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Prophet inequalities for optimal stopping rules with probabilistic recall

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Let X_i, $i = 1, \ldots, n$, be independent random variables, and consider an optimal stopping problem where an observation not chosen in the past is still available i steps later with some probability p_i, $1 \geqslant p_1 \geqslant \ldots \geqslant p_{n-1} \geqslant 0$. Only one object may be chosen. After formulating the general solution to this optimal stopping problem, we consider 'prophet inequalities' for this situation. Let $V_{\mathbf{p}}(X_1, \ldots, X_n)$ be the optimal value to the statistician. We show that for all non-trivial, non-negative X_i and all $n \geqslant 2$, the 'ratio prophet inequality' $\mathrm{E}[\max(X_1, \ldots, X_n)] < (2 - p_{n-1})V_{\mathbf{p}}(X_1, \ldots, X_n)$ holds, and $2 - p_{n-1}$ is the best constant. This generalizes the classical prophet inequality with no recall, in which the best constant is 2. For any $0 \leqslant X_i \leqslant 1$, the 'difference prophet inequality' $\mathrm{E}[\max(X_1, \ldots, X_n)] - V_{\mathbf{p}}(X_1, \ldots, X_n) \leqslant (1 - p_{n-1})[1 - (1 - p_{n-1})^{1/2}]^2/p_{n-1}^2$ holds. Prophet regions are also discussed.

Keywords: backward solicitation; optimal stopping; probabilistic recall; prophet inequalities; prophet region; recall

1. Introduction and summary

Let $X_1, X_2, \ldots$ be a finite or infinite sequence of random variables with finite expectations, which are observed sequentially. Classical optimal stopping theory deals with the existence and structure of an optimal stopping rule t, and its value, $\mathrm{E}X_t$. The 'stopping rule' refers to an integer-valued random variable such that the event $\{t = i\}$ may depend on the already observed random variables $X_1, \ldots, X_i$, and possibly on external variables, but not on the future. When the sequence is infinite the requirement is that $P\{t < \infty\} = 1$, and when the sequence is of length n the requirement is $P\{t \leqslant n\} = 1$. In the latter case an optimal rule always exists, and can be obtained by backward induction (dynamic programming). Chow *et al*. (1971) cover the general theory of optimal stopping.

In the present paper we consider the situation where the optimal stopper has a chance to 'recall' a previously observed X. If the variable X was observed j time units ago, the probability that it is still available is p_j. Hence the term 'probabilistic recall'. The term 'backward solicitation' is also used in the literature, especially in papers where 'recall' is a synonym for 'memory'. We make the realistic assumption that $1 \geqslant p_1 \geqslant \ldots \geqslant p_{n-1} \geqslant 0$. 'Geometric recall' is the special case where $p_i = p^i$, $i = 1, 2, \ldots,$ for some value p,

$0 < p < 1$. Probabilistic recall has been discussed in the literature for specific structures or specific problems. Karni and Schwartz (1977) assume that there is a cost attached to the observation, Ikuta (1988) assumes cost and discounting, and Petrucelli (1981; 1982) considers the 'secretary problem'. Saito (1998) gives a review of the literature. He assumes independent and identically distributed (i.i.d.) observations, each of which carries an associated cost. We are not aware of any work dealing with a completely general model for recall, corresponding to Chow *et al*. (1971).

The model of optimal stopping with probabilistic recall is often more realistic than that with no possibility of recall. Consider, for example, the situation where the X_i are the scores of applicants for a job. An applicant who did not seem sufficiently attractive when first interviewed may later be deemed quite desirable, other candidates having been observed. The applicant may still be available with some probability, typically decreasing as time goes by. A similar situation arises when the X_i are the scores given to prospective houses when a house purchase is being considered.

In Section 2 we describe several models for probabilistic recall, and give a recursion formula for obtaining the optimal rule and its value for one of the models.

Our main interest in the present paper is in deriving 'prophet inequalities' for probabilistic recall in the case where the X_i are independent, non-negative and not all identically zero. In this case, with no recall, the classical ratio prophet inequality states that, for all $n \geqslant 2$,

$$\mathrm{E}\left(\max_{i=1,\dots,n} X_i\right) =: \mathrm{E}[X_1 \vee \ldots \vee X_n] < 2V(X_1, \ldots, X_n); \tag{1.1}$$

see, for example, Krengel and Sucheston (1978) or Hill and Kertz (1981a). Here $V(X_1, \ldots, X_n) = \sup_t \mathrm{E}X_t$. The term 'prophet inequality' stems from the fact that $\mathrm{E}[X_1 \vee \ldots \vee X_n]$ can be thought of as the value to a 'prophet' who can foresee all the X_i and thus pick the largest. (In the present terminology he could also be thought of as having perfect recall.) $V(X_1, \ldots, X_n)$ is the value to the statistician (or mortal) who acts optimally. Inequality (1.1) holds also in the infinite case, and 2 is the best (smallest) constant c for which $\mathrm{E}[X_1 \vee \ldots \vee X_n] \leqslant cV(X_1, \ldots, X_n)$ holds for all sequences of non-negative independent X_i.

