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A technique for exponential change of measure for Markov processes

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We consider a Markov process $X(t)$ with extended generator $\mathbf{A}$ and domain $\mathcal{D}(\mathbf{A})$. Let $\{\mathcal{F}_t\}$ be a right-continuous history filtration and $\mathbb{P}_t$ denote the restriction of $\mathbb{P}$ to $\mathcal{F}_t$. Let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ such that $\mathrm{d}\tilde{\mathbb{P}}_t/\mathrm{d}\mathbb{P}_t = E^h(t)$, where

$$E^h(t) = \frac{h(X(t))}{h(X(0))} \exp\left(-\int_0^t \frac{(\mathbf{A}h)(X(s))}{h(X(s))}\,\mathrm{d}s\right)$$

is a true martingale for a positive function $h \in \mathcal{D}(\mathbf{A})$. We demonstrate that the process $X(t)$ is a Markov process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$, we find its extended generator $\tilde{\mathbf{A}}$ and provide sufficient conditions under which $\mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$. We apply this result to continuous-time Markov chains, to piecewise deterministic Markov processes and to diffusion processes (in this case a special choice of h yields the classical Cameron–Martin–Girsanov theorem).

Keywords: Cameron–Martin–Girsanov theorem; diffusion process; exponential change of measure; extended generator; local martingale; Markov additive process; Markov process; piecewise deterministic Markov process

1. Introduction

The technique of exponential change of measure has been successfully applied in various theories such as large deviations (Ridder and Walrand 1992; Schwartz and Weiss 1995), queues and fluid flows (Asmussen 1994; 1995; Kulkarni and Rolski 1994; Palmowski and Rolski 1996; 1998; Gautam *et al*. 1999), ruin theory (Dassios and Embrechts 1989; Asmussen 1994; 2000; Schmidli 1995; 1996; 1997a; 1997b), simulation (Ridder 1996; Asmussen 1998) and population genetics (Fukushima and Stroock 1986; Ethier and Kurtz 1993). The process of interest is typically Markovian and, under a suitably chosen new probability measure, it is again a Markov process with some 'nicer' desired properties. To know the parameters of the new process we need to find its generator. Although in some cases determining the new generator is straightforward, there are many situations in which it is more difficult. In such situations, a unified theory simplifies the calculations. In this paper we present a detailed account of the change of probability measure technique for cadlag Markov processes. We can also accommodate some non-Markovian processes that are Markovian with a supplementary component, for example, piecewise deterministic Markov processes or Markov additive processes. For diffusion processes, a special case of the theory presented is a Girsanov-type theorem.

Consider a Markov process $X(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ having extended generator $\mathbf{A}$ with domain $\mathcal{D}(\mathbf{A})$. In this paper we use the martingale approach of Stroock and Varadhan (1979) to define generators. For each strictly positive function f, define

$$E^f(t) = \frac{f(X(t))}{f(X(0))} \exp\left(-\int_0^t \frac{(\mathbf{A}f)(X(s))}{f(X(s))} \, \mathrm{d}s\right), \qquad t \geqslant 0. \tag{1.1}$$

If, for some function h, the process $E^h(t)$ is a martingale, then it is said to be an *exponential martingale*. In this case we call h a *good* function. Using this exponential martingale as a likelihood ratio process, we will define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$. In Theorem 4.2 we prove that under some mild assumptions, $X(t)$ is a Markov process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$ having extended generator

$$\tilde{\mathbf{A}}f = \frac{1}{h}[\mathbf{A}(f h) - f\mathbf{A}h] \tag{1.2}$$

and $\mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$. Note that (1.2) can be rewritten using the *opérateur carré du champ* (see Revuz and Yor 1991, Definition 3.2, p. 326) $\tilde{\mathbf{A}}f = \mathbf{A}f + h^{-1}\langle h, f\rangle_{\mathbf{A}}$, where $\langle h, f\rangle_{\mathbf{A}} = \mathbf{A}(hf) - h\mathbf{A}f - f\mathbf{A}h$. Kunita (1969) and Ethier and Kurtz (1986) found that the two generators $\tilde{\mathbf{A}}$ and $\mathbf{A}$ are related by $\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{B}$, where $\mathbf{B}$ is a linear operator. However, the domains in their papers are restricted to bounded functions. Under some additional assumptions, Ethier and Kurtz (1986, Theorems 5.4 and 5.11c) show that $\mathbf{B}$ is the generator of some Markov process and they refer to $\tilde{\mathbf{A}}$ as a perturbation of $\mathbf{A}$. If a good function h is harmonic, that is, $\mathbf{A}h = 0$ (or $\tilde{\mathbf{A}}h^{-1} = 0$), then $h(X(t))/h(X(0))$ is a martingale. In this case $\tilde{\mathbf{A}}f = h^{-1}\mathbf{A}(fh) - f\mathbf{A}h = h^{-1}\mathbf{A}(fh)$ – see Dynkin (1965, Chapter IX, (9.46)) for a similar result expressed in terms of infinitesimal operators, or Doob (1984, Section 2.VI.13), where the idea of h transforms was outlined.

In Section 5 we illustrate the theory, applying it to several cases of special interest. For example, if $X(t)$ is a diffusion, then, choosing the exponential function as a good function, we obtain a Girsanov-type theorem (see Theorem 5.5). We also show that a piecewise deterministic Markov process (PDMP) remains a PDMP under the new measure $\tilde{\mathbb{P}}$, and we find its characteristics (see Theorem 5.3). Other explicit forms of $\tilde{\mathbf{A}}$ are computed for continuous-time Markov chains (CTMCs) in Proposition 5.1, and for Markov additive processes in Proposition 5.6.

Related material on change of measure or Girsanov-type theorems can be found, for example, in Revuz and Yor (1991), Küchler and Sørensen (1997) and Jacod and Shiryaev (1987).

2. Preliminaries

2.1. Basic set-up

Let E be a Borel space. Throughout this paper we denote the space of measurable functions $f : E \to \mathbb{R}$ by $\mathcal{M}(E)$ and the space of continuous functions by $\mathcal{C}(E)$. We add a subscript b to denote the restriction to bounded functions. We consider stochastic process $X(t)$ assuming values in E and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that $\mathcal{F} = \bigvee_{t\geq 0}\mathcal{F}_t$, that is, $\mathcal{F}$ is the smallest σ-field generated by all subsets of $\mathcal{F}_t$ for all $t \geq 0$. We denote by $\mathbb{F}$ the pair $(\mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$. We call $(\Omega, \mathbb{F})$ a *filtered space* and $(\Omega, \mathbb{F}, \mathbb{P})$ a *filtered probability space*. For a probability measure $\mathbb{P}$ we denote by $\mathbb{P}_t$ its restriction to $\mathcal{F}_t$. All stochastic processes considered here are cadlag.

Let $M(t)$ be a positive cadlag martingale defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ such that $\mathbb{E}M(0) = 1$. We define a family of measures $\tilde{\mathbb{P}}_t$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t} = M(t). \tag{2.3}$$

It can be proved that $\tilde{\mathbb{P}}_t$ is a consistent family, that is, for $s \leq t$ we have $\tilde{\mathbb{P}}_t(A) = \tilde{\mathbb{P}}_s(A)$ for all $A \in \mathcal{F}_s$.

We say that the *standard set-up* is satisfied if:

(i) the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ has a right-continuous filtration $\{\mathcal{F}_t\}$;
(ii) there exists a unique probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}}_t = \tilde{\mathbb{P}}_{|\mathcal{F}_t}$, where $\tilde{\mathbb{P}}_t$ is defined by (2.3).

