It follows from (2.11), (2.12) and Theorem II.22 of Protter (1990) that

$$[X, X]_t = X_0, \qquad t \in [0, 1].$$

Hence,

$$[M, M]_t = 0, \qquad t \in [0, 1].$$

Therefore Theorem II.27 of Protter (1990) implies $M_t = 0$, $t \in [0, 1]$. Hence, $X$ is a finite-variation process. This contradicts (2.10). Therefore $X$ cannot be a weak semimartingale. □

**Corollary 2.6.** $\left(B_t^H\right)_{t\in[0,1]}$ *is not a weak semimartingale if* $H \in \left(\frac{1}{2}, 1\right)$.

***Proof.*** It follows from Lemma 2.1(c) that

$$\sum_{j=1}^{n} |\Delta_j^n B^H| \overset{(n\to\infty)}{\longrightarrow} \infty \text{ in probability.}$$

Therefore there exists a sequence $(n_k)_{k=0}^{\infty}$ such that

$$\sum_{j=1}^{n_k} |\Delta_j^{n_k} B^H| \overset{(k\to\infty)}{\longrightarrow} \infty \text{ a.s.}$$

Hence,

$$P\left[\left(B_t^H\right)_{t\in[0,1]} \text{ has finite variation}\right] = 0.$$

On the other hand, Lemma 2.1(b) shows that

$$\sum_{j=1}^{n} (\Delta_j^n B^H)^2 \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } \mathrm{L}^1.$$

Hence, $B^H$ satisfies the assumptions of Proposition 2.5. Therefore it is not a weak semimartingale. □

***Remark 2.7.*** Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ by

$$\mathcal{F}_t = \sigma\left((B_s)_{0\leqslant s\leqslant t}, \left(B_s^H\right)_{0\leqslant s\leqslant t}\right), \qquad t \in [0, 1].$$

Since $B$ is an $\mathbb{F}$-Brownian motion and therefore also an $\mathbb{F}$-weak semimartingale and $B^H$ is not an $\mathbb{F}$-weak semimartingale, $M^{H,\alpha} = B + \alpha B^H$ cannot be an $\mathbb{F}$-weak semimartingale. This does not imply that $M^{H,\alpha}$ is not a weak semimartingale. However, in the next section we show that for $H \in (0, \frac{1}{2})$, $M^{H,\alpha}$ has infinite quadratic variation. Therefore $M^{H,\alpha}$ cannot be a weak semimartingale by Proposition 2.2.

## 3. Proof of Theorem 1.7 for $H \in (0, \frac{1}{2})$

Like $B^H$, $M^{H,\alpha}$ cannot be a weak semimartingale for $H \in (0, \frac{1}{2})$ because it has infinite quadratic variation. To show this we write, for $n \in \mathbb{N}$,

$$\sum_{j=1}^{n}(\Delta_j^n M^{H,\alpha})^2 = \sum_{j=1}^{n}(\Delta_j^n B)^2 + 2\alpha \sum_{j=1}^{n} \Delta_j^n B\, \Delta_j^n B^H + \alpha^2 \sum_{j=1}^{n}(\Delta_j^n B^H)^2.$$

It is known that

$$\sum_{j=1}^{n}\left(\Delta_j^n B\right)^2 \overset{(n\to\infty)}{\longrightarrow} 1 \text{ in } L^2;$$

see, for example, Theorem I.28 of Protter (1990). From

$$\mathrm{E}\left[\left(\sum_{j=1}^{n} \Delta_j^n B\, \Delta_j^n B^H\right)^2\right] = \sum_{j,k=1}^{n} \mathrm{E}\left[\Delta_j^n B\, \Delta_j^n B^H\, \Delta_k^n B\, \Delta_k^n B^H\right]$$

$$= \sum_{j=1}^{n} \mathrm{E}\left[(\Delta_j^n B)^2\right]\mathrm{E}\left[\left(\Delta_j^n B^H\right)^2\right] = n\frac{1}{n}\left(\frac{1}{n}\right)^{2H},$$

it follows that

$$\sum_{j=1}^{n} \Delta_j^n B\, \Delta_j^n B^H \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } L^2.$$

On the other hand, it follows from Lemma 2.1(c) that

$$\sum_{j=1}^{n}\left(\Delta_j^n B^H\right)^2 \overset{(n\to\infty)}{\longrightarrow} \infty \text{ in probability.}$$

Hence,

$$\sum_{j=1}^{n}\left(\Delta_j^n M^{H,\alpha}\right)^2 \overset{(n\to\infty)}{\longrightarrow} \infty \text{ in probability.}$$

In particular,

$$\left\{\sum_{j=1}^{n}(\Delta_j^n M^{H,\alpha})^2 \,|\, n \in \mathbb{N}\right\}$$

is unbounded in $L^0$ and $M^{H,\alpha}$ is not a weak semimartingale by Proposition 2.2.