In the present paper we generalize (1.1) to the case in which recall is allowed. Since the option of recall improves the situation of the statistician, bringing him 'closer' to the prophet, the best constant 2 of (1.1) may in fact be replaced by a smaller constant. Let $\mathbf{p} = (p_1, \ldots, p_{n-1})$, where $1 \geqslant p_1 \geqslant \ldots \geqslant p_{n-1} \geqslant 0$, and let $V_{\mathbf{p}}(X_1, \ldots, X_n)$ be the optimal value to the statistician with probabilistic recall, which is described in detail in Section 2. We prove the following:

Theorem 1.1. *Let $n \geqslant 2$, and let X_i, $i = 1, \ldots, n$, be independent, non-negative random variables with finite expectations, not all identically zero. If $p_{n-1} < 1$, then*

$$\mathrm{E}[X_1 \vee \ldots \vee X_n] < (2 - p_{n-1})V_{\mathbf{p}}(X_1, \ldots, X_n), \tag{1.2}$$

and $2 - p_{n-1}$ is the best constant. If $p_{n-1} = 1$ the inequality in (1.2) *is replaced by equality.*

For an infinite sequence $X_1, X_2, \ldots$ *as above, with* $\lim_{n\to\infty} p_n = p_\infty \geqslant 0$ *and* $\mathrm{E}[\sup X_i] < \infty$,

$$\mathrm{E}\left(\sup_{i\geqslant 1} X_i\right) < (2 - p_\infty)V_{\mathbf{p}}(X_1, X_2, \ldots). \tag{1.3}$$

The corresponding classical 'difference prophet inequality' for bounded independent random variables such that $a \leqslant X_i \leqslant b$ states that

$$\mathrm{E}[X_1 \vee \ldots \vee X_n] - V(X_1, \ldots, X_n) \leqslant \frac{b-a}{4}; \tag{1.4}$$

see Hill and Kertz (1981b). Here we prove the following:

Theorem 1.2. *Let* $n \geqslant 2$, *and let* X_i *be independent,* $a \leqslant X_i \leqslant b$, $i = 1, \ldots, n$. *Then*

$$\mathrm{E}[X_1 \vee \ldots \vee X_n] - V_{\mathbf{p}}(X_1, \ldots, X_n) \leqslant \left\{[1 - (1 - p_{n-1})^{1/2}]^2 \frac{1 - p_{n-1}}{p_{n-1}^2}\right\}(b - a), \tag{1.5}$$

and a corresponding statement is true in the infinite case where p_{n-1} *is replaced by* $p_\infty = \lim p_n$. *In both cases it is a best bound.*

Note that when $p_{n-1} \to 0$ the limiting value of the right-hand side of (1.5) is $(b-a)/4$.

The above theorems are obtained by first considering in Section 3 the 'prophet region' for the case of probabilistic recall. A prophet region S is the collection of points (x, y) such that $x = V(X_1, \ldots, X_n)$ and $y = \mathrm{E}[X_1 \vee \ldots \vee X_n]$, for some $X_1, \ldots, X_n$. For the independent case with $0 \leqslant X_i \leqslant 1$, no recall and any $n \geqslant 2$, Hill (1983) shows that

$$S = \{(x, y) : 0 \leqslant x \leqslant 1, x \leqslant y \leqslant 2x - x^2\}. \tag{1.6}$$

Inequalities (1.1) and (1.4) are easily obtained from (1.6), though chronologically the results were not obtained in this order.

For the present case, we prove in Theorem 3.1 that, for independent X_i with $0 \leqslant X_i \leqslant 1$ and any n where $x = V_{\mathbf{p}}(X_1, \ldots, X_n)$, the prophet region is given by

$$\left\{(x, y); 0 \leqslant x \leqslant 1, x \leqslant y \leqslant \frac{x(2 - p_{n-1} - x)}{1 - p_{n-1}x}\right\}. \tag{1.7}$$

Theorems 1.1 and 1.2 are proved in Section 4 using (1.7). Section 5 includes some additional remarks.

2. Models of probabilistic recall

Consider a general probability vector $\mathbf{p} = (p_1, \ldots, p_{n-1})$, where $1 \geqslant p_1 \geqslant \ldots \geqslant p_{n-1} \geqslant 0$. The geometric case where $p_i = p^i$ for some p, $0 \leqslant p \leqslant 1$, is of special interest, but will not be treated separately here. Since the case where $p_{n-1} = 1$ is trivial, it will not be considered any further. We consider two modes of availability and three modes of stopping, giving six models, some of which coincide.

The modes of availability

(I) If an item X_i is not available at time $k > i$, it will remain unavailable beyond k.
(II) The item may become available, either by some specified probabilistic model, or by probabilistic recall using the vector $\mathbf{p}$ above. In this case the availability of a given item might behave as independent 'coin tossings' with the specified probabilities.

In both modes we naturally assume that the *de facto* availability of a previously observed item becomes known only when an attempt to recall it is made.

Another realistic mode, which will not be discussed here, is where availability of X_i depends not only on the time elapsed since the item was seen, but also on the 'quality' of the item. Thus a large X_i value may have a smaller probability of being available at a later time. (If X_i measures some quality of a candidate for a job, this mode seems especially suitable, due to other offers he/she may receive.) This mode is studied in Petrucelli (1982) in relation to the full-information, best-choice 'secretary problem', with no relation to prophet inequalities. Lee (2001) has studied the case where the recall probabilities depend on the value observed, but not on the time elapsed, for the case of i.i.d. uniform [0,1] random variables.

The three modes of stopping are as follows:

(A) Continue until an item has been selected – either through direct choice at time of observation, or through recalling, or by reaching time n and being forced to stop.
(B) If a recalled item is unavailable one must stop, and the reward is the present observation.
(C) If a recalled item is unavailable one must stop and the reward is 0.