The standard set-up may not be satisfied if the filtration $\{\mathcal{F}_t\}$ is not separable (Parthasarathy 1967, Definition 2.1 and Theorem 4.2); see, for example, the example of Rogers and Williams (1987, p. 83) and Karatzas and Shreve (1988, p. 193) concerning the augmented filtration of Brownian motion.

We now describe a *canonical model* for the standard set-up. Assume that $\Omega \subset E^{[0,\infty)}$ is the space $D_E[0, \infty)$ of cadlag functions from $[0, \infty)$ into E (in some cases we can also use $\mathcal{C}_E[0, \infty)$, the space of continuous functions into E) and consider the canonical process $X(t)$ on $(\Omega, \mathbb{F}, \mathbb{P})$ defined by $X(\omega, t) = \omega(t)$. The filtration is defined by $\{\mathcal{F}_t = \mathcal{F}^X_{t+}\}$, where $\mathcal{F}^X_t = \sigma\{X(s), s \leq t\}$ is the natural filtration generated by the process $X(t)$. We define a new family of probabilities via (2.3). By Kolmogorov's theorem (see Stroock 1987, Theorem 4.2, p. 106; Stroock and Varadhan 1979, Theorem 1.3.5, p. 34; Parthasarathy 1967, Theorem 4.2, p. 143), there exists a unique probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathbb{F})$ such that $\tilde{\mathbb{P}}_t = \tilde{\mathbb{P}}_{|\mathcal{F}_t}$, where $\tilde{\mathbb{P}}_t$ satisfies (2.3). The above considerations are, of course, true if we take the natural filtration $\{\mathcal{F}^X_t\}$ instead of $\{\mathcal{F}^X_{t+}\}$, but in applications one often needs the right-continuity assumption to define certain stopping times (e.g. moment of ruin in risk theory).

2.2. Markov processes

The following theorem was proved by Kunita and Watanabe (1963, Propositions 3 and 5). See also Dynkin (1965) and Küchler and Sørensen (1997, Proposition 6.3.5).

Theorem 2.1. *Let $X(t)$ be a Markov process on $(\Omega, \mathbb{F}, \mathbb{P})$. Assume that standard set-up is satisfied. We define a new probability measure $\tilde{\mathbb{P}}$ by* (2.3)*, where the martingale $M(t)$ is a non-negative multiplicative functional for which $\mathbb{E}(M(0)) = 1$. Then on the new filtered probability space $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$, the process $X(t)$ is Markovian.*

For a Markov process $X(t)$ with values in E we define a *full* (*extended*) *generator* $\mathbf{A}$ by

$$\mathbf{A} = \{(f, f^*) \in \mathcal{M}(E) \times \mathcal{M}(E) : D^f(t) \text{ is a (local) martingale}\},$$

where $D^f(t) = f(X(t)) - \int_0^t f^*(X(s))\mathrm{d}s$, $t \geqslant 0$, is *Dynkin's* (*local*) *martingale* and the function $s \to f^*(X(s))$ is integrable $\mathbb{P}$-almost surely on $[0, t]$ for all $t \geqslant 0$. From now on we will identify all versions of the function f^* and we denote all these versions by $\mathbf{A}f$ if $(f, f^*) \in \mathbf{A}$.

By $\mathcal{D}(\mathbf{A})$ we denote the set of measurable functions $f \in \mathcal{M}(E)$ such that $\mathbf{A}f \in \mathcal{M}(E)$ exists, and such that $\int_0^t |\mathbf{A}f(X(s))|\, \mathrm{d}s < \infty$ $\mathbb{P}$-a.s. for all $t \geqslant 0$. Thus

$$D^f(t) = f(X(t)) - \int_0^t \mathbf{A}f(X(s))\mathrm{d}s \tag{2.4}$$

is an $\mathbb{F}$-(local) martingale for $f \in \mathcal{D}(\mathbf{A})$. The space $\mathcal{D}(\mathbf{A})$ is called the *domain of the full* (*extended*) *generator* $\mathbf{A}$. If we restrict the domain of the extended or the full generator to a subset $\mathcal{D} \subset \mathcal{D}(\mathbf{A})$, then to avoid cumbersome notation we also denote this subset by $\mathcal{D}(\mathbf{A})$.

Remark 2.1. We will also use the notation $D^f(t)$ for the process given in (2.4) for any linear operator $\mathbf{A} : \mathcal{M}(E) \to \mathcal{M}(E)$ for which $s \to \mathbf{A}f(X(s))$ is integrable $\mathbb{P}$-a.s. on $[0, t]$ for all $t \geqslant 0$.

Remark 2.2. If $D^f(t)$ is a local martingale with respect to the augmented $\{\mathcal{F}_t^{\mathbb{P}}\}$, and $\{\mathcal{F}_t^{\mathbb{P}}\}$-stopping times in a fundamental sequence are $\{\mathcal{F}_{t+}^X\}$-stopping times, then $D^f(t)$ is also an $\{\mathcal{F}_{t+}^X\}$-martingale. In this case, the extended generator and its domain do not change under augmentation.

Remark 2.3. Note that for $f \in \mathcal{D}(\mathbf{A})$ the process $\int_0^t \mathbf{A}f(X(s))\mathrm{d}s$ is a continuous process of finite variation. Thus the process $f(X(t))$ is a semimartingale. In particular, since the local martingale $D^f(t)$ is cadlag this process is also cadlag.

3. Exponential martingale

Consider a Markov process $X(t)$ defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For a linear operator $\mathbf{A} : \mathcal{M}(E) \to \mathcal{M}(E)$ we define

$$\mathcal{M}^*(\mathbf{A}) = \Big\{ f \in \mathcal{M}(E) : f(x) \neq 0 \text{ for all } x,$$

$$\int_0^t \left| \frac{(\mathbf{A}f)(X(s))}{f(X(s))} \right| \mathrm{d}s < \infty \text{ and } \int_0^t |\mathbf{A}f(X(s))| \, \mathrm{d}s < \infty \text{ for all } t \geqslant 0, \mathbb{P}\text{-a.s.} \Big\}.$$

Moreover, for an extended generator $\mathbf{A}$, we let

$$\mathcal{D}^*(\mathbf{A}) = \mathcal{M}^*(\mathbf{A}) \cap \mathcal{D}(\mathbf{A}).$$

Thus, $E^f(t)$ defined in (1.1) makes sense for $f \in \mathcal{M}^*(\mathbf{A})$. The following result is slight extension of Proposition 3.2 in Ethier and Kurtz (1986, p. 175).