## 4. Proof of Theorem 1.7 for $H \in (\frac{1}{2}, \frac{3}{4}]$

For $H \in (\frac{1}{2}, \frac{3}{4}]$, the key in the proof of Theorem 1.7 is Lemma 4.2 below. It is based on Theorem 1 of Stricker (1984). Before we can formulate Lemma 4.2, we must specify our notion of a quasimartingale. As we did in the case of weak semimartingale, we call a stochastic process $X$ a quasimartingale if it is a quasimartingale with respect to $\mathcal{F}^X$.

***Definition 4.1.*** *A stochastic process* $(X_t)_{t\in[0,1]}$ *is a quasimartingale if*

$$X_t \in L^1 \qquad \text{for all } t \in [0, 1],$$

*and*

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathrm{E}\left[X_{t_{j+1}} - X_{t_j} | \mathcal{F}^X_{t_j}\right] \right\|_1 < \infty,$$

*where* $\tau$ *is the set of all finite partitions* $0 = t_0 < t_1 < \ldots < t_n = 1$ *of* $[0, 1]$.

**Lemma 4.2.** *If* $M^{H,\alpha}$ *is not a quasimartingale, it is not a weak semimartingale.*

***Proof.*** Let us assume that $M^{H,\alpha}$ is a weak semimartingale. Then Theorem 1 of Stricker (1984) implies that $I_{M^{H,\alpha}}(\beta(\mathbb{F}^{M^{H,\alpha}}))$ is bounded in $L^2$. Therefore it is also bounded in $L^1$. For any partition $0 = t_0 < t_1 \ldots < t_n = 1$,

$$\sum_{j=0}^{n-1} \mathrm{sgn}\Big(\mathrm{E}\Big[M^{H,\alpha}_{t_{j+1}} - M^{H,\alpha}_{t_j} | \mathcal{F}_{t_j}\Big]\Big) 1_{(t_j, t_{j+1}]} \in \beta\Big(\mathbb{F}^{M^{H,\alpha}}\Big),$$

and

$$\begin{aligned} &\left\| I_{M^{H,\alpha}}\left(\sum_{j=0}^{n-1} \mathrm{sgn}\left(\mathrm{E}\left[M^{H,\alpha}_{t_{j+1}} - M^{H,\alpha}_{t_j} \middle| \mathcal{F}^{M^{H,\alpha}}_{t_j}\right]\right) 1_{(t_j,t_{j+1}]}\right) \right\|_1 \\ &\geqslant \mathrm{E}\left[ I_{M^{H,\alpha}}\left(\sum_{j=0}^{n-1} \mathrm{sgn}\left(\mathrm{E}\left[M^{H,\alpha}_{t_{j+1}} - M^{H,\alpha}_{t_j} \middle| \mathcal{F}^{M^{H,\alpha}}_{t_j}\right]\right) 1_{(t_j,t_{j+1}]}\right) \right] \\ &= \sum_{j=0}^{n-1} \left\| \mathrm{E}\left[M^{H,\alpha}_{t_{j+1}} - M^{H,\alpha}_{t_j} \middle| \mathcal{F}^{M^{H,\alpha}}_{t_j}\right] \right\|_1 . \end{aligned}$$

It follows that $M^{H,\alpha}$ is a quasimartingale. Hence, if $M^{H,\alpha}$ is not a quasimartingale, it cannot be a weak semimartingale. □

It remains to prove that $M^{H,\alpha}$ is not a quasimartingale if $H \in (\frac{1}{2}, \frac{3}{4}]$. We do this in the next two lemmas.