In Section 3 we consider non-negative X_i only. For this case clearly the combination (I, C) is the least favourable for the statistician, and hence the prophet inequalities will be most extreme for this case. It turns out, however, that the prophet inequalities and region obtained in later sections for this case are valid in all six models considered.

We first describe the value and the optimal stopping rule for the combination (I, C). Let $p_0 = 1$, and for $k = 1, \ldots, n$ let $\mathcal{F}_k$ be the σ-field generated by $X_1, \ldots, X_k$ and possibly some unrelated variables needed for randomization, when considering randomized stopping rules. It will be convenient to let $\mathcal{F}_0$ be the trivial σ-field consisting only of the whole space and the empty set. In the present section there is no need to assume that the X_i are independent. Define the optimal value, denoted $V_{\mathbf{p}}(X_1, \ldots, X_n)$, as follows. For $j \geqslant i$ let $Z_{i,j}$ be the indicator of the event that the item observed at time i is still available at time j. Thus $Z_{i,j}$ takes the values 0 and 1 with $P(Z_{i,j} = 1) = p_{j-i}$. (Note that in some of the models, for fixed i, the $Z_{i,j}$ are dependent.) For model (I, C) define

$$V_{\mathbf{p}}(X_1, \ldots, X_n) = \sup \mathrm{E}\left[\bigvee_{i=1}^{n}(\mathrm{E}(Z_{i,t})X_i)\right],$$

with the supremum taken over all stopping rules t. An optimal stopping rule may be described recursively as follows. Let

$$W_n^n(X_1, \ldots, X_n) = \bigvee_{i=1}^{n} (p_{n-i} X_i). \tag{2.1}$$

Note that $W_n^n(X_1, \ldots, X_n)$ represents the 'actual optimal payoff' if one has reached time n without stopping earlier, the quotation marks denoting that $Z_{i,j}$ has been replaced by its expected value. Next, let

$$W_{n-1}^*(X_1, \ldots, X_{n-1}) = \mathrm{E}[W_n^n(X_1, \ldots, X_n) | \mathcal{F}_{n-1}] \tag{2.2}$$

and

$$W_{n-1}^n(X_1, \ldots, X_{n-1}) = \max\left\{ \bigvee_{i=1}^{n-1} p_{n-1-i} X_i,\ W_{n-1}^*(X_1, \ldots, X_{n-1}) \right\}.$$

Generally define, for $k = n-1, n-2, \ldots, 1$,

$$W_k^*(X_1, \ldots, X_k) = \mathrm{E}[W_{k+1}^n(X_1, \ldots, X_{k+1}) | \mathcal{F}_k]. \tag{2.3}$$

$W_k^*(X_1, \ldots, X_k)$ is the optimal payoff which can be expected in the future, having seen $X_1, \ldots, X_k$ and not having stopped. Let

$$W_k^n(X_1, \ldots, X_k) = \max\left\{ \bigvee_{i=1}^{k} p_{k-i} X_i,\ W_k^*(X_1, \ldots, X_k) \right\}. \tag{2.4}$$

Then it is clear by backward induction that the optimal value is

$$V_{\mathbf{p}}(X_1, \ldots, X_n) = W_0^* = \mathrm{E}[W_1^n(X_1) | \mathcal{F}_0], \tag{2.5}$$

and an optimal rule (which stops as early as possible) is

$$t = \inf\left\{ k : W_k^n(X_1, \ldots, X_k) = \bigvee_{i=1}^{k} p_{k-i} X_i \right\}.$$

When $t = k$ the $j \leq k$ actually chosen is any j for which $p_{k-j} X_j = \bigvee_{i=1}^{k} p_{k-i} X_i$.

Although this description is simple in principle, the evaluation and implementation of this rule is usually not straightforward in practice.

A referee has kindly pointed out to us that the optimal stopping problem (I, C) is equivalent to the optimal stopping problem for the *dependent* random variables $Y_1, \ldots, Y_n$ where $Y_i = \max\{p_{i-1} X_1, \ldots, p_1 X_{i-1}, X_i\}$. Prophet inequalities involving dependent random variables, and in particular random variables that are certain functions of independent random variables, have been discussed in the past by Choi and Klass (1997), Wittmann (1995) and Brunel and Krengel (1979), among others.

It should be noted that the p_i can also be thought of as 'discounting factors'. Thus the value X_i, when chosen only j time periods later, is worth only $p_j X_i$. This interpretation is, however, different from the usual discounting concept, where right from outset the values in the future are discounted. Thus the usual discounting forces the optimal stopper to stop 'sooner' than with no discounting. Clearly the possibility of recall causes the optimal stopper to stop 'later' than with no recall.

The special case where $p_1 = \ldots = p_{n-1}$ is worth mentioning. Though possibly not very realistic, the optimal rule simplifies considerably in this case, since clearly no recall should be attempted (in situation (I, C)) before reaching time $t = n$.

The 'indifference value' μ satisfying $\mu = W_1^*(\mu)$ will play a special role in the next section. Clearly it is unique. When $X_1 \equiv \mu$ one will be indifferent between picking μ right at time 1, on the one hand, and not picking it and continuing optimally, on the other. It is clear (and can be shown by induction) that when the X_i are independent, and when using an optimal rule, if one has not stopped earlier, and the present observation X_i satisfies $X_i > \mu$, one should stop and pick it. (One might also stop for smaller values of X_i.)

3. Construction of extremal distributions

We now return to the case of independent X_i. In the present section we consider bounded random variables, and for convenience take $0 \leqslant X_i \leqslant 1$, excluding the trivial case where $X_2 = \ldots = X_n = 0$. Since for non-negative random variables model (I, C) is the worst for the statistician, we shall at present consider this situation only. Later it is shown that the results obtained are valid more generally.