Lemma 3.1. *Let $f \in \mathcal{M}^*(\mathbf{A})$. Then the process $\{D^f(t), t \geqslant 0\}$ is a local martingale if and only if $\{E^f(t), t \geqslant 0\}$ is a local martingale.*

Proof. Assume that the process $D^f(t)$ is a local martingale. For $f \in \mathcal{M}^*(\mathbf{A})$, the process $\int_0^t (\mathbf{A}f)(X(s))/f(X(s)) \, \mathrm{d}s$ is continuous of finite variation. Hence by Jacod and Shiryaev (1987, Proposition 2.1.3, p. 75)

$$\left[f(X(t)), \exp\left(-\int_0^t \frac{(\mathbf{A}f)(X(s))}{f(X(s))} \, \mathrm{d}s \right) \right]_t = 0,$$

where $[\cdot, \cdot]_t$ is a quadratic covariation. Using the integration by parts formula for semimartingales, we have

$$\mathrm{d}E^f(t) = \frac{1}{f(X(0))} \exp\left(-\int_0^t \frac{(\mathbf{A}f)(X(s))}{f(X(s))} \, \mathrm{d}s \right) \mathrm{d}D^f(t). \tag{3.5}$$

Thus, by Dellacherie and Meyer (1982, Theorem VIII.3.e, p. 314), the process $E^f(t)$ is a local martingale. Now, assume that $E^f(t)$ is a local martingale. By (3.5) we also have that

$$\mathrm{d}D^f(t) = f(X(0)) \exp\left(\int_0^t \frac{(\mathbf{A}f)(X(s))}{f(X(s))} \, \mathrm{d}s \right) \mathrm{d}E^f(t),$$

which completes the proof. □

Proposition 3.2. *Suppose that either one of the following two conditions holds for a positive function $h \in \mathcal{D}(\mathbf{A})$:*

(M1) $h \in \mathcal{M}_b(E)$ *and* $h^{-1}\mathbf{A}h \in \mathcal{M}_b(E)$;
(M2) $h, \mathbf{A}h \in \mathcal{M}_b(E)$ *and* $\inf_x h(x) > 0$.

Then h is a good function.

Proof. A criterion of Protter (1990, Theorem 47, p. 35), and Lemma 3.1 justify the above statement. □

4. Main theorem

In this section we consider the Markov process from Section 3. Throughout this section, h is a good function and we define a new family of probabilities $\{\tilde{\mathbb{P}}_t, t \geqslant 0\}$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t} = E^h(t), \qquad t \geqslant 0.$$

We now assume that the standard set-up holds and thus there exists a unique probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}}_t = \tilde{\mathbb{P}}_{|\mathcal{F}_t}$. The Markovian property of $X(t)$ on the filtered probability space $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$ follows from Theorem 2.1.

We now quote the following useful lemma (Liptser and Shiryaev 1986, Lemma 4.5.3, p. 188):

Lemma 4.1. *An adapted process $Z(t)$ on a filtered space $(\Omega, \mathbb{F})$ is a $\tilde{\mathbb{P}}$-local martingale if $Z(t)E^h(t)$ is a $\mathbb{P}$-local martingale.*

For the operator $\tilde{\mathbf{A}}$ defined in (1.2), let

$$\mathcal{D}_{\mathbf{A}} = \left\{ f \in \mathcal{D}(\mathbf{A}) : hf \in \mathcal{D}^*(\mathbf{A}) \text{ and } \int_0^t |\tilde{\mathbf{A}} f(X(s))| \mathrm{d}s < \infty \text{ for all } t \geqslant 0, \tilde{\mathbb{P}}\text{-a.s.} \right\}.$$

For $f \in \mathcal{D}_{\mathbf{A}}$ we have $f(x) \neq 0$. Note also that if $f \in \mathcal{D}_{\mathbf{A}}$, then

$$\int_0^t \frac{(\tilde{\mathbf{A}} f)(X(s))}{f(X(s))} \mathrm{d}s = \int_0^t \frac{(\mathbf{A}(fh))(X(s))}{fh(X(s))} \mathrm{d}s - \int_0^t \frac{(\mathbf{A}h)(X(s))}{h(X(s))} \mathrm{d}s$$

is well defined. Thus for $f \in \mathcal{D}_{\mathbf{A}}$ we can define the processes

$$\tilde{E}^f(t) = \frac{f(X(t))}{f(X(0))} \exp\left(-\int_0^t \frac{(\tilde{\mathbf{A}} f)(X(s))}{f(X(s))} \mathrm{d}s \right) \tag{4.6}$$

and

$$\tilde{D}^f(t) = f(X(t)) - \int_0^t (\tilde{\mathbf{A}} f)(X(s)) \mathrm{d}s.$$

We have

$$\begin{aligned} \tilde{E}^f(t) E^h(t) &= \frac{f(X(t))h(X(t))}{f(X(0))h(X(0))} \exp\left(-\int_0^t \left[\frac{(\tilde{\mathbf{A}} f)(X(s))}{f(X(s))} + \frac{(\mathbf{A}h)(X(s))}{h(X(s))} \right] \mathrm{d}s \right) \\ &= \frac{fh(X(t))}{fh(X(0))} \exp\left(-\int_0^t \frac{(\mathbf{A}(fh))(X(s))}{fh(X(s))} \mathrm{d}s \right). \end{aligned}$$

Applying Lemma 3.1 to $f \in \mathcal{D}_{\mathbf{A}}$ and Lemma 4.1 to $Z(t) = \tilde{E}^f(t)$, we have that $\tilde{D}^f(t)$ is a local martingale. Thus $\tilde{\mathbf{A}}$ is an extended generator of the process $X(t)$ on $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$ and $\mathcal{D}_{\mathbf{A}} \subset \mathcal{D}(\tilde{\mathbf{A}})$. Unfortunately, with this approach, we cannot easily identify the domain $\mathcal{D}(\tilde{\mathbf{A}})$ of the extended generator $\tilde{\mathbf{A}}$.

Let

$$\mathcal{D}_{\mathbf{A}}^{h} = \left\{ f \in \mathcal{D}(\mathbf{A}) : fh \in \mathcal{D}(\mathbf{A}) \text{ and } \int_0^t |\tilde{\mathbf{A}} f(X(s))| \mathrm{d}s < \infty \text{ for all } t \geqslant 0, \tilde{\mathbb{P}}\text{-a.s.} \right\}$$

and

$$\mathcal{D}_{\tilde{\mathbf{A}}}^{h^{-1}} = \left\{ f \in \mathcal{D}(\tilde{\mathbf{A}}) : fh^{-1} \in \mathcal{D}(\tilde{\mathbf{A}}) \text{ and } \int_0^t |\mathbf{A} f(X(s))| \mathrm{d}s < \infty \text{ for all } t \geqslant 0, \mathbb{P}\text{-a.s.} \right\}.$$

Remark 4.1. Note that if $\inf_x h(x) > 0$ or $h \in \mathcal{C}(E)$, then by the inequality

$$|\tilde{\mathbf{A}} f| \leqslant \frac{|\mathbf{A}(fh)|}{|h|} + \frac{|\mathbf{A}h|}{|h|},$$

we have

$$\mathcal{D}_{\mathbf{A}}^{h} = \{ f \in \mathcal{D}(\mathbf{A}) : fh \in \mathcal{D}(\mathbf{A}) \}.$$

Similarly, if $h \in \mathcal{M}_{\mathrm{b}}(E)$ or $h \in \mathcal{C}(E)$, then

$$\mathcal{D}_{\mathbf{A}}^{h^{-1}} = \{ f \in \mathcal{D}(\tilde{\mathbf{A}}) : fh^{-1} \in \mathcal{D}(\tilde{\mathbf{A}}) \}.$$

Theorem 4.2. *Suppose that* $\mathcal{D}_{\mathbf{A}}^{h} = \mathcal{D}(\mathbf{A})$ *and* $\mathcal{D}_{\mathbf{A}}^{h^{-1}} = \mathcal{D}(\tilde{\mathbf{A}})$. *Then on the filtered probability space* $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$ *the process* $X(t)$ *is Markovian with extended generator* $\tilde{\mathbf{A}}$ *and* $\mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$.