**Lemma 4.3.** *If* $H \in (\frac{1}{2}, \frac{3}{4})$, $M^{H,\alpha}$ *is not a quasimartingale.*

***Proof.*** Since conditional expectation is a contraction with respect to the $L^1$-norm, we have, for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n-1$,

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} \middle| \mathcal{F}^{M^{H,\alpha}}_{j/n}\right] \right\|_1 \geqslant \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha}\right] \right\|_1 . \tag{4.1}$$

Moreover,

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} \middle| \Delta^n_j M^{H,\alpha}\right] \right\|_1 = \sqrt{\frac{2}{\pi}} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} \middle| \Delta^n_j M^{H,\alpha}\right] \right\|_2 \tag{4.2}$$

because $\mathrm{E}[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha}]$ is a centred Gaussian random variable. Using (4.1) and (4.2), we obtain

$$\begin{aligned}
\sum_{j=0}^{n-1} \left\| \mathrm{E}\left[ \Delta^n_{j+1} M^{H,\alpha} \middle| \mathcal{F}^{M^{H,\alpha}}_{j/n} \right] \right\|_1 &\geqslant \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \mathrm{E}\left[ \Delta^n_{j+1} M^{H,\alpha} \middle| \Delta^n_j M^{H,\alpha} \right] \right\|_2 \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \frac{\operatorname{cov}(\Delta^n_{j+1} M^{H,\alpha}, \Delta^n_j M^{H,\alpha})}{\operatorname{cov}(\Delta^n_j M^{H,\alpha}, \Delta^n_j M^{H,\alpha})} \Delta^n_j M^{H,\alpha} \right\|_2 \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \frac{\operatorname{cov}\left(\Delta^n_{j+1} M^{H,\alpha}, \Delta^n_j M^{H,\alpha}\right)}{\sqrt{\operatorname{cov}\left(\Delta^n_j M^{H,\alpha}, \Delta^n_j M^{H,\alpha}\right)}} \\
&= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \frac{\alpha^2 n^{-2H} \left( \frac{2^{2H}}{2} - 1 \right)}{\sqrt{\frac{1}{n} + \alpha^2 n^{-2H}}} \\
&\geqslant \sqrt{\frac{2}{\pi}} \alpha^2 \left( \frac{2^{2H}}{2} - 1 \right) \sum_{j=1}^{n-1} n^{-2H} \Big/ \sqrt{1/n + \alpha^2 \frac{1}{n}} \\
&= \sqrt{\frac{2}{\pi}} \left( \frac{2^{2H}}{2} - 1 \right) \frac{\alpha^2}{\sqrt{1+\alpha^2}} \sum_{j=1}^{n-1} n^{1/2-2H} \\
&= \sqrt{\frac{2}{\pi}} \left( \frac{2^{2H}}{2} - 1 \right) \frac{\alpha^2}{\sqrt{1+\alpha^2}} (n-1) n^{1/2-2H} \to \infty, \qquad \text{as } n \to \infty.
\end{aligned}$$

This proves the lemma. □

**Lemma 4.4.** $M^{3/4,\alpha}$ *is not a quasimartingale.*

***Proof.*** In this case the estimate (4.1) is not good enough. Now we need that, for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n-1$,

$$\left\| \mathrm{E}\left[ \Delta^n_{j+1} M^{3/4,\alpha} | \mathcal{F}^{M^{3/4,\alpha}}_{j/n} \right] \right\|_1 \geqslant \left\| \mathrm{E}\left[ \Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha} \right] \right\|_1,$$

which follows, like (4.1) from the fact that conditional expectation is a contraction with respect to the $L^1$-norm. Since

$$\mathrm{E}\left[ \Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha} \right]$$

is centred Gaussian,

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] \right\|_1 = \sqrt{\frac{2}{\pi}} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] \right\|_2 .$$

Hence,

$$\sum_{j=0}^{n-1} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \mathcal{F}^{M^{3/4,\alpha}}_{j/n}\right] \right\|_1 \geqslant \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] \right\|_2$$

and the lemma is proved if we can show that

$$\sum_{j=1}^{n-1} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] \right\|_2 \to \infty, \qquad \text{as } n \to \infty. \tag{4.3}$$

