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Concentration and deviation inequalities in infinite dimensions via covariance representations

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Concentration and deviation inequalities are obtained for functionals on Wiener space, Poisson space or more generally for normal martingales and binomial processes. The method used here is based on covariance identities obtained via the chaotic representation property, and provides an alternative to the use of logarithmic Sobolev inequalities. It enables the recovery of known concentration and deviation inequalities on the Wiener and Poisson space (including those given by sharp logarithmic Sobolev inequalities), and extends results available in the discrete case, i.e. on the infinite cube $\{-1, 1\}^\infty$.

Keywords: chaotic representation property; Clark formula; concentration inequalities; covariance identities; deviation inequalities

1. Introduction

The purpose of the present paper is to further explore topics in concentration and deviation inequalities, in particular in infinite-dimensional settings. Deviation and concentration have attracted a lot of attention in recent years, well summarized in Ledoux (1996b, 1999) where the reader will find up-to-date information and references. Among the various methods used to obtain these results, one that we would like to emphasize is based on covariance representations. This method has already been used in the Gaussian or more generally infinitely divisible cases in Bobkov *et al.* (2001a) and Houdré (2002). Here we tackle the infinite-dimensional case in a similar fashion, recovering the results recently obtained in Bobkov and Ledoux (1998) and Ané and Ledoux (2000) using (modified) logarithmic Sobolev inequalities, and also the stronger results of Wu (2000) obtained from sharp logarithmic Sobolev inequalities (see Corollaries 4.3 and 5.1). We also show that our method covers the discrete cube and extends the concentration inequalities of Bobkov and Ledoux (1998) to infinite dimensions (see Proposition 7.8 and Corollary 7.7).

In the next section, we briefly review the notion of the normal martingale and recall elements of its structure theory. Section 3 is devoted to concentration inequalities for normal martingales having the chaos representation property. This is then specialized in

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Section 4 to 'deterministic' structure equations that simultaneously cover the Poisson and Wiener cases. The general case of Poisson random measure on a metric space is treated in Section 5, and the gradient of Carlen and Pardoux (1990) is also used in Section 6 for the Poisson process on $\mathbb{R}_+$. Section 7 is devoted to the case of the binomial process, and includes functionals on the infinite discrete cube under non-symmetric Bernoulli measures.

2. Preliminaries: normal martingales

Let $(M_t)_{t\in\mathbb{R}_+}$ be a normal martingale, i.e. one with deterministic angle bracket $\mathrm{d}\langle M_t, M_t\rangle = \mathrm{d}t$. Let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by $(M_t)_{t\in\mathbb{R}_+}$ and let $\mathcal{F} = \bigvee_{t\in\mathbb{R}_+}\mathcal{F}_t$. The multiple stochastic integral $I_n(f_n)$ is then defined as

$$I_n(f_n) = n!\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1, \ldots, t_n)\mathrm{d}M_{t_1}\cdots\mathrm{d}M_{t_n}, \qquad f_n \in L^2(\mathbb{R}_+)^{\circ n},\ n \geqslant 1,$$

where $L^2(\mathbb{R}_+)^{\circ n}$ is the set of symmetric square-integrable functions on $\mathbb{R}_+^n$, with

$$\mathrm{E}[I_n(f_n)I_m(g_m)] = n!\mathbf{1}_{\{n=m\}}\langle f_n, g_m\rangle_{L^2(\mathbb{R}_+)^{\circ n}}. \tag{2.1}$$

We assume that $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property, i.e. every $F \in L^2(\Omega, \mathcal{F}, P)$ has a decomposition as $F = \sum_{n=0}^\infty I_n(f_n)$. Let $D\colon \mathrm{Dom}(D) \to L^2(\Omega\times\mathbb{R}_+, \mathrm{d}P\times\mathrm{d}t)$ denote the closable gradient operator defined as

$$D_tF = \sum_{n=1}^\infty nI_{n-1}(f_n(*, t)), \qquad \mathrm{d}P\times\mathrm{d}t\text{-a.e.},$$

with $F = \sum_{n=0}^\infty I_n(f_n)$. The Clark formula is a consequence of the chaos representation property for $(M_t)_{t\in\mathbb{R}_+}$, and states that any $F \in \mathrm{Dom}(D) \subset L^2(\Omega, \mathcal{F}, P)$ has a representation

$$F = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[D_tF \mid \mathcal{F}_t]\mathrm{d}M_t. \tag{2.2}$$

It admits a simple proof via the chaos expansion of F:

$$\begin{aligned} F &= \mathrm{E}[F] + \sum_{n=1}^\infty n!\int_0^\infty\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1, \ldots, t_n)\mathrm{d}M_{t_1}\cdots\mathrm{d}M_{t_n} \\ &= \mathrm{E}[F] + \sum_{n=1}^\infty n\int_0^\infty I_{n-1}(f_n(*, t_n)\mathbf{1}_{\{*<t_n\}})\mathrm{d}M_{t_n} = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[D_tF \mid \mathcal{F}_t]\mathrm{d}M_t. \end{aligned}$$

Let $(P_t)_{t\in\mathbb{R}_+}$ denote the Ornstein–Uhlenbeck semigroup, defined as

$$P_tF = \sum_{n=0}^\infty \mathrm{e}^{-nt}I_n(f_n),$$

with $F = \sum_{n=0}^\infty I_n(f_n)$.