Proof. Let $f \in \mathcal{D}(\mathbf{A})$. Thus $f \in \mathcal{D}_{\mathbf{A}}^{h}$ and the process $\tilde{D}^f(t)$ is well defined. Moreover, it is cadlag because, by Remark 2.3, the process $f(X(t))$ is cadlag. We will show that the process $\tilde{D}^f(t)$ is a $\tilde{\mathbb{P}}$-local martingale. In that case we also prove the inclusion $\mathcal{D}(\mathbf{A}) \subset \mathcal{D}(\tilde{\mathbf{A}})$. To prove the reverse inclusion, it suffices to change the measure by

$$\frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\tilde{\mathbb{P}}_t} = E^h(t)^{-1},$$

which can be done since h is positive. Note that $h\tilde{\mathbf{A}}h^{-1} = -h^{-1}\mathbf{A}h$ and hence

$$\tilde{E}^{h^{-1}}(t) = \frac{h^{-1}(X(t))}{h^{-1}(X(0))} \exp\left\{ -\int_0^t h\tilde{\mathbf{A}}h^{-1}(X(s))\mathrm{d}s \right\} = (E^h(t))^{-1} \tag{4.7}$$

is a $\tilde{\mathbb{P}}$-martingale. Note also that $f = 1 \in \mathcal{D}(\tilde{\mathbf{A}})$. So by the equality $\mathcal{D}_{\tilde{\mathbf{A}}}^{h^{-1}} = \mathcal{D}(\tilde{\mathbf{A}})$, we have that $h^{-1} \in \mathcal{D}^*(\tilde{\mathbf{A}})$.

We now prove that the process $\tilde{D}^f(t)$ is a local martingale. Following Lemma 4.1, it suffices to show that the process $E^h(t)\tilde{D}^f(t)$ is a $\mathbb{P}$-local martingale. By the integration by parts formula for semimartingales, we obtain

$$\tilde{D}^f(t)E^h(t) = \int_0^t \tilde{D}^f(s-)\mathrm{d}E^h(s) + \int_0^t E^h(s-)\mathrm{d}\tilde{D}^f(s) + [E^h, \tilde{D}^f]_t.$$

However, from the definition of the process $\tilde{D}^f(t)$ and the operator $\tilde{\mathbf{A}}$, we have

$$\tilde{D}^f(t) = D^f(t) - \int_0^t \frac{\langle f,\, h\rangle_{\mathbf{A}}(X(s))}{h(X(s))}\,\mathrm{d}s.$$

Hence

$$\tilde{D}^f(t)E^h(t) = \int_0^t \tilde{D}^f(s-)\mathrm{d}E^h(s) + \int_0^t E^h(s-)\mathrm{d}D^f(s) + [E^h,\, \tilde{D}^f]_t - \int_0^t E^h(s-)\frac{\langle f,\, h\rangle_{\mathbf{A}}(X(s))}{h(X(s))}\,\mathrm{d}s. \tag{4.8}$$

The first two components are local martingales. We now consider the third component. Let X^{c} be the continuous component, and X^{d} be the pure-jump component of the process $X(t)$, and let $\Delta f(X(t)) = f(X(t)) - f(X(t-))$. Then we have

$$\mathrm{d}[E^h,\, \tilde{D}^f]_t = \mathrm{d}[f(X),\, E^h]_t - \mathrm{d}\left[\int_0^t \tilde{\mathbf{A}} f(X(s))\mathrm{d}s,\, E^h(t)\right]_t.$$

Note that $\int_0^t \tilde{\mathbf{A}} f(X(s))\mathrm{d}s$ is continuous of finite variation and hence that $[\int_0^t \tilde{\mathbf{A}} f(X(s))\mathrm{d}s,\, E^h(t)]_t = 0$. Thus $\mathrm{d}[E^h,\, \tilde{D}^f]_t = \mathrm{d}[f(X),\, E^h]_t$, which by (3.5) is equal to

$$\mathrm{d}\left[f(X(t)),\, \int_0^t \frac{E^h(s-)}{h(X(s-))}\,\mathrm{d}h(X(s))\right]_t - \mathrm{d}\left[f(X(t)),\, \int_0^t \frac{E^h(s-)}{h(X(s-))}\mathbf{A}h(X(s))\mathrm{d}s\right]_t. \tag{4.9}$$

Note that

$$\int_0^t \left| \frac{1}{h(X(s-))} E^h(s-)\mathbf{A}h(X(s))\right| \mathrm{d}s \leqslant \frac{1}{h(X(0))} \exp\left\{\int_0^t \left|\frac{\mathbf{A}h(X(s))}{h(X(s))}\right| \mathrm{d}s\right\} \int_0^t |\mathbf{A}h(X(s))|\,\mathrm{d}s,$$

which is finite $\mathbb{P}$-a.s. for all $t \geqslant 0$. Thus process $\int_0^t (1/h(X(s-)))E^h(s-)\mathbf{A}h(X(s))\mathrm{d}s$ is continuous of finite variation and the second component of (4.9) is equal to 0. We have

$$\begin{aligned}
\mathrm{d}[E^h,\, \tilde{D}^f]_t &= \mathrm{d}\left[f(X^{\mathrm{c}}(t)) + f(X^{\mathrm{d}}(t)),\, \int_0^t \frac{1}{h(X(s-))} E^h(s-)\mathrm{d}(h(X^{\mathrm{c}}(s)) + h(X^{\mathrm{d}}(s)))\right]_t \\
&= \frac{1}{h(X(t-))} E^h(t-)\mathrm{d}[f(X^{\mathrm{d}}(t)),\, h(X^{\mathrm{d}}(t))]_t \\
&= \frac{1}{h(X(t-))} E^h(t-)\mathrm{d}[f(X(t)),\, h(X(t))]_t.
\end{aligned}$$

Using the integration by parts formula for semimartingales again, one obtains

$\mathrm{d}[E^h,\, \tilde{D}^f]_t$

$$
\begin{aligned}
&= \frac{1}{h(X(t-))} E^h(t-)\mathrm{d}\left(f(X(t))h(X(t)) - \int_0^t f(X(s-))\mathrm{d}h(X(s)) - \int_0^t h(X(s-))\mathrm{d}f(X(s)) \right) \\
&= \frac{1}{h(X(t-))} E^h(t-)\mathrm{d}D^{fh}(t) - \frac{1}{h(X(t-))} E^h(t-)f(X(t-))\mathrm{d}D^h(t) \\
&\quad - \frac{1}{h(X(t-))} E^h(t-)h(X(t-))\mathrm{d}D^f(t) \\
&\quad + \frac{1}{h(X(t-))} E^h(t-)((\mathbf{A}(fh))(X(t)) - f(X(t-))(\mathbf{A}h)(X(t)) \\
&\quad - h(X(t-))(\mathbf{A}f)(X(t)))\mathrm{d}t.
\end{aligned}
$$

The first three components above are all local martingales since they are the integrals with respect to local martingales. The last component is a Riemann differential, and is equal to

$$
\frac{\langle f, h\rangle_{\mathbf{A}}(X(t))}{h(X(t))} E^h(t-)\mathrm{d}t,
$$

which completes the proof in view of (4.8). □

5. Examples

5.1. Continuous-time Markov chain

The simplest case we can analyse is when $X(t)$ is a CTMC. We will only consider CTMCs with a finite state space. The modification to a countable state space is straightforward, but it would require many additional assumptions.

Let $\mathbf{Q} = (q_{ij})_{i,j=1,\ldots,\ell}$ be the intensity matrix of the process, which, in the case of the finite state space, is also the generator of the process. Here, functions f, h are column vectors. We define $fh = (f_1 h_1, \ldots, f_\ell h_\ell)^{\mathrm{T}}$. Let h be positive. Now we have the following proposition (see also Rolski *et al*. 1999, Lemma 12.3.3).