For $n \in \mathbb{N}$ and $j \in \{1, \ldots, n-1\}$,

$$\left(\Delta^n_{j+1} M^{3/4,\alpha}, \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right)$$

is a Gaussian vector. Therefore,

$$\mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] = \sum_{k=1}^{j} b_k \Delta^n_k M^{3/4,\alpha}, \tag{4.4}$$

where the vector $b = (b_1, \ldots, b_j)^{\mathrm{T}}$ solves the system of linear equations

$$m = Ab, \tag{4.5}$$

in which $m$ is a $j$-vector whose $k$th component $m_k$ is

$$\mathrm{cov}\left(\Delta^n_{j+1} M^{3/4,\alpha}, \Delta^n_k M^{3/4,\alpha}\right)$$

and $A$ is the covariance matrix of the Gaussian vector

$$\left(\Delta^n_1 M^{3/4,\alpha}, \ldots, \Delta^n_j M^{3/4,\alpha}\right).$$

Note that $A$ is symmetric and, since the random variables $\Delta^n_1 M^{3/4,\alpha}, \ldots, \Delta^n_j M^{3/4,\alpha}$ are linearly independent, also positive definite. It follows from (4.4) and (4.5) that

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{3/4,\alpha} | \Delta^n_j M^{3/4,\alpha}, \ldots, \Delta^n_1 M^{3/4,\alpha}\right] \right\|_2^2 = b^{\mathrm{T}} A b = m^{\mathrm{T}} A^{-1} m \geqslant \left\| m \right\|_2^2 \lambda^{-1}, \tag{4.6}$$

where $\lambda$ is the largest eigenvalue of the matrix $A$. Since

$$A = \frac{1}{n} \mathrm{id} + \alpha^2 C,$$

where $C$ is the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta^n_1 B^{3/4}, \ldots, \Delta^n_j B^{3/4}\right),$$

we have

$$\lambda = \frac{1}{n} + \alpha^2 \mu,$$

where $\mu$ is the largest eigenvalue of $C$. As

$$C_{kl} = n^{-3/2}\tfrac{1}{2}\Big((|k-l|+1)^{3/2} - 2|k-l|^{3/2} + \||k-l|-1|^{3/2}\Big), \qquad k,\ l = 1, \ldots, j,$$

it follows from the Gershgorin circle theorem (see Golub and Van Loan 1989) and the special form of $C$ that

$$\mu \leqslant \max_{k=1,\ldots,j} \sum_{l=1}^{j} |C_{kl}| \leqslant 2\sum_{l=1}^{j} |C_{1l}|$$
$$= 2n^{-3/2}\tfrac{1}{2}\sum_{l=0}^{j-1}\Big((l+1)^{3/2} - 2l^{3/2} + |l-1|^{3/2}\Big) = n^{-3/2}\Big(1 + j^{3/2} - (j-1)^{3/2}\Big).$$

Furthermore,

$$n^{-3/2}\Big(1 + j^{3/2} - (j-1)^{3/2}\Big) \leqslant \frac{1}{n} + n^{-3/2}\frac{\partial}{\partial j} j^{3/2}$$
$$= \frac{1}{n} + n^{-\frac{3}{2}}\frac{3}{2} j^{1/2} \leqslant \frac{1}{n} + n^{-\frac{3}{2}}\frac{3}{2} n^{1/2} \leqslant 3\frac{1}{n}.$$

Hence,

$$\lambda \leqslant \frac{1}{n} + \alpha^2 3\frac{1}{n} = \left(1 + 3\alpha^2\right)\frac{1}{n}$$

and

$$\lambda^{-1} \geqslant \frac{n}{1+3\alpha^2}. \tag{4.7}$$

On the other hand,

$$\|m\|_2^2 = \sum_{k=1}^{j}\Big(\mathrm{cov}\Big(\Delta^n_{j+1} M^{3/4,\alpha},\ \Delta^n_k M^{3/4,\alpha}\Big)\Big)^2 = \alpha^4 \sum_{k=1}^{j}\Big(\mathrm{cov}\Big(\Delta^n_{j+1} B^{3/4}\Delta^n_k B^{3/4}\Big)\Big)^2$$
$$= \alpha^4\tfrac{1}{4}n^{-3}\sum_{k=1}^{j}((k+1)^{3/2} - 2k^{3/2} + (k-1)^{3/2})^2.$$

Since the function $x \mapsto x^{3/2}$ is analytic on the set $\{x \in \mathbb{C} \mid \mathrm{Re}\, x > 0\}$,

$$(k+1)^{3/2} - 2k^{3/2} + (k-1)^{3/2} = \sum_{m=1}^{\infty}\left(\frac{1}{m!}\frac{\partial^m}{\partial k^m} k^{3/2} + (-1)^m \frac{1}{m!}\frac{\partial^m}{\partial k^m} k^{3/2}\right)$$
$$\geqslant \frac{\partial^2}{\partial k^2} k^{3/2} = \frac{3}{4} k^{-1/2}, \qquad k = 2, \ldots j.$$