Proposition 2.1. *Let $F, G \in \mathrm{Dom}(D)$. Then*

$$\mathrm{cov}(F,\ G) = \mathrm{E}\left[\int_0^\infty D_t F\, \mathrm{E}[D_t G \mid \mathcal{F}_t]\mathrm{d}t\right], \tag{2.3}$$

and

$$\mathrm{cov}(F,\ G) = \mathrm{E}\left[\int_0^\infty \int_0^\infty \mathrm{e}^{-s} D_u F P_s D_u G \mathrm{d}u \mathrm{d}s\right]. \tag{2.4}$$

Proof. The first identity is a consequence of the Clark formula. By orthogonality of multiple integrals of different orders and continuity of P_s on $L^2(\Omega)$, it suffices to prove the second identity for $F = I_n(f_n)$ and $G = I_n(g_n)$. But

$$\begin{aligned}\mathrm{E}[I_n(f_n)I_n(g_n)] = n!\langle f_n,\ g_n\rangle_{L^2(\mathbb{R}_+^n)} &= \frac{1}{n}\mathrm{E}\left[\int_0^\infty D_u F D_u G \mathrm{d}u\right] \\ &= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-s} \int_0^\infty D_u F P_s D_u G \mathrm{d}u \mathrm{d}s\right].\end{aligned}$$

□

If $(M_t)_{t\in\mathbb{R}_+}$ is in $L^4(\Omega, \mathcal{F}, P)$ then the chaos representation property implies that there exists a square-integrable predictable process $(\phi_t)_{t\in\mathbb{R}_+}$ such that

$$\mathrm{d}[M_t,\ M_t] = \mathrm{d}t + \phi_t \mathrm{d}M_t, \qquad t \in \mathbb{R}_+. \tag{2.5}$$

This last equation is called a structure equation; see Émery (1989). Let $i_t = \mathbf{1}_{\{\phi_t=0\}}$ and $j_t = 1 - i_t = \mathbf{1}_{\{\phi_t\neq 0\}}$, $t \in \mathbb{R}_+$. The continuous part of $(M_t)_{t\in\mathbb{R}_+}$ is given by $\mathrm{d}M_t^c = i_t \mathrm{d}M_t$ and the eventual jump of $(M_t)_{t\in\mathbb{R}_+}$ at time $t \in \mathbb{R}_+$ is given by $\Delta M_t = \phi_t$ on $\{\Delta M_t \neq 0\}$, $t \in \mathbb{R}_+$ (Émery 1989, p. 70). The following are examples of normal martingales with the chaos representation property (Émery 1989):

(a) $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic. Then $(M_t)_{t\in\mathbb{R}_+}$ can be presented as

$$\mathrm{d}M_t = i_t \mathrm{d}B_t + \phi_t(\mathrm{d}N_t - \lambda_t \mathrm{d}t), \qquad t \in \mathbb{R}_+,\ M_0 = 0, \tag{2.6}$$

with $\lambda_t = (1 - i_t)/\phi_t^2$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $(N_t)_{t\in\mathbb{R}_+}$ a Poisson process independent of $(B_t)_{t\in\mathbb{R}_+}$, with intensity $\nu_t = \int_0^t \lambda_s \mathrm{d}s$, $t \in \mathbb{R}_+$.

(b) Azéma martingales where $\phi_t = \beta M_t$, $\beta \in [-2, 0)$.

If $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function, then $i_t D_t$ is still a derivation operator, and we have the product rule

$$D_t(FG) = FD_t G + GD_t F + \phi_t D_t F D_t G, \qquad t \in \mathbb{R}_+; \tag{2.7}$$

see Proposition 1.3 in Privault (1999). In fact D_t can be written as

$$D_t = \frac{j_t}{\phi_t}\Delta_t^\phi + i_t D_t, \tag{2.8}$$

where Δ_t^ϕ is the finite difference operator defined on random functionals by addition at time t of a jump of height ϕ_t to $(M_t)_{t\in\mathbb{R}_+}$. If $\phi_t \neq 0$, this implies

$$D_t \mathrm{e}^F = \frac{\mathrm{e}^F}{\phi_t}(\mathrm{e}^{\phi_t D_t F} - 1), \tag{2.9}$$

and in the limit $\phi_t \to 0$, D_t becomes a derivation: $D_t\mathrm{e}^F = \mathrm{e}^F D_t F$.

In the deterministic case, an Ornstein–Uhlenbeck process $(X_t)_{t\in\mathbb{R}_+}$ can be associated with the semigroup $(P_s)_{s\in\mathbb{R}_+}$, and this implies the continuity of P_s.