Consider the prophet region

$$S_{\mathbf{p}}^n = \{(x, y) : x = V_{\mathbf{p}}(X_1, \ldots, X_n),\ y = \mathrm{E}[X_1 \vee \ldots \vee X_n] \text{ for some } 0 \leqslant X_i \leqslant 1\}. \quad (3.1)$$

Our aim in the present section is to construct this region. Clearly $0 < x \leqslant 1$, and for a given value of x we show that $x \leqslant y \leqslant y(x)$, where

$$y(x) = \max\{\mathrm{E}[X_1 \vee \ldots \vee X_n] : V_{\mathbf{p}}(X_1, \ldots, X_n) = x\} \quad (3.2)$$

and $\mathbf{p}$ and n are considered fixed. The function $y(x)$ satisfies $y(0) = 0$, $y(1) = 1$, and $S_{\mathbf{p}}^n$ is completely characterized by $y(x)$.

A vector of random variables $(X_1, \ldots, X_n)$, and its distribution, will be called *extremal* if $V_{\mathbf{p}}(X_1, \ldots, X_n) = x$ for some $0 < x < 1$, and $\mathrm{E}[X_1 \vee \ldots \vee X_n] = y(x)$, and a point $(x, y(x))$ will be called an *extremal point*.

Using standard arguments, consider the problem of maximizing $D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n)$, defined by

$$D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) =: \mathrm{E}[X_1 \vee \ldots \vee X_n] - \alpha V_{\mathbf{p}}(X_1, \ldots, X_n), \quad (3.3)$$

over $(X_1, \ldots, X_n)$ where $\alpha > 0$. Let $\sup_{X_1,\ldots,X_n} D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) = d(\alpha; \mathbf{p})$, and let

$$\alpha^* = \sup\{\alpha : d(\alpha; \mathbf{p}) > 0\}. \quad (3.4)$$

Let

$$R_{\mathbf{p}}(X_1, \ldots, X_n) = \frac{\mathrm{E}[X_1 \vee \ldots \vee X_n]}{V_{\mathbf{p}}(X_1, \ldots, X_n)}. \quad (3.5)$$

We have the following proposition:

Proposition 3.1.

$$\sup_{X_1,\ldots,X_n} R_{\mathbf{p}}(X_1, \ldots, X_n) = \alpha^*. \tag{3.6}$$

Proof. Suppose $R_{\mathbf{p}}(X_1, \ldots, X_n) = \tilde{\alpha} > \alpha^*$, for some $X_1, \ldots, X_n$. Then $\mathrm{E}[X_1 \vee \ldots \vee X_n] = \tilde{\alpha} V_{\mathbf{p}}(X_1, \ldots, X_n)$, i.e. $D_{\mathbf{p}}^{\alpha_0}(X_1, \ldots, X_n) > 0$ for $\tilde{\alpha} > \alpha_0 > \alpha^*$, contrary to the definition of α^*. Now suppose $\sup_{X_1,\ldots,X_n} R_{\mathbf{p}}(X_1, \ldots, X_n) = \hat{\alpha} < \alpha^*$. Then for all $(X_1, \ldots, X_n)$ the inequality $\mathrm{E}[X_1 \vee \ldots \vee X_n] \leqslant \hat{\alpha} V_{\mathbf{p}}(X_1, \ldots, X_n)$ holds, i.e. $D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) \leqslant 0$ for all $\alpha > \hat{\alpha}$, again contradicting the definition of α^*. □

We note in passing that necessarily

$$\alpha^* \leqslant \frac{1}{p_{n-1}}, \tag{3.7}$$

since the rule which waits to the end and then recalls the largest X_i has a value of at least $p_{n-1}\mathrm{E}[X_1 \vee \ldots \vee X_n]$, and thus $V_{\mathbf{p}}(X_1, \ldots, X_n) \geqslant p_{n-1}\mathrm{E}[X_1 \vee \ldots \vee X_n]$.

In the sequel we shall make use of dilation. Dilation of X in the interval $[a, b]$ creates a new variable, say X_a^b, such that

$$X_a^b = \begin{cases} X & \text{when } X \notin [a, b], \\ b & \text{with probability } \mathrm{E}\{[X - a]I(a \leqslant X \leqslant b)\}/(b - a), \\ a & \text{with probability } \mathrm{E}\{[b - X]I(a \leqslant X \leqslant b)\}/(b - a). \end{cases} \tag{3.8}$$

X_a^b has the following properties:

$$P(a \leqslant X \leqslant b) = P(a \leqslant X_a^b \leqslant b); \tag{3.9a}$$

$$\mathrm{E}X = \mathrm{E}X_a^b; \tag{3.9b}$$

$$Eh(X_a^b) \geqslant Eh(X), \qquad \text{for any convex function } h. \tag{3.9c}$$

In the following series of lemmas we shall stepwise replace one set of variables $(X_1, \ldots, X_n)$ by another, thereby increasing $D_{\mathbf{p}}^{\alpha}$ for $0 < \alpha \leqslant 1/p_{n-1}$. This will lead to a set of extremal distributions.

Our first step is to show that, for any given $X_2, \ldots, X_n$, the value of $D_{\mathbf{p}}^{\alpha}$ is maximal when X_1 is replaced by a constant.