Proposition 5.1. *The new generator is* $\tilde{\mathbf{Q}} = (\tilde{q}_{ij})$, *where*

$$
\tilde{q}_{ij} = \begin{cases} q_{ij}\dfrac{h_j}{h_i}, & i \neq j, \\ -\displaystyle\sum_{k\neq i} q_{ik}\dfrac{h_k}{h_i}, & i = j. \end{cases}
$$

Proof. By inspection. □

Corollary 5.2. *If* $\mathbf{Q}$ *is irreducible and reversible with the stationary probability vector* π, *then the stationary vector* $\tilde{\pi}$ *for* $\tilde{\mathbf{Q}}$ *is given by*

$$\tilde{\pi}_i = C\frac{\pi_i h_i}{\prod_{j\neq i} h_j},$$

where C is a constant such that $\sum_{j=1}^{\ell}\tilde{\pi}_j = 1$.

5.2. Piecewise deterministic Markov process

We follow Davis's (1993) description of PDMPs. Let E be a state space consisting of pairs $x = (\nu, z)$, where ν assumes a finite number of values from a set $\mathcal{I}$ and z belongs to an open subset $\mathcal{O}_\nu$ of $\mathbb{R}^{d(\nu)}$ (note that E is a Borel space). The process $X(t)$ is determined (see Davis 1993, p. 62) by the following:

(i) $\mathcal{X} = \sum_{i=1}^{d_\nu} g^{\nu,i}(x)\partial/\partial z_i$, a vector field determining the flow $\phi_\nu(t, z)$ (we assume that $g^{\nu,i}$ are locally Lipschitz continuous);
(ii) $\lambda(\cdot)$, a force of transitions;
(iii) $Q(\cdot, \cdot)$, a transition kernel.

Denote by $\partial\mathcal{O}_\nu$ the boundary of $\mathcal{O}_\nu$ and let

$$\partial^*\mathcal{O}_\nu = \{z \in \partial\mathcal{O}_\nu : z = \phi_\nu(t, z') \text{ for some } (t, z') \in \mathbb{R}_+ \times \mathcal{O}_\nu\},$$

$$\Gamma = \{(\nu, z) \in \partial E : \nu \in \mathcal{I},\, z \in \partial^*\mathcal{O}_\nu\},$$

$$t^*(\nu, z) = \sup\{t > 0 : \phi_\nu(t, z) \text{ exists and } \phi(t, z) \in \mathcal{O}_\nu\}.$$

The set Γ is called an *active boundary.* It is the set of boundary points of E, which can be reached from E via integral curves in finite time. For each point (ν, z), $t^*(\nu, z)$ is the time needed to reach the boundary from (ν, z). We will assume that $\phi_\nu(t^*(\nu, z), z) \in \Gamma$ if $t^*(\nu, z) < \infty$, which means that the integral curves cannot 'disappear' inside E. Assume that

$$\lim_{n\to\infty} T_n = \infty, \qquad \mathbb{P}\text{-a.s.,} \tag{5.10}$$

where T_n denotes the consecutive jumps of the process $X(t)$. To obtain (5.10) we can assume that λ is bounded and that one of the following conditions is satisfied: $t^*(x) = \infty$ for each $x = (\nu, z) \in E$, that is, there are no active boundary points (e.g. $\Gamma = \varnothing$), or for some $\epsilon > 0$, we have $Q(x, B_\epsilon) = 1$ for all $x \in \Gamma$, where $B_\epsilon = \{x \in E : t^*(x) \geqslant \epsilon\}$. The last condition means that the minimal distance between consecutive boundary hitting times is not smaller than ϵ (see Davis 1993, Proposition 24.6, p. 60).

From Davis (1993, Proposition 26.14, p. 69), the formula for the extended generator is

$$\mathbf{A}f(x) = \mathcal{X}f(x) + \lambda(x)\int_E (f(y) - f(x))Q(x, \mathrm{d}y),$$

where $x \in E$ and the domain $\mathcal{D}(\mathbf{A})$ consists of every function f that is the restriction to E of a measurable function $\bar{f} : E \cup \Gamma \to \mathbb{R}$ satisfying three conditions:

(i) For each $(\nu, z) \in E$ the function $t \to \bar{f}(\nu, \phi_\nu(t, z))$ is absolutely continuous on $(0, t^*(\nu, z))$.

(ii) For each $x \in \Gamma$,

$$\bar{f}(x) = \int_E \bar{f}(y) Q(x, \mathrm{d}y). \tag{5.11}$$

(iii) For $n = 1, 2, \ldots,$

$$\mathbb{E}\left(\sum_{i=1}^{n} |\bar{f}(X(T_i)) - \bar{f}(X(T_i-))| \right) < \infty. \tag{5.12}$$

From (1.2) we have

$$(\tilde{\mathbf{A}} f)(x) = (\mathcal{X} f)(x) + \frac{\lambda(x) H(x)}{h(x)} \int_E (f(y) - f(x)) \frac{h(y)}{H(x)} Q(x, \mathrm{d}y),$$

where $H(x) = \int_E h(y) Q(x, \mathrm{d}y)$. We assume that

$$\lim_{n \to \infty} T_n = \infty, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{5.13}$$

This condition holds if, for example, $\lambda(x)$ is bounded, $\inf_x h(x) > 0$ and $\Gamma = \varnothing$. We can now conclude with the following theorem.

Theorem 5.3. *Assume that h is a good function satisfying $H(x) < \infty$ for all $x \in E$. Suppose that* (5.13) *holds, and that $\mathcal{D}^h_{\mathbf{A}} = \mathcal{D}(\mathbf{A})$ and $\mathcal{D}^{h^{-1}}_{\tilde{\mathbf{A}}} = \mathcal{D}(\tilde{\mathbf{A}})$. Then on the new probability space $(\mathcal{D}_E[0, \infty), \mathbb{F}, \tilde{\mathbb{P}})$, the process $X(t)$ is a PDMP with the unchanged differential operator $\mathcal{X}$ and the following jump intensity and transition kernel:*

$$\tilde{\lambda}(x) = \frac{\lambda(x) H(x)}{h(x)}, \qquad \tilde{Q}(x, \mathrm{d}y) = \frac{h(y)}{H(x)} Q(x, \mathrm{d}y). \tag{5.14}$$

The domain does not change: $\mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$.

Remark 5.1. Assume that condition (M1) or (M2) of Proposition 3.2 holds. Then h is a good function and $H(x) < \infty$ for all x. Moreover, if, for each $f \in \mathcal{D}(\mathbf{A})$, the function fh satisfies condition (5.11) (e.g. if there are no active boundary points: $\Gamma = \varnothing$), then $\mathcal{D}^h_{\mathbf{A}} = \mathcal{D}(\mathbf{A})$ and $\mathcal{D}^{h^{-1}}_{\tilde{\mathbf{A}}} = \mathcal{D}(\tilde{\mathbf{A}})$ by Remark 4.1. Thus, if (5.13) holds additionally, then all assumptions of Theorem 5.3 are satisfied. In particular, this is the case when $\Gamma = \varnothing$, $\lambda(x)$ is bounded and h is an absolutely continuous, bounded function such that $\inf_x h(x) > 0$.