Lemma 2.2. *Assume that $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function. For $F \in \mathrm{Dom}(D)$, we have*

$$\|P_t DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \leqslant \|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+. \tag{2.10}$$

Proof. Let $(M_t)_{t\in\mathbb{R}_+}$ be defined as in (2.6) on the product space $\Omega = \Omega_1 \times \Omega_2$ of independent Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and Poisson process $(N_t)_{t\in\mathbb{R}_+}$. The exponential vector

$$\varepsilon(f) = \sum_{n=0}^\infty \frac{1}{n!} I_n(f^{\circ n}),$$

$f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, has the probabilistic interpretation

$$\varepsilon(f) = \exp\left(\int_0^\infty i_s f(s)\mathrm{d}B(s) + \int_0^\infty j_s \log(1+\phi(s)f(s))\mathrm{d}N(s) - \frac{1}{2}\int_0^\infty i_s f(s)\mathrm{d}s - \int_0^\infty j_s \frac{f(s)}{\phi(s)}\mathrm{d}s\right).$$

Let $(X_1^t)_{t\in\mathbb{R}_+}$ and $(X_2^t)_{t\in\mathbb{R}_+}$ be respectively the classical Ornstein–Uhlenbeck process on Wiener space, and the Ornstein–Uhlenbeck process on Poisson space (Surgailis 1984). We have

$$\begin{aligned}
&\mathrm{E}[\varepsilon(f)(X_1^t, X_2^t) \mid (X_1^0, X_2^0)] \\
&\quad = \mathrm{E}\Bigg[\exp\left(\int_0^\infty i_s f(s)\mathrm{d}X_1^t(s) + \int_0^\infty j_s \log(1+\phi(s)f(s))\mathrm{d}X_2^t(s)\right. \\
&\qquad \left. - \frac{1}{2}\int_0^\infty i_s f(s)\mathrm{d}s - \int_0^\infty j_s \frac{f(s)}{\phi(s)}\mathrm{d}s\right) \mid (X_1(0), X_2(0))\Bigg] \\
&\quad = \exp\left(\int_0^\infty i_s \mathrm{e}^{-t} f(s)\mathrm{d}X_1^0(s) + \int_0^\infty j_s \log(1+\mathrm{e}^{-t}\phi(s)f(s))\mathrm{d}X_2^0(s)\right. \\
&\qquad \left. - \frac{1}{2}\int_0^\infty i_s \mathrm{e}^{-t} f(s)\mathrm{d}s - \int_0^\infty j_s \mathrm{e}^{-t}\frac{f(s)}{\phi(s)}\mathrm{d}s\right) \\
&\quad = \varepsilon(\mathrm{e}^{-t}f)(X_1^0, X_2^0) = P_t\varepsilon(f).
\end{aligned}$$

This identity extends to linear combinations of exponential vectors by linearity, and to $L^2(\Omega)$ by density and continuity of P_t. This implies that

$$\|P_t DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leqslant \|P_t |DF|_{L^2(\mathbb{R}_+)}\|_{L^\infty(\Omega)} \leqslant \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+,$$

for all $F \in \mathrm{Dom}(D)$. □

Before proceeding to general concentration inequalities for normal martingales with the chaos representation property, we note that some infinite-dimensional inequalities can be obtained from their finite-dimensional analogues. For example, if $(M_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, then D is a derivation operator whose action on cylindrical functionals of the form $F = f(I_1(e_1), \ldots, (I_1(e_n)),\ e_1, \ldots, e_n \in L^2(\mathbb{R}_+),\ f$ bounded and continuously differentiable on $\mathbb{R}^n$, is given by

$$D_t F = \sum_{i=1}^{i=n} e_i(t)\partial_i f(I_1(e_1), \ldots, I_1(e_n)), \qquad t \in \mathbb{R}_+.$$

We also have the relations

$$\|DF\|_{L^2(\mathbb{R}_+)} = |\nabla f|(I_1(e_1), \ldots, I_1(e_n)) \text{ almost surely,}$$

and

$$\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} = \|f\|_{\mathrm{Lip}}.$$

Applying the Gaussian isoperimetric inequality of Borell (1975) and Sudakov and Tsirel'son (1974) to $F = f(I_1(e_1), \ldots, I_1(e_n))$ with $\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leqslant 1$ leads to concentration inequalities. By density of the cylindrical functionals this result extends to Wiener functions F in the domain of D and satisfying the condition $\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leqslant 1$. In a similar way, the Gaussian concentration inequalities obtained in Pisier (1986), Ledoux (1999) or Bobkov *et al.* (2001a) extend to infinite dimensions.

3. Concentration inequalities in the general case

In this section we work in the general framework of normal martingales with the chaos representation property. To do so we extend some arguments due to Houdré (2002).

Lemma 3.1. *Let* $F \in \mathrm{Dom}(D)$ *be such that* $\mathrm{E}[\mathrm{e}^{t_0|F|}] < \infty$, *and* $\mathrm{e}^{sF} \in \mathrm{Dom}(D)$, $0 < s \leqslant t_0$, *for some* $t_0 > 0$. *Then*

$$\mathrm{E}[\mathrm{e}^{t(F-\mathrm{E}[F])}] \leqslant \exp\left(\int_0^t h(s)\mathrm{d}s\right), \qquad 0 \leqslant t \leqslant t_0, \tag{3.1}$$

where h *is defined as*

$$h(s) = \int_0^\infty \|D_u F\|_\infty \|\mathrm{e}^{-sF} D_u \mathrm{e}^{sF}\|_\infty \mathrm{d}u, \qquad s \in [0,\ t_0]. \tag{3.2}$$