Lemma 3.1. *Let $n \geqslant 2$ and $X_2, \ldots, X_n$ be given, not all identically* 0. *Then for every α there exists a constant c_α such that, for any X_1,*

$$D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) \leqslant D_{\mathbf{p}}^{\alpha}(c_\alpha, X_2, \ldots, X_n). \tag{3.10}$$

Proof. Let

$$\max_{0 \leqslant x \leqslant 1} D_{\mathbf{p}}^{\alpha}(x, X_2, \ldots, X_n) = D_{\mathbf{p}}^{\alpha}(c_\alpha, X_2, \ldots, X_n).$$

(The maxima are attained since the corresponding functions are continuous in x.) Now let X_1 be any random variable, independent of $X_2, \ldots, X_n$, satisfying $0 \leqslant X_1 \leqslant 1$, with cdf F. Then

$$D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) = \int_0^1 \mathrm{E}(x \vee X_2 \vee \ldots \vee X_n)\mathrm{d}F(x) - \alpha \int_0^1 V_{\mathbf{p}}(x, X_2, \ldots, X_n)\mathrm{d}F(x)$$

$$\leqslant D_{\mathbf{p}}^{\alpha}(c_\alpha, X_2, \ldots, X_n). \qquad \square$$

(Note that the proof utilizes the fact that the value of X_1 will be known before a decision to stop with $t = 1$, or to continue, must be made. Hence $V_{\mathbf{p}}(X_1, \ldots, X_n) = \int V_{\mathbf{p}}(x, X_2, \ldots, X_n)\mathrm{d}F(x)$. A similar conditioning on the value of X_j for $j \geqslant 2$ is impossible.)

Lemma 3.2. *Let* $0 \leqslant c \leqslant 1$ *and, for* $i = 2, \ldots, n$, *let* $X_i^{(c)} = X_i I(X_i > c)$. *Then*

$$D_{\mathbf{p}}^{\alpha}(c, X_2, \ldots, X_n) \leqslant D_{\mathbf{p}}^{\alpha}\left(c, X_2^{(c)}, \ldots, X_n^{(c)}\right).$$

Proof. Clearly $\mathrm{E}[c \vee X_2 \vee \ldots \vee X_n] = \mathrm{E}[c \vee X_2^{(c)} \vee \ldots \vee X_n^{(c)}]$, whereas $V_{\mathbf{p}}(c, X_2, \ldots, X_n) \geqslant V_{\mathbf{p}}(c, X_2^{(c)}, \ldots, X_n^{(c)})$. $\square$

Since for given p_{n-1} the value to the optimal stopper is worst when $p_1 = \ldots = p_{n-1}$, we shall henceforth assume this, and denote the common value by p. Clearly this assumption renders $V_{\mathbf{p}}$ the smallest, and hence $D_{\mathbf{p}}^{\alpha}$ the largest. Note that with this assumption an optimal rule need never recall before time n (if at all).

Lemma 3.3. *Let* $p_1 = \ldots = p_{n-1} = p$, *and suppose* $X_i = X_i I(X_i > c)$, $i = 2, \ldots, n$, *for some* $0 < c \leqslant \mu$, *where* μ *is the indifference value for* $X_2, \ldots, X_n$, *i.e. satisfies* $\mu = W_1^*(\mu)$. *Then, for all* $0 < \alpha \leqslant 1/p$,

$$D_{\mathbf{p}}^{\alpha}(c, X_2, \ldots, X_n) \leqslant D_{\mathbf{p}}^{\alpha}(\mu, X_2, \ldots, X_n). \tag{3.11}$$

Proof. Let $\gamma = P(X_2 = \ldots = X_n = 0) \geqslant 0$. Then

$$\mathrm{E}[c \vee X_2 \vee \ldots \vee X_n] = \mathrm{E}[X_2 \vee \ldots \vee X_n] + \gamma c \tag{3.12}$$

and

$$V_{\mathbf{p}}(c, X_2, \ldots, X_n) = V_{\mathbf{p}}(0, X_2, \ldots, X_n) + \gamma p c; \tag{3.13}$$

(3.13) follows since the optimal rule will recall (if at all) only at the nth stage, and then will recall c only if all the other X_i are 0. Thus

$$D_{\mathbf{p}}^{\alpha}(c, X_2, \ldots, X_n) = \mathrm{E}[X_2 \vee \ldots \vee X_n] - \alpha V_{\mathbf{p}}(0, X_2, \ldots, X_n) + c\gamma(1 - p\alpha),$$

which is non-decreasing in c for all $0 < \alpha \leqslant 1/p$ (which by (3.7) are the only values of interest). $\square$

Lemma 3.4. *Let* $(c, X_2, \ldots, X_n)$ *be given,* $p_1 = \ldots = p_{n-1} = p$, *and suppose that* $X_i = X_i I(X_i > c)$, *where* $\mu \leqslant c < 1$ *and* μ *is the indifference value for* $X_2, \ldots, X_n$. *Let*

$$\hat{X}_i = \begin{cases} 1 & \text{with probability } \mathrm{E}[X_i - c]^+/(1-c) = \beta_i, \\ 0 & \text{with probability } 1 - \beta_i. \end{cases} \tag{3.14}$$

Then

$$D_{\mathbf{p}}^{\alpha}(c, X_2, \ldots, X_n) \leqslant D_{\mathbf{p}}^{\alpha}(c, \hat{X}_2, \ldots, \hat{X}_n). \tag{3.15}$$