Remark 5.2. Consider a positive function $h \in \mathcal{D}(\mathbf{A})$ which is harmonic (typically unbounded so that conditions (M1) and (M2) of Proposition 3.2 do not hold) and which belongs to the domain of the full generator of the process $X(t)$. That is, h is a good function and by Davis (1993, Remark 26.17, p. 70), it satisfies a condition stronger than (5.12), namely

$$\mathbb{E}\left(\sum_{T_n \leqslant t} |\bar{h}(X(T_i)) - \bar{h}(X(T_i-))| \right) < \infty, \tag{5.15}$$

for all $t \geqslant 0$. Assume that (5.13) holds, $H(x) < \infty$ for all $x \in E$ and fh satisfies (5.11) (e.g.

$\Gamma = \emptyset$). We restrict the domains of the extended generators $\mathbf{A}$ and $\tilde{\mathbf{A}}$ to the bounded measurable functions satisfying (i) and (ii) and we also denote these sets by $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\tilde{\mathbf{A}})$, respectively. Then by Theorem 5.3, the process $X(t)$ on $(\mathcal{D}_E[0, \infty), \mathbb{F}, \tilde{\mathbb{P}})$ is a PDMP with the parameters given in (5.14) and $\mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$. Such a framework is often used in risk theory (see Rolski *et al*. 1999, p. 460).

5.3. Diffusion processes

Let $\mathbf{a} = (a_{ij})_{i,j=1,\dots,d}$ and $b = (b_1, \dots, b_d)^{\mathrm{T}}$ be functions of $x \in \mathbb{R}^d$ satisfying the following conditions:

(i) $b_i \in \mathcal{C}(\mathbb{R}^d)$ and, for some $L > 0$,

$$|b_i(x)| \leqslant L(1 + \|x\|), \qquad x \in \mathbb{R}^d. \tag{5.16}$$

(ii) $\mathbf{a}(x)$ is a strictly positive definite matrix and

$$a_{ij} \in \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d). \tag{5.17}$$

In this subsection we assume that $X(t)$ is a Markov diffusion process on the state space $E = \mathbb{R}^d$ with a given initial state $X(0)$. We consider here $\Omega = \mathcal{C}_{\mathbb{R}^d}[0, \infty)$ and $X(\omega, t) = \omega(t)$ with the right-continuous filtration $\{\mathcal{F}_t = \mathcal{F}^X_{t+}\}_{t \geqslant 0}$. The extended generator $\mathbf{A}$ of the diffusion process is

$$(\mathbf{A}f)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}, \tag{5.18}$$

where $x \in \mathbb{R}^d$ and the family of twice continuously differentiable functions $f \in \mathcal{C}^2(\mathbb{R}^d)$ is included in the domain of this generator $\mathcal{D}(\mathbf{A})$; see Karatzas and Shreve (1988, Proposition 4.2, p. 312) or Rogers and Williams (1987, Theorem 24.1, p. 170). We will write $\mathcal{D}(\mathbf{A}) = \mathcal{C}^2(\mathbb{R}^d)$. Let $c(x) = (c_1(x_1), \dots, c_d(x_d))^{\mathrm{T}} : \mathbb{R}^d \to \mathbb{R}^d$, where

$$c_i \in \mathcal{C}^1_{\mathrm{b}}(\mathbb{R}), \qquad i = 1, \dots, d. \tag{5.19}$$

For $x_0 = (x_{0,1}, \dots, x_{0,d})$, $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$, we define

$$\int_{x_0}^{x} c(y)\mathrm{d}y = \sum_{j=1}^{d} \int_{x_{0,j}}^{x_j} c_j(y_j)\mathrm{d}y_j.$$

We use the following function $h : \mathbb{R}^d \to \mathbb{R}_+$:

$$h(x) = \exp\left(\int_{x_0}^{x} c(y)\mathrm{d}y\right). \tag{5.20}$$

For $d = 1$, we have $h(x) = \exp(\int_{x_0}^{x} c(y)\mathrm{d}y)$. Note that the function $h(x)$ satisfies

$$\frac{1}{h(x)}\frac{\partial h(x)}{\partial x_j} = c_j(x_j). \tag{5.21}$$

Moreover, we have that $h \in \mathcal{D}(\mathbf{A})$ and $h(x) \neq 0$. Hence $h \in \mathcal{D}^*(\mathbf{A})$ and thus by Lemma 3.1, the process $E^h(t)$ is a local martingale. However, a stronger result is also true. We adopt the convention that

$$(c^{\mathrm{T}}b)(x) = c(x)^{\mathrm{T}}b(x), \qquad (\mathbf{a}c)(x) = \mathbf{a}(x)c(x), \qquad (c^{\mathrm{T}}\mathbf{a}c)(x) = c(x)^{\mathrm{T}}\mathbf{a}(x)c(x).$$

Lemma 5.4. *We have*

$$E^h(t) = \exp\left(\int_0^t c(X(s))\mathrm{d}X(s) - \int_0^t (c^{\mathrm{T}}b)(X(s))\mathrm{d}s - \frac{1}{2}\int_0^t (c^{\mathrm{T}}\mathbf{a}c)(X(s))\mathrm{d}s\right), \tag{5.22}$$

where

$$\int_0^t c(X(s))\mathrm{d}X(s) = \sum_{j=1}^d \int_0^t c_j(X_j(s))\mathrm{d}X_j(s).$$

Proof. From (5.18) and (5.21) we have

$$\frac{\mathbf{A}h(x)}{h(x)} = \frac{1}{2}\sum_{i=1}^d a_{ii}(x)c_i'(x_i) + \frac{1}{2}(c^{\mathrm{T}}\mathbf{a}c)(x) + (c^{\mathrm{T}}b)(x), \tag{5.23}$$

where

$$c_i'(t) = \frac{\mathrm{d}}{\mathrm{d}z}c_i(z)_{|z=t}.$$

Then

$$E^h(t) = \exp\left(\int_{X(0)}^{X(t)} c(x)\mathrm{d}x\right)\exp\left(-\frac{1}{2}\int_0^t \sum_{i=1}^d a_{ii}(X(s))c_i'(X_i(s))\mathrm{d}s\right)$$
$$\times \exp\left(-\frac{1}{2}\int_0^t (c^{\mathrm{T}}\mathbf{a}c)(X(s))\mathrm{d}X(s) - \int_0^t (c^{\mathrm{T}}b)(X(s))\mathrm{d}s\right).$$

Thus it suffices to show that

$$\sum_{i=1}^d \int_{X_i(0)}^{X_i(t)} c_i(z)\mathrm{d}z = \sum_{i=1}^d \int_0^t c_i(X_i(s))\mathrm{d}X_i(s) + \frac{1}{2}\int_0^t \sum_{i=1}^d a_{ii}(X(s))c_i'(X_i(s))\mathrm{d}s. \tag{5.24}$$

From Stroock (1987, p. 75), we have

$$\mathrm{d}\left\langle X_i(t) - \int_0^t b_i(X(s))\mathrm{d}s\right\rangle_t = a_{ii}(X(t))\mathrm{d}t.$$