That

$$(k+1)^{\frac{3}{2}} \ - \ 2k^{\frac{3}{2}} \ + \ (k-1)^{\frac{3}{2}} \geqslant \frac{3}{4} k^{-\frac{1}{2}}$$

also holds for $k = 1$, can be checked directly. It follows that

$$\|m\|_2^2 \geqslant \alpha^4 \frac{1}{4} n^{-3} \frac{9}{16}\sum_{k=1}^{j}\frac{1}{k} \geqslant \alpha^4 \frac{9}{64} n^{-3}\int_1^j \frac{1}{x}\mathrm{d}x = \alpha^4 \frac{9}{64} n^{-3}\log j. \tag{4.8}$$

Putting (4.6), (4.7) and (4.8) together, we obtain

$$\sum_{j=1}^{n-1}\left\|\mathrm{E}\left[\Delta_{j+1}^{n}M^{3/4,\alpha}|\Delta_{j}^{n}M^{3/4,\alpha},\ldots,\Delta_{1}^{n}M^{3/4,\alpha}\right]\right\|_{2}\geqslant\frac{3}{8}\frac{\alpha^{2}}{\sqrt{1+3\alpha^{2}}}\frac{1}{n}\sum_{j=1}^{n-1}\sqrt{\log j}\to\infty$$

as $n\to\infty$.

Hence, (4.3) holds and the lemma is proved. □

# 5. Proof of Theorem 1.7 for $H\in(\frac{3}{4},1]$

To show that, for $H\in(\frac{3}{4},1]$, $M^{H,\alpha}$ is equivalent to Brownian motion, we use the concept of relative entropy. The following definition and all results on relative entropy that we need in this section can be found in Chapter 6 of Hida and Hitsuda (1976).

***Definition 5.1.*** *Let $Q_1$ and $Q_2$ be probability measures on a measurable space $(\Omega,\mathcal{E})$ and denote by $\mathcal{P}$ all finite partitions,*

$$\Omega=\bigcup_{j=1}^{n}E_j,$$

*of $\Omega$, where $E_j\in\mathcal{E}$ and $E_j\cap E_k=\varnothing$ if $j\neq k$. The entropy of $Q_1$ relative to $Q_2$ is given by*

$$H(Q_1|Q_2):=\sup_{\mathcal{P}}\sum_{j=1}^{n}\log\left(\frac{Q_1[E_j]}{Q_2[E_j]}\right)Q_1[E_j],$$

*where we assume $\frac{0}{0}=0\log 0=0$.*

For all $n\in\mathbb{N}$, we define $Y_n:C[0,1]\to\mathbb{R}^n$ by

$$Y_n(\omega)=\left(\omega\left(\frac{1}{n}\right)-\omega(0),\,\omega\left(\frac{2}{n}\right)-\omega\left(\frac{1}{n}\right),\ldots,\omega(1)-\omega\left(\frac{n-1}{n}\right)\right)^{\mathrm{T}},$$

and $\mathcal{B}_n=\sigma(Y_n)$. Note that $\bigvee_{n=1}^{\infty}\mathcal{B}_n$ is equal to the $\sigma$-algebra $\mathcal{B}$ generated by the cylinder sets. We denote by $Q_{M^{H,\alpha}}$ the measure induced by $M^{H,\alpha}$ on $(C[0,1],\mathcal{B})$ and by $Q_W$ Wiener measure on $(C[0,1],\mathcal{B})$. Further, we let, for all $n\in\mathbb{N}$, $Q^n_{M^{H,\alpha}}$ and $Q^n_W$ be the restrictions of $Q_{M^{H,\alpha}}$ and $Q_W$, respectively, to $\mathcal{B}_n$.

To show that $M^{H,\alpha}$ is equivalent to Brownian motion, we make use of the following lemma.