Proof. Consider first the case $c = \mu$. Here the optimal rule for the X_i either picks μ at time $t = 1$, or otherwise stops (without recall) with the smallest i (if any) such that $X_i \geqslant \mu$. If no such i exists, it will recall μ at time n. Now dilate X_i in the interval $[\mu, 1]$. This yields a random variable $\tilde{X}_i$ taking values 1, μ and 0, with probabilities β_i, $\mathrm{E}[(1 - X_i)I(X_i \geqslant \mu)]/(1-\mu)$ and $P(X_i = 0)$ respectively, where β_i is defined in (3.14) for $c = \mu$. By (3.9), $P(X_i \geqslant \mu) = P(\tilde{X}_i \geqslant \mu)$ and $\mathrm{E}\tilde{X}_i I(\tilde{X}_i \geqslant \mu) = \mathrm{E}X_i I(X_i \geqslant \mu)$, and hence the optimal stopping values for the corresponding sequences satisfy

$$V_{\mathbf{p}}(\mu, X_2, \ldots, X_n) = V_{\mathbf{p}}(\mu, \tilde{X}_2, \ldots, \tilde{X}_n). \tag{3.16}$$

Since dilation can only increase the expected value of the maximum, it follows that $\mathrm{E}[\mu \vee X_2 \vee \ldots \vee X_n] \leqslant \mathrm{E}[\mu \vee \tilde{X}_2 \vee \ldots \vee \tilde{X}_n]$. Using Lemma 3.2, $\tilde{X}_i$ can now be replaced by $\tilde{X}_i I(\tilde{X}_i > \mu) = \hat{X}_i$, yielding (3.15).

Now consider $c > \mu$. Here the optimal rule must pick c at time $t = 1$. Dilating X_i in $[c, 1]$ to yield a three-valued random variable $\tilde{X}_i$ will yield a possibly new indifference value $\tilde{\mu}$ for the $\tilde{X}_i$, with $\tilde{\mu} \leqslant c$. Again using Lemma 3.2 to replace $\tilde{X}_i$ by $\tilde{X}_i I(\tilde{X}_i > c) = \hat{X}_i$ yields (3.15) in this case. □

Lemma 3.5. *Let* $n \geqslant 2$, *let* $\mathbf{p}$ *be given with* $p_1 \geqslant \ldots \geqslant p_{n-1}$ *and fix* α, $0 < \alpha \leqslant 1/p_{n-1}$. *Then, for any* $(X_1, \ldots, X_n)$, *there exist* $(X_1^*, \ldots, X_n^*)$ *of the form* $X_1^* = c$, $X_2^* = \ldots = X_{n-1}^* = 0$ *and*

$$X_n^* = \begin{cases} 1 & \text{with probability } \delta, \\ 0 & \text{with probability } 1 - \delta, \end{cases}$$

such that

$$D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) \leqslant D_{\mathbf{p}}^{\alpha}(X_1^*, \ldots, X_n^*). \tag{3.17}$$

Proof. Using Lemmas 3.1–3.4, it follows that X_1 can be replaced by a constant, and, for $\mathbf{p}^* = (p, \ldots, p)$ where $p = p_{n-1}$, all other random variables can be replaced by binary random variables $\hat{X}_i$, $i = 2, \ldots, n$.

Let $\delta = P(\hat{X}_i = 1 \text{ for some } i = 2, \ldots, n)$, and let $X_n^* = 1$ and 0 with probabilities δ and $1 - \delta$, respectively. Let $X_i^* \equiv 0$ for $i = 2, \ldots, n-1$. Then clearly $\mathrm{E}[c \vee \hat{X}_2 \vee \ldots \vee \hat{X}_n] = \mathrm{E}[c \vee X_2^* \vee \ldots \vee X_n^*]$ and $V_{\mathbf{p}^*}(c, \hat{X}_2, \ldots, \hat{X}_n) = V_{\mathbf{p}^*}(c, \tilde{X}_2^*, \ldots, X_n^*)$. For $\mathbf{p}$ and any $X_1, \ldots, X_n$, clearly $V_{\mathbf{p}}(X_1, \ldots, X_n) \geqslant V_{\mathbf{p}^*}(X_1, \ldots, X_n)$, whereas

$V_{\mathbf{p}}(c, X_2^*, \ldots, X_n^*) = V_{\mathbf{p}^*}(c, X_2^*, \ldots, X_n^*)$, thus generally $D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) \leqslant D_{\mathbf{p}^*}^{\alpha}(X_1, \ldots, X_n)$, but for the random variables $X_1^*, \ldots, X_n^*$ we have

$$D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n) \leqslant D_{\mathbf{p}^*}^{\alpha}(X_1, \ldots, X_n) \leqslant D_{\mathbf{p}^*}^{\alpha}(X_1^*, \ldots, X_n^*) = D_{\mathbf{p}}^{\alpha}(X_1^*, \ldots, X_n^*). \qquad \square$$

Theorem 3.1. *For $n \geqslant 2$ and any vector $\mathbf{p}$ with $p_1 \geqslant \ldots \geqslant p_{n-1}$, a collection of extremal random variables is $X_1 = x$, $X_2 = \cdots = X_{n-1} = 0$ and*

$$X_n = \begin{cases} 1 & \text{with probability } x(1-p_{n-1})/(1-p_{n-1}x), \\ 0 & \text{with probability } (1-x)/(1-p_{n-1}x), \end{cases} \tag{3.18}$$

where $0 < x \leqslant 1$, and the prophet region $S_{\mathbf{p}}^n$ of (3.1) *depends only on $p_{n-1} = p$ (and not on the rest of the vector $\mathbf{p}$, nor on n) and is given by*

$$S_p = \left\{(x, y) : 0 \leqslant x \leqslant 1, x \leqslant y \leqslant \frac{x(2-p-x)}{1-px}\right\}. \tag{3.19}$$