Proof. Let us first assume that $\mathrm{E}[F] = 0$. We have

$$\begin{aligned}
\mathrm{E}[F\mathrm{e}^{sF}] &= \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_u F \mid \mathcal{F}_u]\mathrm{E}[D_u \mathrm{e}^{sF} \mid \mathcal{F}_u]\mathrm{d}u\right] \\
&= \mathrm{E}\left[\int_0^\infty D_u \mathrm{e}^{sF}\mathrm{E}[D_u F \mid \mathcal{F}_u]\mathrm{d}u\right] \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}]\int_0^\infty \|D_u F\|_\infty \|\mathrm{e}^{-sF} D_u \mathrm{e}^{sF}\|_\infty \mathrm{d}u, \qquad 0 \leqslant s \leqslant t_0.
\end{aligned}$$

In the general case, letting $L(s) = \mathrm{E}[\mathrm{e}^{s(F-\mathrm{E}[F])}]$, we have

$$\log(\mathrm{E}[\mathrm{e}^{t(F-\mathrm{E}[F])}]) = \int_0^t \frac{L'(s)}{L(s)}\mathrm{d}s \leqslant \int_0^t \frac{\mathrm{E}[(F-\mathrm{E}[F])\mathrm{e}^{s(F-\mathrm{E}[F])}]}{\mathrm{E}[\mathrm{e}^{s(F-\mathrm{E}[F])}]}\mathrm{d}s, \qquad 0 \leqslant t \leqslant t_0.$$

□

Given $F \in L^2(\Omega)$ we denote by η_F the process

$$\eta_F(t) = \mathrm{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,$$

i.e. we have

$$F = \mathrm{E}[F] + \int_0^\infty \eta_F(t)\mathrm{d}M_t.$$

A modification of the above proof to

$$\begin{aligned}
\mathrm{E}[F\mathrm{e}^{sF}] = \mathrm{E}\left[\int_0^\infty D_u \mathrm{e}^{sF}\eta_F(u)\mathrm{d}u\right] &\leqslant \mathrm{E}[\mathrm{e}^{sF}\|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^2(\mathbb{R}_+)}\|\eta_F\|_{L^2(\mathbb{R}_+)}] \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}]\|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}\|\eta_F\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))},
\end{aligned}$$

also shows that (3.1) holds with

$$h(s) = \|\eta_F\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}\|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}.$$

Various deviation inequalities can be obtained from this function; however, it will not be used any further since it does not directly involve the norm of DF.

In the next lemma we apply the semigroup correlation identity (2.4). We refer to Ledoux (2000) for other applications of semigroups, in particular to logarithmic Sobolev inequalities.

Lemma 3.2. *Let* $(P_t)_{t\in\mathbb{R}_+}$ *satisfy* (2.10). *Let* $F \in \mathrm{Dom}(D)$ *be such that* $\mathrm{E}[\mathrm{e}^{t_0|F|}] < \infty$, *and* $\mathrm{e}^{sF} \in \mathrm{Dom}(D)$, $0 < s \leqslant t_0$, *for some* $t_0 > 0$. *Then*

$$\mathrm{E}[\mathrm{e}^{t(F-\mathrm{E}[F])}] \leqslant \exp\left(\int_0^t h(s)\mathrm{d}s\right), \qquad 0 \leqslant t \leqslant t_0, \tag{3.3}$$

where h is any of the functions

$$h(s) = \|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}\|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}, \qquad s \in [0,\ t_0], \tag{3.4}$$

$$h(s) = \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad s \in [0,\ t_0]. \tag{3.5}$$

Proof. Again assume first that $\mathrm{E}[F] = 0$. If the Ornstein–Uhlenbeck semigroup satisfies (2.10), then

$$\begin{aligned}
\mathrm{E}[F\mathrm{e}^{sF}] &= \left[\int_0^\infty \mathrm{e}^{-v} \int_0^\infty D_u \mathrm{e}^{sF} P_v D_u F \,\mathrm{d}u\mathrm{d}v\right] \\
&\leqslant \mathrm{E}\left[\mathrm{e}^{sF} \|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^2(\mathbb{R}_+)} \int_0^\infty \mathrm{e}^{-v} \|P_v DF\|_{L^2(\mathbb{R}_+)} \mathrm{d}v\right] \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \left\| \int_0^\infty \mathrm{e}^{-v} P_v \|DF\|_{L^2(\mathbb{R}_+)} \mathrm{d}v \right\|_\infty \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \int_0^\infty \mathrm{e}^{-v} \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \mathrm{d}v \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}.
\end{aligned}$$

A similar argument shows that

$$\begin{aligned}
\mathrm{E}[F\mathrm{e}^{sF}] &= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-v} \int_0^\infty D_u \mathrm{e}^{sF} P_v D_u F \,\mathrm{d}u\mathrm{d}v\right] \\
&\leqslant \mathrm{E}\left[\mathrm{e}^{sF} \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \int_0^\infty \mathrm{e}^{-v} \|DFP_v DF\|_{L^1(\mathbb{R}_+)} \mathrm{d}v\right] \\
&\leqslant \mathrm{E}\left[\mathrm{e}^{sF} \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \int_0^\infty \mathrm{e}^{-v} \|DF\|_{L^2(\mathbb{R}_+)} \|P_v DF\|_{L^2(\mathbb{R}_+)} \mathrm{d}v\right] \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \left\| \int_0^\infty \mathrm{e}^{-v} P_v \|DF\|_{L^2(\mathbb{R}_+)} \mathrm{d}v \right\|_\infty \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \int_0^\infty \mathrm{e}^{-v} \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \mathrm{d}v \\
&\leqslant \mathrm{E}[\mathrm{e}^{sF}] \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}.
\end{aligned}$$