**Lemma 5.2.** *If*

$$\sup_n H(Q^n_{M^{H,\alpha}}|Q^n_W)<\infty, \tag{5.1}$$

*then $Q_{M^{H,\alpha}}$ and $Q_W$ are equivalent.*

***Proof.*** From (5.1) it follows by Lemma 6.3 of Hida and Hitsuda (1976) that $Q_{M^{H,\alpha}}$ is absolutely continuous with respect to $Q_W$. But two Gaussian measures on $(C[0, 1], \mathcal{B})$ can only be equivalent or singular – see, for example, Theorem 6.1 of Hida and Hitsuda (1976). Therefore $Q_{M^{\alpha,H}}$ and $Q_W$ must be equivalent. □

In the following lemma we show that (5.1) holds.

**Lemma 5.3.**

$$\sup_n H(Q^n_{M^{H,\alpha}}|Q^n_W) < \infty.$$

***Proof.*** For all $n \in \mathbb{N}$, $Y_n$ is a centred Gaussian vector under both measures $Q^n_{M^{H,\alpha}}$ and $Q^n_W$. The covariance matrices of $Y_n$ under $Q^n_{M^{H,\alpha}}$ and $Q^n_W$ are

$$\mathrm{E}_{Q^n_{M^{H,\alpha}}}\left[Y_n Y_n^{\mathrm{T}}\right] = \frac{1}{n}\mathrm{id} + \alpha^2 C_n,$$

where $C_n$ is the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta_1^n B^H, \ldots, \Delta_n^n B^H\right)$$

and

$$\mathrm{E}_{Q^n_W}\left[Y_n Y_n^{\mathrm{T}}\right] = \frac{1}{n}\mathrm{id}.$$

Since $C_n$ is symmetric, there exists an orthogonal $n \times n$ matrix $U_n$ such that $U_n C_n U_n^{\mathrm{T}}$ is a diagonal matrix $D_n = \mathrm{diag}\left(\lambda_1^n, \ldots \lambda_n^n\right)$. $X_n = \sqrt{n} U_n Y_n$ is still a centred Gaussian vector under both measures $Q^n_{M^{H,\alpha}}$ and $Q^n_W$. The covariance matrices of $X_n$ under these two measures are

$$\mathrm{E}_{Q^n_{M^{H,\alpha}}}\left[X_n X_n^{\mathrm{T}}\right] = \mathrm{id} + n\alpha^2 D_n$$

and

$$\mathrm{E}_{Q^n_W}\left[X_n X_n^{\mathrm{T}}\right] = \mathrm{id}.$$

Through $X_n$, $Q^n_{M^{H,\alpha}}$ and $Q^n_W$ induce measures $R^n_{M^{H,\alpha}}$ and $R^n_W$ on $\mathbb{R}^n$. It can easily be seen from Definition 5.1 that

$$H\left(Q^n_{M^{H,\alpha}}|Q^n_W\right) = H\left(R^n_{M^{H,\alpha}}|R^n_W\right).$$

Since both measures $R^n_{M^{H,\alpha}}$ and $R^n_W$ are non-degenerate Gaussian measures on $\mathbb{R}^n$, they are equivalent. We denote by $\varphi_n$ the Radon–Nikodym derivative of $R^n_{M^{H,\alpha}}$ with respect to $R^n_W$. Using Lemma 6.1 of Hida and Hitsuda (1976), one finds that

$$H\left(R^n_{M^{H,\alpha}}|R^n_W\right) = \mathrm{E}_{R^n_{M^{H,\alpha}}}[\log \varphi_n] = \tfrac{1}{2}\sum_{j=1}^{n}\left(n\alpha^2\lambda_j^n - \log\left(1 + n\alpha^2\lambda_j^n\right)\right).$$

For all $x \geqslant 0$, we have

$$x - \log(1+x) = \int_0^x \frac{u}{1+u}\,\mathrm{d}u \leqslant \int_0^x u\,\mathrm{d}u = \tfrac{1}{2}x^2.$$

Therefore,

$$H\left(R^n_{M^{H,\alpha}} | R^n_W\right) \leqslant \tfrac{1}{4} n^2 \alpha^4 \sum_{j=1}^{n} \left(\lambda_j^n\right)^2.$$

Hence, the lemma is proved if we can show that

$$\sup_n n^2 \sum_{j=1}^{n} \left(\lambda_j^n\right)^2 < \infty, \tag{5.2}$$

where $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta_1^n B^H, \ldots, \Delta_n^n B^H\right).$$