Proof. Using Lemma 3.5, it follows that, for each fixed value of α, one must determine $\delta = P(X_n^* = 1)$ and x in such a way that $D_{\mathbf{p}}^{\alpha}(x, 0, \ldots, 0, X_n^*)$ is maximal. Now

$$D_{\mathbf{p}}^{\alpha}(x, 0, \ldots, 0, X_n^*) = [\delta + (1-\delta)x] - \alpha \max(x, \delta + p(1-\delta)x).$$

Simple arithmetic shows that for fixed α and p this is maximal for some pair (x, δ) for which $x = \delta + p(1-\delta)x$ holds. Thus x turns out to be the indifference value. Hence $V_{\mathbf{p}}(x, 0, \ldots, 0, X_n^*) = x$ and $\delta = x(1-p)/(1-px)$, yielding the variables in (3.18). For the latter variables the value of $\mathrm{E}[X_1 \vee \ldots \vee X_n]$ is $x(2-p-x)/(1-px)$, which is the value $y(x)$ in (3.19). Note that $x(2-p-x)/(1-px)$ is a strictly concave function. Hence maximizing $D_{\mathbf{p}}^{\alpha}(X_1, \ldots, X_n)$ for all $\alpha > 0$ yields the same curve that is obtained by maximizing, for each $0 < x < 1$, the value of $\mathrm{E}[X_1 \vee \ldots \vee X_n]$ over all $X_1, \ldots, X_n$ for which $V_{\mathbf{p}}(X_1, \ldots, X_n) = x$.

To see that all values (x, y) as described in (3.19) are attainable, let (x_0, y_0) be a point with $0 < x_0 < 1$ and $x_0 \leqslant y_0 < y(x_0)$. We shall show that $(x_0, y_0) \in S_p$. Let $X_1 = x_0$, $X_2 = \ldots = X_{n-1} = 0$ and

$$X_n = \begin{cases} 1 & \text{with probability } (y_0 - x_0)/(1-x_0), \\ 0 & \text{with probability } (1-y_0)/(1-x_0). \end{cases} \tag{3.20}$$

Then $V_{\mathbf{p}}(X_1, \ldots, X_n) = x_0$ and $\mathrm{E}[X_1 \vee \ldots \vee X_n] = y_0$.

Note that since for the random variables in (3.18) and (3.20) the values of $p_1, \ldots, p_{n-2}$ are irrelevant, the result holds for any vector $\mathbf{p}$. $\square$

It is worthwhile to note that the extremal cases obtained here reduce to those obtained in the references mentioned when there is no possibility of recall, i.e. when $p_{n-1} = 0$. In particular, note that (3.19) becomes (1.6) when $p = 0$.

Corollary 3.1. *Theorem* 3.1 *is true for all six options of probabilistic recall defined in Section* 2.

Proof. We have seen that for non-negative X_i the case (I, C) is the least favourable for the statistician, thus the prophet region for all other cases must be a subset of S_p given in (3.19). But for the variables given in (3.18) (and also for those of (3.20)) the value $V_{\mathbf{p}}(X_1, \ldots, X_n)$ is the same in all six options considered. □

Remark 3.1. Note that when $p < q$ the relation $S_p \supset S_q$ holds.

4. Proofs of the ratio and difference prophet inequalities

It is now a simple matter to prove Theorem 1.2. Note that for the variables in (3.18) one has

$$D^1_{\mathbf{p}}(X_1, \ldots, X_n) = x\left(\frac{2 - p - x}{1 - px} - 1\right), \tag{4.1}$$

where we have written p instead of p_{n-1}. Maximizing (4.1) over x, $0 \leqslant x \leqslant 1$, for a fixed p, one finds that the maximum is attained at

$$x_p = \frac{1 - (1 - p)^{1/2}}{p} \tag{4.2}$$

and yields

$$D^1_{\mathbf{p}}(X_1, \ldots, X_n) \leqslant \frac{[1 - (1 - p)^{1/2}]^2(1 - p)}{p^2}, \tag{4.3}$$

where the value on the right-hand side of (4.3) is a best bound, obtained for the variables in (3.18) with x_p given in (4.2). When $a \leqslant X_i \leqslant b$ one can define $X_i' = (X_i - a)/(b - a)$, and then $0 \leqslant X_i' \leqslant 1$, for which (4.3) holds. Converting back yields Theorem 1.2 for the original X_i. In stopping mode (C) of Section 2, the reward when a recalled item is unavailable must here be taken as the minimal possible value, i.e. a. Note that by Corollary 3.1 it follows that Theorem 1.2 holds for all six cases considered.

For an infinite sequence $X_1, X_2, \ldots$ one has

$$\mathrm{E}\left(\sup_i X_i\right) = \lim_{n\to\infty} \mathrm{E}[X_1 \vee \ldots \vee X_n], \qquad V_{\mathbf{p}}(X_1, X_2, \ldots) = \lim_{n\to\infty} V_{\mathbf{p}}(X_1, \ldots, X_n). \tag{4.4}$$

If $\lim_{n\to\infty} p_n = p_\infty$ then, by Remark 3.1, it follows that the prophet region for the infinite case, S^∞, contains S_{p_n} for all n, thus is the limiting set S_{p_∞} except possibly for the upper boundary $x(2 - p_\infty - x)/(1 - p_\infty x)$, which may not be attainable for $0 < x < 1$. Thus a statement corresponding to (4.3) holds for $D^1_{\mathbf{p}}(X_1, X_2, \ldots)$, where on the right-hand side of (4.3) p_∞ must be substituted for p. It will still be a best bound, except that equality may not be attainable in all cases.