The remainder of the proof is as in Lemma 3.1. □

From these lemmas a general concentration inequality follows:

Proposition 3.3. *Let* $F \in \mathrm{Dom}(D)$ *be such that* $\mathrm{E}[\mathrm{e}^{t_0|F|}] < \infty$, *and* $\mathrm{e}^{sF} \in \mathrm{Dom}(D)$,

$0 < s \leqslant t_0$, for some $t_0 > 0$. Let h be the function defined either in (3.2), *or* (*if* $(\phi_t)_{t\in\mathbb{R}_+}$ *is deterministic*) *in* (3.4) *or* (3.5). *Then*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\int_0^x h^{-1}(s)\mathrm{d}s\right), \qquad 0 < x < h(t_0),$$

where h^{-1} is the inverse of h.

Proof. From Lemma 3.1 we have, for all $x \in \mathbb{R}_+$

$$\mathrm{e}^{tx} P(F - \mathrm{E}[F] \geqslant x) \leqslant \mathrm{E}[\mathrm{e}^{t(F-\mathrm{E}[F])}] \leqslant \mathrm{e}^{H(t)}, \qquad 0 \leqslant t \leqslant t_0,$$

with

$$H(t) = \int_0^t h(s)\mathrm{d}s, \qquad 0 \leqslant t \leqslant t_0.$$

For any $0 < t < t_0$ we have $\mathrm{d}(H(t) - tx)\mathrm{d}t = h(t) - x$, hence

$$\min_{0<t<t_0} (H(t) - tx = H(h^{-1}(x)) - xh^{-1}(x) = \int_0^{h^{-1}(x)} h(s)\mathrm{d}s - xh^{-1}(x)$$

$$= \int_0^x s\mathrm{d}h^{-1}(s) - xh^{-1}(x) = -\int_0^x h^{-1}(s)\mathrm{d}s.$$

□

4. Concentration and deviation inequalities for deterministic structure

In this section we work with $(\phi_t)_{t\in\mathbb{R}_+}$ a deterministic function, i.e. $(M_t)_{t\in\mathbb{R}_+}$ is written as in (2.6). This covers the Gaussian case for $\phi = 0$, and also the general Poisson case, as shown Section 5.

Proposition 4.1. *Let $F \in \mathrm{Dom}(D)$ be such that $\mathrm{E}[\mathrm{e}^{t_0|F|}] < \infty$, for some $t_0 > 0$. Then*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\int_0^x h^{-1}(s)\mathrm{d}s\right), \qquad 0 < x < h(t_0),$$

where h^{-1} is the inverse of any of the following functions:

$$h(t) = \int_0^\infty \frac{j_u}{|\phi_u|} \|D_u F\|_\infty (\mathrm{e}^{t|\phi_u| \, \|D_u F\|_\infty} - 1)\mathrm{d}u + t\int_0^\infty i_u \|D_u F\|_\infty^2 \mathrm{d}u, \tag{4.1}$$

$$h(t) = \|DF\|_{L^2(\Omega, L^2(\mathbb{R}_+))} \|\phi^{-1}(\mathrm{e}^{t|\phi DF|} - 1)\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \tag{4.2}$$

$$h(t) = \left\| \frac{1}{\phi DF}(\mathrm{e}^{t\phi DF} - 1) \right\|_\infty \|D_u F\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in [0, \, t_0]. \tag{4.3}$$

Proof. In the deterministic case, $\mathrm{e}^{-tF}D\mathrm{e}^{tF} \in L^2(\Omega \times \mathbb{R}_+)$, with

$$\mathrm{e}^{-tF}D_u\mathrm{e}^{tF} = \frac{j_u}{\phi_u}(\mathrm{e}^{t\phi_u D_u F} - 1) + i_u t D_u F, \qquad u \in \mathbb{R}_+, \tag{4.4}$$

which can also be written as

$$\mathrm{e}^{-tF}D_u\mathrm{e}^{tF} = \frac{1}{\phi_u}(\mathrm{e}^{t\phi_u D_u F} - 1), \tag{4.5}$$

by replacing $\phi_u^{-1}(\mathrm{e}^{t\phi_u D_u F} - 1)$ with its limit as $\phi_u \to 0$, i.e. tD_uF, if $\phi_u = 0$. All that remains is to apply Proposition 3.3. □

Note that the inequalities given by (4.1), (4.2) and (4.3) are not comparable. Using the bound

$$|\phi_u^{-1}(\mathrm{e}^{t\phi_u D_u F} - 1)| \leqslant t|D_uF|\mathrm{e}^{t|\phi_u D_u F|},$$

for all values of $\phi_u \in \mathbb{R}$, Proposition 4.1 also holds for the functions

$$h(t) = t\int_0^\infty \|D_uF\|_\infty^2 \|\mathrm{e}^{t|\phi_u D_u F|}\|_\infty \mathrm{d}u$$

and

$$h(t) = t\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}\|\mathrm{e}^{t|\phi DF|}DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in [0,\ t_0].$$

We will show in the rest of this paper many instances where we can estimate h and h^{-1}.