Since orthogonal transformation leaves the Hilbert–Schmidt norm of a matrix invariant,

$$\sum_{j=1}^{n} (\lambda_j^n)^2 = \sum_{j,k=1}^{n} \operatorname{cov}\left(\Delta_j^n B^H, \Delta_k^n B^H\right)^2.$$

As fractional Brownian motion has stationary increments,

$$\begin{aligned}\sum_{j,k=1}^{n} \operatorname{cov}\left(\Delta_j^n B^H, \Delta_k^n B^H\right)^2 &\leqslant 2n \sum_{k=1}^{n} \operatorname{cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 \\ &= 2nn^{-4H}\left(1 + \left(\frac{2^{2H}}{2} - 1\right)^2\right) + 2n \sum_{k=3}^{n} \operatorname{cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2.\end{aligned}$$

Since, for $H \in (\frac{3}{4}, 1]$,

$$n^2 2nn^{-4H}\left(1 + \left(\frac{2^{2H}}{2} - 1\right)^2\right) \to 0, \qquad \text{as } n \to \infty,$$

it is enough to show

$$\sup_n n^3 \sum_{k=3}^{n} \operatorname{cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 < \infty \tag{5.3}$$

to prove (5.2). For all $k \geqslant 3$, we have

$$\begin{aligned}\operatorname{cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right) &= n^{-2H} \tfrac{1}{2}\left(k^{2H} - 2(k-1)^{2H} + (k-2)^{2H}\right) \\ &\leqslant n^{-2H} \tfrac{1}{2}\left(\frac{\partial}{\partial k} k^{2H} - \frac{\partial}{\partial k}(k-2)^{2H}\right) = Hn^{-2H}\left(k^{2H-1} - (k-2)^{2H-1}\right) \\ &\leqslant Hn^{-2H} 2 \frac{\partial}{\partial k}(k-2)^{2H-1} = 2H(2H-1)n^{-2H}(k-2)^{2H-2}.\end{aligned}$$

Using this, we obtain

$$
\begin{aligned}
n^3 \sum_{k=3}^{n} \operatorname{cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 &\leqslant 4H^2(2H-1)^2 n^{3-4H} \sum_{k=1}^{n-2} k^{4H-4} \\
&\leqslant 4H^2(2H-1)^2 n^{3-4H} \int_0^{n-2} x^{4H-4}\, \mathrm{d}x \\
&= \frac{4H^2(2H-1)^2}{4H-3} n^{3-4H}(n-2)^{4H-3} \\
&\leqslant \frac{4H^2(2H-1)^2}{4H-3}.
\end{aligned}
$$

Hence, (5.3) holds and the lemma is proved. □

***Remark 5.4.*** In this section we have shown that, for $H \in (\frac{3}{4}, 1]$, $Q_{M^{H,\alpha}}$ and $Q_W$ are equivalent. But our method of proof has not given us the Radon–Nikodym derivative, nor have we found the semimartingale decomposition of $M^{H,\alpha}$. These problems will be addressed in future work.

## 6. Mixed fractional Brownian motion and option pricing

Theorem 1.7 enables us to present an example that calls into question a current practice in mathematical finance.

Let us consider a market that consists of a bank account and a stock that pays no dividends. There are no transaction costs. Borrowing and short-selling are allowed. The borrowing and the lending rate are both equal to a constant $r$ and the discounted stock price follows a stochastic process $(S_t)_{t \in [0,1]}$.

We are interested in the time-zero price $C_0$ of a European call option on $S$ with strike price $K$ and maturity $T = 1$. Its discounted pay-off is $(S_1 - \mathrm{e}^{-r}K)^+$. To exclude trivial arbitrage strategies, $C_0$ must be in the interval

$$((S_0 - \mathrm{e}^{-r}K)^+, S_0).$$

Samuelson (1965) proposed modelling the discounted stock price as follows:

$$S_t = S_0 \exp(\nu t + \sigma B_t), \qquad t \in [0, 1],$$

where $\nu$, $\sigma$ are constants and $B$ is a Brownian motion. In this model Black and Scholes (1973) derived an explicit formula for $C_0$. For given $S_0$, $r$, $K$ and maturity $T = 1$, the Black–Scholes price BS of a European call option depends only on the volatility $\sigma$ and not on $\nu$. As a function of $\sigma$, BS is continuous, increasing and bijective from $(0, \infty)$ to $((S_0 - K\mathrm{e}^{-r})^+, S_0)$.