Now (5.24) follows from Itô's formula applied to the function $g(x) = \int_{x_0}^x c_i(z)\mathrm{d}z$:

$$\int_{X_i(0)}^{X_i(t)} c_i(z)\mathrm{d}z = \int_0^t c_i(X_i(s))\mathrm{d}X_i(s) + \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}x} c_i'(X_i(s))\mathrm{d}\langle X_i(s)\rangle_s$$

$$= \int_0^t c_i(X_i(s))\mathrm{d}X_i(s) + \frac{1}{2}\int_0^t c_i'(X_i(s))\mathrm{d}\left\langle X_i(s) - \int_0^s b_i(X(v))\mathrm{d}v \right\rangle_s \tag{5.25}$$

$$= \int_0^t c_i(X_i(s))\mathrm{d}X_i(s) + \frac{1}{2}\int_0^t a_{ii}(X(s))c_i'(X_i(s))\mathrm{d}s,$$

where (5.25) follows from the fact that $\int_0^t b_i(X(s))\mathrm{d}s$ has a finite variation ($b_i \in \mathcal{C}(\mathbb{R}^d)$). □

Note that by (5.17), (5.19) and Stroock (1987, p. 75) we have

$$\left\langle \int_0^t c(X(s))\mathrm{d}X(s) - \int_0^t c(X(s))b(X(s))\mathrm{d}s \right\rangle_t = \int_0^t (c^{\mathrm{T}}\mathbf{a}c)(X(s))\mathrm{d}s \leqslant K_t, \tag{5.26}$$

for some constant K_t. Thus from Rogers and Williams (1987, Theorem 37.8, p. 77), the process $E^h(t)$ is a true martingale, that is, h is a good function.

As an illustration of our Theorem 4.2, we give another proof of the Cameron–Martin–Girsanov theorem. In this theorem, we consider the new probability measure $\tilde{\mathbb{P}}$ defined by the martingale $\mathrm{d}\tilde{\mathbb{P}}_t/\mathrm{d}\mathbb{P}_t = E^h(t)$, where h is a good function of the form (5.20). We have

$$(\tilde{\mathbf{A}}f)(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(b_i(x) + \sum_{j=1}^d a_{ij}(x)\frac{1}{h(x)}\frac{\partial h(x)}{\partial x_j}\right)\frac{\partial f(x)}{\partial x_i}. \tag{5.27}$$

Theorem 5.5. *Suppose that h is a good function of the form* (5.20). *On the new probability space* ($\mathcal{C}_{\mathbb{R}^d}[0, \infty)$, $\mathbb{F}$, $\tilde{\mathbb{P}}$)*, the process $X(t)$ is a diffusion process with parameters*

$$\tilde{\mathbf{a}} = \mathbf{a}, \qquad \tilde{b} = b + \mathbf{a}c.$$

Proof. The assumptions of Theorem 4.2 are satisfied, that is, $\mathcal{D}_{\mathbf{A}}^h = \mathcal{D}(\mathbf{A}) = \mathcal{D}_{\tilde{\mathbf{A}}}^{h^{-1}} = \mathcal{D}(\tilde{\mathbf{A}}) = \mathcal{C}^2(\mathbb{R}^d)$ and we deduce $\tilde{\mathbf{a}}$ and $\tilde{b}$ from the form of the generator given in (5.27). □

Example 5.1. Take $d = 1$ and $c_i(x) = -\alpha$. In this case a diffusion process with infinitesimal variance function $a(x)$ and infinitesimal drift function $b(x)$ has new parameters $\tilde{a}(x) = a(x)$ and $\tilde{b}(x) = b(x) - \alpha a(x)$ after the exponential change of measure. In particular, if we consider Brownian motion $B(t)$, then after the exponential change of measure the process $B(t)$ is Brownian motion with drift $-\alpha$ (see Itô and Watanabe 1965).

Example 5.2. The second example concerns the so-called Gauss–Markov process $X(t)$ (see Karatzas and Shreve 1988, p. 355), which is the solution of the stochastic differential equation

$$\mathrm{d}X(t) = \mathbf{m}X(t)\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}B(t),$$

where $\mathbf{m}$, $\boldsymbol{\sigma}$ are $d \times d$ matrices and $B(t)$ is d-dimensional Brownian motion. We assume that all eigenvalues of $\mathbf{m}$ have strictly negative real parts. Then by Stroock (1987, Theorem 2.6, p. 91) the process $X(t)$ is a diffusion with

$$\mathbf{a}(x) = \boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}, \qquad b(x) = \mathbf{m}x,$$

where $x = (x_1, \ldots, x_d)^{\mathrm{T}}$. After the exponential change of measure, the process $X(t)$ is a diffusion with parameters $\tilde{\mathbf{a}}(x) = \mathbf{a}(x)$ and $\tilde{b}(x) = \mathbf{m}x + \mathbf{a}c(x)$, where $c(x) = (c_1(x_1), \ldots, c_d(x_d))^{\mathrm{T}}$ and $c_i(x_i) \in \mathcal{C}_{\mathrm{b}}^1(\mathbb{R})$. In particular, when $d = 1$, $\mathbf{m} = -\alpha \in \mathbb{R}$ and $\boldsymbol{\sigma} = \sqrt{a} \in \mathbb{R}_+$, then the Gauss–Markov process is a one-dimensional Ornstein–Uhlenbeck process. Taking $c(x) = c \in \mathbb{R}$, after the exponential change of measure, the process $X(t)$ is the diffusion with infinitesimal drift changed from $-\alpha x$ to $-\alpha x + ac$; see Kulkarni and Rolski (1994).

5.4. Markov additive process

Following Asmussen and Kella (2000), we consider a Markov additive process $X(t)$, where $X(t) = X^{(1)}(t) + X^{(2)}(t)$, and the independent processes $X^{(1)}(t)$ and $X^{(2)}(t)$ are specified by the characteristics q_{ij}, G_i, σ_i, a_i, $\nu_i(\mathrm{d}x)$ which will be defined below. Let $J(t)$ be a right-continuous, irreducible, finite state space CTMC, with $\mathcal{I} = \{1, \ldots, N\}$, and with the intensity matrix $\mathbf{Q} = (q_{ij})$. We denote the jumps of the process $J(t)$ by $\{T_i\}$ (with $T_0 = 0$). Let $\{U_n^i\}$ be i.i.d. random variables with distribution function $G_i(\cdot)$. Define the jump process by

$$X^{(1)}(t) = \sum_{n \geqslant 1} \sum_i U_n^i \mathbf{1}_{\{J(T_{n-1}) = i, T_n \leqslant t\}}.$$

For each $i \in \mathcal{I}$, let $X^i(t)$ be a Lévy process such that

$$\log \mathbb{E}\mathrm{e}^{\alpha X^i(1)} = \psi_i(\alpha) = a_i\alpha + \frac{\sigma_i^2\alpha^2}{2} + \int_{-\infty}^{\infty} (\mathrm{e}^{\alpha y} - 1 - \alpha y \mathbf{1}_{[0,1]}(|y|))\nu_i(\mathrm{d}y),$$

where $\int_{-\infty}^{\infty} \mathrm{e}^{\alpha y} y \mathbf{1}_{[0,1]}(|y|)\nu_i(\mathrm{d}y) < \infty$. Note that the process $X^i(t)$ has the Lévy characteristics $(\alpha_i t, \sigma_i^2 t, \nu_i(\mathrm{d}x))$. By $X^{(2)}(t)$ we denote the process which behaves in law like $X^i(t)$, when $J(t) = i$. Note that $(X^{(1)}(t), J(t))$ is a PDMP with extended generator

$$(\mathbf{A}^{(1)}f)(x, i) = \sum_{k=1}^N q_{ik} \int_{-\infty}^{\infty} (f(x+y, k) - f(x, i))\mathrm{d}G_i(y) = \sum_{k=1}^N q_{ik} \int_{-\infty}^{\infty} f(x+y, k)\mathrm{d}G_i(y), \tag{5.28}$$