Now consider the ratio prophet inequality. When $0 \leqslant X_i \leqslant 1$ it follows from (3.19) that, for a given value $0 < x = V_{\mathbf{p}}(X_1, \ldots, X_n)$,

$$R_{\mathbf{p}}(X_1, \ldots, X_n) \leqslant \frac{2 - p_{n-1} - x}{1 - p_{n-1}x}, \tag{4.5}$$

where equality holds for the variables of (3.18). Now the right-hand side of (4.5) is decreasing in x, and tends to $2 - p_{n-1}$ as $x \to 0$, but this value is not attainable. This proves Theorem 1.1 for $0 \leqslant X_i \leqslant 1$.

Now suppose $0 \leqslant X_i \leqslant B$, $i = 1, \ldots, n$. Then $X_i^B = X_i/B$ satisfy $0 \leqslant X_i^B \leqslant 1$, and clearly $R_{\mathbf{p}}(X_1, \ldots, X_n) = R_{\mathbf{p}}(X_1^B, \ldots, X_n^B)$, thus (1.2) holds for any non-negative bounded random variables. But when $0 \leqslant X_i < \infty$ with $\mathrm{E}X_i < \infty$, then

$$\mathrm{E}(X_1 \vee \ldots \vee X_n) = \lim_{B \to \infty} \mathrm{E}[X_1^B \vee \ldots \vee X_n^B]$$

and

$$V_{\mathbf{p}}(X_1, \ldots, X_n) = \lim_{B \to \infty} V_{\mathbf{p}}(X_1^B, \ldots, X_n^B) = x > 0$$

(if the X_i are not identically 0), thus

$$R_{\mathbf{p}}(X_1, \ldots, X_n) = \lim_{B \to \infty} R_{\mathbf{p}}(X_1^B, \ldots, X_n^B) \leqslant \frac{2 - p_{n-1} - x}{1 - p_{n-1}x},$$

and therefore (1.2) holds by letting $x \to 0$. Clearly $2 - p_{n-1}$ is the best constant. For the infinite case (1.3) follows, using (4.4) and (4.5), since the stopping region for the infinite case, S^∞, equals S_{p_∞}, except possibly for the upper boundary of S_{p_∞}.

5. Additional remarks

Remark 5.1. As with the prophet inequalities without recall, the present prophet inequalities also have an order selection interpretation (see Hill 1983). This says that if the random variables $X_1, \ldots, X_n$ are presented in any known order – even in the order least favourable to the optimal stopper – the optimal stopper can guarantee to obtain at least $(2 - p_{n-1})^{-1}$ times the value to the prophet (which is order-independent).

Remark 5.2. All the results obtained clearly remain valid if the probability of recalling item i at time j, $1 \leqslant i < j$, is p_{ji}, i.e. is time-dependent, as long as $p_{ji} \geqslant p_{n1}$. It should be noted that nowhere in our proofs have we utilized $p_1 \geqslant \ldots \geqslant p_{n-1}$. This assumption was made only to make the model attractive. The only inequality really needed is $p_j \geqslant p_{n-1}$ for $j = 1, \ldots, n-2$.

Remark 5.3. It should be noted that, for $n = 2$, inequality (1.2) also holds for non-negative dependent random variables. This can be seen as follows. Let $p_1 = p$, $0 < p < 1$. Then

$$\begin{aligned} \mathrm{E}[X_1 \vee X_2] &= \mathrm{E}X_1 + \mathrm{E}[X_2 - X_1]^+ \leqslant \mathrm{E}X_1 + \mathrm{E}[X_2 - pX_1]^+ \qquad (5.1) \\ &= (1 - p)\mathrm{E}X_1 + \mathrm{E}[X_2 \vee pX_1] \end{aligned}$$

with strict inequality unless $X_1 \equiv 0$ or $\mathrm{E}[X_2 - pX_1]^+ = 0$, i.e. $\mathrm{E}[X_1 \vee X_2] = \mathrm{E}X_1$. Both these exceptional cases are uninteresting, since for both $R_p(X_1, X_2) = 1$. Further,

$$\begin{aligned} V_p(X_1, X_2) &= \mathrm{E}[\max\{X_1, \mathrm{E}(X_2 \vee pX_1|X_1)\}] \\ &\geqslant \max\{\mathrm{E}X_1, \mathrm{E}[\mathrm{E}(X_2 \vee pX_1|X_1)]\} \\ &= \max\{\mathrm{E}X_1, \mathrm{E}[X_2 \vee pX_1]\}. \end{aligned} \tag{5.2}$$

There are two cases to consider.

Case (i): $\mathrm{E}X_1 > \mathrm{E}[X_2 \vee pX_1]$. Using (5.1) and (5.2) for this case yields

$$R_p(X_1, X_2) < \frac{(1-p)\mathrm{E}X_1 + \mathrm{E}X_1}{\mathrm{E}X_1} = 2 - p. \tag{5.3}$$

Case (ii): $0 < \mathrm{E}X_1 \leqslant \mathrm{E}[X_2 \vee pX_1]$. Then (5.3) holds when, instead of $\mathrm{E}X_1$, one writes $\mathrm{E}[X_2 \vee pX_1]$ everywhere. The strict inequality in (5.3) for this case holds since strict inequality must hold in (5.1).

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