Proposition 4.2. *Let $F \in \mathrm{Dom}(D)$ be such that $\mathrm{E}[\mathrm{e}^{t_0|F|}] < \infty$, for some $t_0 > 0$, and $\phi_u D_u F \leqslant K(u)$ a.s., $u \in \mathbb{R}_+$, for some function $K : \mathbb{R}_+ \to \mathbb{R}$. Then*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\int_0^x h^{-1}(s)\mathrm{d}s\right), \qquad 0 < x < h(t_0),$$

where h^{-1} is the inverse of

$$h(t) = \left\|\frac{1}{K(\cdot)}(\mathrm{e}^{tK(\cdot)} - 1)\right\|_\infty \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in [0,\ t_0].$$

Proof. Since the function $x \mapsto (\mathrm{e}^x - 1)/x$ is positive and increasing on $\mathbb{R}$, we have

$$0 \leqslant \frac{\mathrm{e}^{-tF}D_u\mathrm{e}^{tF}}{D_uF} = \frac{1}{\phi_u D_u F}(\mathrm{e}^{t\phi_u D_u F} - 1) \leqslant \frac{1}{K(u)}(\mathrm{e}^{tK(u)} - 1), \qquad u \in \mathbb{R}_+,$$

and

$$\left|\frac{\mathrm{e}^{-tF}D_u\mathrm{e}^{tF}}{D_uF}\right| \leqslant \frac{1}{K(u)}(\mathrm{e}^{tK(u)} - 1), \qquad u \in \mathbb{R}_+.$$

All that remains is to apply Proposition 3.3 and Lemma 3.2. □

The following corollary is the main result of this section. It unifies the Poisson and Brownian cases, and in particular enables the recovery of the classical inequality (4.7) in the case $\phi = 0$, i.e. on Wiener space – see Pisier (1986) and Proposition 3.1 of Wu (2000), which is proved from sharp logarithmic Sobolev inequalities on Poisson space.

Corollary 4.3. *Let $F \in \mathrm{Dom}(D)$ be such that $\phi DF \leqslant K$ a.s. for some $k \geqslant 0$ and $\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} < \infty$. Then, for $x \geqslant 0$,*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right) \right)$$

$$\leqslant \exp\left(-\frac{x}{2k} \log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right) \right), \tag{4.6}$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geqslant 0$. If $K = 0$ (decreasing functionals) we have

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{2\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right). \tag{4.7}$$

Proof. We first assume that $F \in \mathrm{Dom}(D)$ is a bounded random variable. The function h defined in Proposition 4.2 satisfies

$$h(t) \leqslant \frac{1}{K}(\mathrm{e}^{tK} - 1)\|DF\|^{-2}_{L^\infty(\Omega, L^2(\mathbb{R}_+))},$$

hence

$$-\int_0^x h^{-1}(t)\mathrm{d}t \leqslant -\frac{1}{k}\int_0^x \log\left(1 + tK\|DF\|^{-2}_{L^\infty(\Omega, L^2(\mathbb{R}_+))}\right)\mathrm{d}t$$

$$= -\frac{1}{K}\left(\left(x + \frac{1}{K}\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \right) \log\left(1 + xK\|DF\|^{-2}_{L^\infty(\Omega, L^2(\mathbb{R}_+))}\right) - x \right),$$

and (4.6) holds for all $x \geqslant 0$ since F is bounded. If $K = 0$, the above proof remains valid if all terms are replaced by their limits as $K \to 0$. If $F \in \mathrm{Dom}(D)$ is not bounded the conclusion holds for $F_n = \max(-n, \min(F, n)) \in \mathrm{Dom}(D)$, $n \geqslant 1$, and $(F_n)_{n\in\mathbb{N}}$ and $(DF_n)_{n\in\mathbb{N}}$ converge to F and DF in $L^2(\Omega)$ and $L^2(\Omega \times \mathbb{R}_+)$ respectively, with $\|DF_n\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leqslant \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}$. □

The bounds (4.6) and (4.7) respectively imply $\mathrm{E}[\mathrm{e}^{\alpha|F|\log_+|F|}] < \infty$ for some $\alpha > 0$, and $\mathrm{E}[\mathrm{e}^{\alpha F^2}] < \infty$ for all $\alpha < (2\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))})^{-1}$. In particular, if F is $\mathcal{F}_T$-measurable with $DF \leqslant K$ for some $K \geqslant 0$, and if moreover $\phi_t = \phi \in \mathbb{R}_+$ is constant in $t \in \mathbb{R}_+$, then

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{T}{\phi^2} g\left(\frac{\phi x}{KT}\right) \right) \leqslant \exp\left(-\frac{x}{2K\phi} \log\left(1 + \frac{\phi x}{KT}\right) \right),$$

since $\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leqslant KT$. This improves (as in Wu 2000) the inequality

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x}{4\phi K}\log\left(1 + \frac{\phi x}{2KT}\right)\right) \tag{4.8}$$

obtained from Proposition 6.1 in Ané and Ledoux (2000) which relies on modifed (and not sharp) logarithmic Sobolev inequalities on Poisson space.