and domain $\mathcal{D}(\mathbf{A}^{(1)})$ consisting of absolutely continuous functions for which the above integrals are finite. By Çinlar *et al*. (1980, Theorem 7.14, p. 211) and Jacod and Shiryaev (1987, Theorem 2.42, p. 86), the process $X^i(t)$ has extended generator

$$\mathbf{A}^i f(x) = a_i f'(x) + \frac{\sigma_i}{2} f''(x) + \int_{-\infty}^{\infty} (f(x+y) - f(x) - \mathbf{1}_{[0,1]}(|y|) y f'(x))\nu_i(\mathrm{d}y),$$

with domain $\mathcal{C}^2(\mathbb{R}) \subset \mathcal{D}(\mathbf{A}^i)$. By Palmowski (2002, Theorem 2.1), the process $(X^{(2)}(t), X^{(1)}(t), J(t))$ has extended generator $\mathbf{A}$ such that, for $h(x, y, i) = g(x)p(y, i)$,

$$\mathbf{A}(gp)(x,\ y,\ i) = g(x)\mathbf{A}^{(1)}p(y,\ i) + p(y,\ i)\mathbf{A}^{i}g(x),$$

for $g \in \mathcal{C}^2(\mathbb{R}) \subset \mathcal{D}(\mathbf{A}^i)$ and $p \in \mathcal{D}(\mathbf{A}^{(1)})$ (i.e., $\mathbf{A} = \mathbf{A}^{(1)} \oplus \mathbf{A}^i$ and $\mathcal{D}(\mathbf{A}^{(1)}) \otimes \mathcal{C}^2(\mathbb{R}) \subset \mathcal{D}(\mathbf{A})$). Letting $\mathbf{Q} \circ \hat{\mathbf{G}}(\alpha) = (q_{ij}\hat{G}_i(\alpha))$, where $\hat{G}_i(\alpha) = \mathbb{E}\exp(\alpha U_i)$ is assumed to be finite, we define

$$\mathbf{F}(\alpha) = \mathbf{Q} \circ \hat{\mathbf{G}}(\alpha) + \mathrm{diag}(\psi_1(\alpha),\ \dots,\ \psi_N(\alpha)).$$

The Perron–Frobenius eigenvalue and the corresponding right eigenvector of $\mathbf{F}(\alpha)$ are denoted by $\lambda(\alpha)$ and $\mathbf{h}(\alpha)$, respectively. Note that $\lambda(\alpha)$ is real and $\mathbf{h}(\alpha)$ is positive. As a good function we propose $h(x,\ y,\ i) = g(x)f(y,\ i)$, where $g(x) = \mathrm{e}^{\alpha x}$ and $p(y,\ i) = \mathrm{e}^{\alpha y}h_i(\alpha)$. Then

$$(\mathbf{A}h)(x,\ y,\ i) = \mathrm{e}^{\alpha(x+y)}(\psi_i(\alpha)h_i(\alpha) + (\mathbf{Q} \circ \hat{\mathbf{G}}(\alpha)\mathbf{h}(\alpha))_i) = \lambda(\alpha)h(x,\ y,\ i)\mathrm{e}^{\alpha(x+y)}. \tag{5.29}$$

Proposition 5.6. *The process*

$$E^h(t) = \mathrm{e}^{\alpha X(t) - \lambda(\alpha)t}\,\frac{h_{J(t)}(\alpha)}{h_{J(0)}(\alpha)}, \qquad t \geqslant 0, \tag{5.30}$$

is a mean-one martingale. Furthermore, the process $X(t)$, after the exponential change of measure, has the following characteristics:

$$\begin{aligned}
\tilde{q}_{ij} &= \frac{h_j(\alpha)}{h_i(\alpha)}\,q_{ij}\tilde{G}_i(\alpha),\\
\tilde{G}_i(\mathrm{d}x) &= \frac{\mathrm{e}^{\alpha x}}{\hat{G}_i(\alpha)}\,G_i(\mathrm{d}x),\\
\tilde{\sigma}_i &= \sigma_i,\\
\tilde{\alpha}_i &= \alpha + \alpha\sigma_i^2 - \int_{-\infty}^{\infty} |y|\mathbf{1}_{[0,1]}(|y|)(1 - \mathrm{e}^{\alpha y})\nu_i(\mathrm{d}y),\\
\tilde{\nu}_i(\mathrm{d}x) &= \mathrm{e}^{\alpha x}\nu_i(\mathrm{d}x).
\end{aligned} \tag{5.31}$$

Proof. By Lemma 3.1, $E^h(t)$ is a local martingale. Following Asmussen and Kella (2000), we have $\mathbb{E}E^h(t) = 1$ for all $t \geqslant 0$ and hence $E^h(t)$ is a true martingale. To prove (5.31), we write

$$\begin{aligned}
\mathbf{A}(fh)(x,\ y,\ i) = \mathrm{e}^{\alpha(x+y)}h_i(\alpha)\sum_{k=1}^{N}\tilde{q}_{ik}\int_{-\infty}^{\infty}[f(x,\ y+z,\ k) - f(x,\ y,\ i)]\mathrm{d}\tilde{G}_i(z)\\
+ \mathrm{e}^{\alpha(x+y)}h_i(\alpha)\tilde{\mathbf{A}}^i f(x,\ y,\ i) + \mathrm{e}^{\alpha(x+y)}\sum_{k=1}^{N} q_{ik}f(x,\ y,\ i)h_k(\alpha)\hat{G}_i(\alpha)\\
+ \mathrm{e}^{\alpha(x+y)}h_i(\alpha)f(x,\ y,\ i)\psi_i(\alpha),
\end{aligned}$$

which completes the proof in view of (5.29) and the formula $\tilde{\mathbf{A}}f = h^{-1}(\mathbf{A}(fh) - f\mathbf{A}h)$. $\square$

For a one-dimensional Lévy process ($U_n^i = 0$ and $N = 1$), taking $h(x,\ y,\ i) = h(y) = \mathrm{e}^{\alpha y}$, $h_i(\alpha) = 1$ and $\lambda(\alpha) = \psi(\alpha)$, we obtain the well-known Wald martingale

$$\mathbb{E}^h(t) = \mathrm{e}^{\alpha X(t) - \psi(\alpha)t}. \tag{5.32}$$

After the exponential change of measure, the Lévy process changes its Lévy characteristics from (a, σ, ν) to $(\tilde{a}, \sigma, \tilde{\nu})$, where

$$\tilde{a} = a + \alpha\sigma^2 - \int_{-\infty}^{\infty} |y| \mathbf{1}_{[0,1]}(|y|)(1 - e^{\alpha y})\nu(\mathrm{d}y) \tag{5.33}$$

$$\tilde{\nu}(dx) = \mathrm{e}^{\alpha x}\nu_i(\mathrm{d}x)$$

(see Küchler and Sørensen 1997, Proposition 2.1.3, p. 11).

For a multidimensional Lévy process $(X_1(t), \ldots, X_d(t))$ we take $h(x) = \mathrm{e}^{\alpha^{\mathrm{T}} x}$, where $\alpha, x \in \mathbb{R}^d$. This yields the martingale $E^h(t) = \prod_{k=1}^d \mathrm{e}^{\alpha_k X^k - \psi_k(\alpha_k)t}$; for further generalizations of this result see Küchler and Sørensen (1997, Theorem A.10.1, p. 294), Jacod and Mémin (1976, Theorem 3.3, p. 13) and Jacod and Shiryaev (1987, Theorem 3.24, p. 159).

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