The Samuelson model has several deficiencies and up to now there have been many efforts to build better models, including several attempts to remedy some shortcomings of the Samuelson model with the help of fractional Brownian motion (for a discussion, see Cutland *et al*., 1995).

For our example let us assume that empirical data suggest that the discounted price of the stock should be modelled as

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right), \qquad t \in [0, 1], \tag{6.1}$$

for constants $\nu$, $\sigma$ and a fractional Brownian motion $B^H$. In Cheridito (2000) it is shown that for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ such a model admits arbitrage. However, if $H \in \left(\frac{3}{4}, 1\right)$, we can exclude all arbitrage strategies by regularizing fractional Brownian motion in the following way:

If $(B_t)_{t\in[0,1]}$ is a Brownian motion independent of $B^H$, Theorem 1.7 implies that, for all $\varepsilon > 0$,

$$\left(\varepsilon B_t + B_t^H\right)_{t\in[0,1]} \text{ is equivalent to } (\varepsilon B_t)_{t\in[0,1]}.$$

We observe that

$$\operatorname{cov}\left(\varepsilon B_t + B_t^H, \varepsilon B_s + B_s^H\right) = \varepsilon^2 (t \wedge s) + \operatorname{cov}\left(B_t^H, B_s^H\right), \qquad t, s \in [0, 1].$$

Hence, $\left(\varepsilon B_t + B_t^H\right)_{t\in[0,1]}$ is an a.s. continuous centred Gaussian process that has, up to $\varepsilon^2$, the same covariance structure as $\left(B_t^H\right)_{t\in[0,1]}$. This shows that if the model (6.1) fits empirical data, then so does

$$S_t = S_0 \exp\left\{\nu t + \sigma\left(\varepsilon B_t + B_t^H\right)\right\}, \qquad t \in [0, 1], \tag{6.2}$$

for $\varepsilon > 0$ small enough. But in contrast to (6.1), and like the Samuelson model, (6.2) has a unique equivalent martingale measure $Q^\varepsilon$. This implies that the model (6.2) is arbitrage-free and complete. According to current practice in mathematical finance, in such a framework options are priced by taking the expected value under the equivalent martingale measure of the option's discounted pay-off. In the model (6.2) this leads to the following option price:

$$C_0(\varepsilon) = \mathrm{E}_{Q^\varepsilon}\left[\left(S_0 \exp\left\{\nu + \sigma\left(\varepsilon B_1 + B_1^H\right)\right\} - \mathrm{e}^{-r} K\right)^+\right] = \mathrm{BS}(\varepsilon\sigma). \tag{6.3}$$

By the above-mentioned properties of the function BS, $C_0(\varepsilon)$ in (6.3) is close to $(S_0 - \mathrm{e}^{-r}K)^+$ when $\varepsilon > 0$ is small. The deeper reason why $C_0(\varepsilon)$ is so low in this situation is that (6.3) gives the initial capital necessary to replicate the pay-off of the call option with a predictable trading strategy satisfying certain admissibility conditions, and this strategy seems to exploit small movements of the stochastic process (6.2) over very short time intervals.

In reality a seller of the option can only carry out finitely many transactions to hedge the option. Moreover, he cannot buy and sell within nanoseconds. Therefore he will demand a higher price than $\mathrm{BS}(\varepsilon\sigma) \approx (S_0 - \mathrm{e}^{-r}K)^+$.

To find a reasonable option price, one should introduce a waiting time $h > 0$ and restrict trading strategies to the class $\Theta^h\left(\mathbb{F}^S\right)$ of strategies that can buy and sell at $\mathbb{F}^S$-stopping times, but after each transaction there must be a waiting period of minimal length $h$ before the next. For small $\varepsilon > 0$, the discounted gain process of such a strategy is similar in both models (6.1) and (6.2), as should be the case. Moreover, it is shown in Cheridito (2000) that the model (6.1) has no arbitrage in $\Theta^h\left(\mathbb{F}^S\right)$. Hence, if we confine the strategies to the class $\Theta^h(\mathbb{F}^S)$, we can return to the model (6.1) to value the option. Since (6.1) with the strategies $\Theta^h(\mathbb{F}^S)$ is an incomplete model, one has to decide in which sense the pay-off of the option should be approximated and then search for an optimal strategy. It is not clear whether the regularization (6.2) is of any use in such a procedure.

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