Corollary 4.4. *Let $\phi_t = \phi \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, be constant. Let $F \in \mathrm{Dom}(D)$ be such that $\|DF\|_\infty \leqslant K$ and $\|DF\|_{L^1(\mathbb{R}_+, L^\infty(\Omega))} < \infty$. Then*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{\|DF\|_{L^1(\mathbb{R}_+, L^\infty(\Omega))}}{\phi^2 K} g\left(\frac{x\phi}{\|DF\|_{L^1(\mathbb{R}_+, L^\infty(\Omega))}}\right)\right)$$

$$\leqslant \exp\left(-\frac{x}{2\phi K}\log\left(1 + \frac{x;\phi}{\|DF\|_{L^1(\mathbb{R}_+, L^\infty(\Omega))}}\right)\right),$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geqslant 0$, and we have $\mathrm{E}[\mathrm{e}^{\lambda|F|\log_+|F|}] < \infty$ for some $\lambda > 0$. If $\phi_t = 0$, $t \in \mathbb{R}_+$, and $F \in \mathrm{Dom}(D)$ is such that $\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))} < \infty$, then

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{2\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))}}\right). \tag{4.9}$$

Proof. The function defined in (4.1) of Proposition 4.1 satisfies

$$h(t) \leqslant \phi^{-1}(\mathrm{e}^{t\phi K} - 1)\|DF\|_{L^1(\mathbb{R}_+, L^\infty(\Omega))},$$

which allows the proof of Corollary 4.3 to work. In the limiting case $\phi = 0$, (4.1) gives $h(t) = t\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))}$, hence $-h^{-1}(t) = -t\|DF\|^{-2}_{L^2(\mathbb{R}_+, L^\infty(\Omega))}$. Again we may first obtain (4.9) when F is bounded and treat the general case via an approximation argument. □

Corollary 4.4 is weaker than Corollary 4.3 – however, it only relies on the Clark formula (i.e. on (4.1) and Lemma 3.1), not on the use of semigroups. For this reason it can be stated for any derivation operator D which can be used in the Clark formula. In particular, it transfers immediately to the Poisson space for the operator $\tilde{D}$; see Section 6.

5. Difference operator on Poisson space

Let X be a σ-compact metric space and let Ω^X denote the set of Radon measures

$$\Omega^X = \left\{\omega = \sum_{i=1}^{i=N} \epsilon_{t_i} : (t_i)_{i=1}^{i=N} \subset X,\ t_i \neq t_j,\ \forall i \neq j,\ N \in \mathbb{N} \cup \{\infty\}\right\},$$

where ϵ_t denotes the Dirac measure at $t \in X$. Given $A \in \mathcal{B}(X)$, let $\mathcal{F}_A = \sigma(\omega(B) : B \in \mathcal{B}(X), B \subset A)$. Let σ be a diffuse Radon measure on X, let P denote the Poisson measure with intensity σ on Ω^X and let $L^2_\sigma(X) = L^2(X, \sigma)$. The multiple Poisson stochastic integral $I_n(f_n)$ is then defined as

$$I_n(f_n)(\omega) = \int_{\Delta n} f_n(t_1, \ldots, t_n)(\omega(\mathrm{d}t_1) - \sigma(\mathrm{d}t_1)) \cdots (\omega(\mathrm{d}t_n) - \sigma(\mathrm{d}t_n)), \qquad f_n \in L^2_\sigma(X)^{\circ n},$$

with $\Delta_n = \{(t_1, \ldots, t_n) \in X^n : t_i \neq t_j, \forall i \neq j\}$, and the isometry formula

$$\mathrm{E}[I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2_\sigma(X)^{\circ n}}$$

holds true (see Nualart and Vives 1995). Moreover, every square-integrable random variable $F \in L^2(\Omega^X, P)$ admits the Wiener–Poisson decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

in series of multiple stochastic integrals. The linear closable operator

$$D: L^2(\Omega^X, P) \to L^2(\Omega^X \times X, P \otimes \sigma)$$

is defined via

$$D_t I_n(f_n)(\omega) = n I_{n-1}(f_n(*, t))(\omega), \qquad P(\mathrm{d}\omega) \otimes \sigma(\mathrm{d}t)\text{-a.e.}, \ n \in \mathbb{N}.$$

It is known – see Ito (1988) or Proposition 1 in Nualart and Vives (1995) –

$$D_t F(\omega) = F(\omega \cup \{t\}) - F(\omega), \qquad \mathrm{d}P \times \mathrm{d}t\text{-a.e.}, \ F \in \mathrm{Dom}(D),$$

where by convention we identify $\omega \in \Omega^X$ with its support. Since there exists a measurable map $\tau: X \to \mathbb{R}_+$, bijective almost everywhere, such that the Lebesgue measure is the image of σ by τ (Dieudonné 1968, p. 192), Corollaries 4.3 and 4.4 can be restated. Again we recover Proposition 3.1 of Wu (2000) in the setting of Poisson random measures on a metric space, without using (sharp) logarithmic Sobolev inequalities:

Corollary 5.1. *Let $F \in \mathrm{Dom}(D)$ be such that $DF \leqslant K$ a.s., for some $K \geqslant 0$, and $\|DF\|_{L^\infty(\Omega, L^2(X))} < \infty$. Then*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right)\right)$$

$$\leqslant \exp\left(-\frac{x}{2K} \log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right)\right),$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geqslant 0$. If $k = 0$ (decreasing functionals) we have

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{2\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right). \tag{5.1}$$

In particular, if $F = \int_X f(y)\omega(\mathrm{d}y)$ then $\|DF\|_{L^\infty(\Omega,L^2(X))} = \|f\|_{L^2(X)}$, and if $f \geqslant K$ a.s. then

$$P\left(\int_X f(y)(\omega(\mathrm{d}y) - \sigma(\mathrm{d}y)) \geqslant x\right) \leqslant \exp\left(-\frac{\int_X f^2(y)\sigma(\mathrm{d}y)}{K^2}\, g\left(\frac{xK}{\int_X f^2(y)\sigma(\mathrm{d}y)}\right)\right),$$

which covers Proposition 2 in Reynaud-Bouret (2001). If $f \leqslant 0$ a.s., then

$$P\left(\int_X f(y)(\omega(\mathrm{d}y) - \sigma(\mathrm{d}y)) \geqslant x\right) \leqslant \exp\left(-\frac{x^2}{2\int_X f^2(y)\sigma(\mathrm{d}y)}\right).$$

If $F = \int_X f(y)\omega(\mathrm{d}y)$, then $\|DF\|_{L^1(X, L^\infty(\Omega))} = \|f\|_{L^1(X)}$, and we obtain

$$P\left(\int_X f(y)(\omega(\mathrm{d}y) - \sigma(\mathrm{d}y)) \geqslant x\right) \leqslant \exp\left(-\frac{\int_X |f(y)|\sigma(\mathrm{d}y)}{\|f\|_\infty}\, g\left(\frac{x}{\int_X |f(y)|\sigma(\mathrm{d}y)}\right)\right).$$

If $f \geqslant 0$ a.s., this can be written as

$$P\left(\int_X f(y)(\omega(\mathrm{d}y) - \sigma(\mathrm{d}y)) \geqslant x\right) \leqslant \exp\left(-\frac{\mathrm{E}[F]}{\|f\|_\infty}\, g\left(\frac{x}{\mathrm{E}[F]}\right)\right).$$

By way of an application, we consider as in Reynaud-Bouret (2001) a family $(\Psi_a)_{a\in\mathbb{N}} \subset L^2(X)$ of functions with values in $[0, K]$, with $\sigma(X) < \infty$, and the functional

$$F = \sup_{a\in\mathbb{N}} \int_X \Psi_a(x)\omega(\mathrm{d}x).$$

Then

$$0 \leqslant D_x F = \sup_{a\in\mathbb{N}}\left(\int_X \Psi_a(x)\omega(\mathrm{d}x) + \Psi_a(x)\right) - \sup_{a\in\mathbb{N}} \int_X \Psi_a(x)\omega(\mathrm{d}x),$$

hence

$$0 \leqslant D_x F \leqslant \sup_{a\in\mathbb{N}} \Psi_a(x) \leqslant K,$$

and

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\sigma(X) g\left(\frac{x}{K\sigma(X)}\right)\right).$$

Moreover,

$$\begin{aligned}
\mathrm{E}[F] &= \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\sigma(X)}}{n!} \int_{X^n} \sup_{a\in\mathbb{N}} (\Psi_a(x_1) + \ldots + \Psi_a(x_n))\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_n) \\
&\geqslant \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\sigma(X)}}{n!} \int_{X^n} \sup_{a\in\mathbb{N}} \Psi_a(x_1)\sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_n) \\
&\geqslant \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\sigma(X)}}{n!} \int_{X^n} \|D_{x_1}F\|_\infty \sigma(\mathrm{d}x_1)\cdots\sigma(\mathrm{d}x_n) \\
&\geqslant \|DF\|_{L^1(X,L^\infty(\Omega))} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\sigma(X)}}{n!} (\sigma(X))^{n-1} \\
&\geqslant \frac{1}{\sigma(X)} \|DF\|_{L^1(X,L^\infty(\Omega))}(1-\mathrm{e}^{-\sigma(X)}).
\end{aligned}$$

Hence

$$\|DF\|_{L^1(X,L^\infty(\Omega))} \leqslant \frac{\sigma(X)}{1-\mathrm{e}^{-\sigma(X)}}\mathrm{E}[F]$$

and

$$P(F-\mathrm{E}[F]\geqslant x) \leqslant \exp\left(-\frac{\sigma(X)}{K(1-\mathrm{e}^{-\sigma(X)})}\mathrm{E}[F] g\left(\frac{x(1-\mathrm{e}^{-\sigma(X)})}{\sigma(X)\mathrm{E}[F]}\right)\right).$$

6. Local gradient on Poisson space

In the Poisson case, if $X = \mathbb{R}_+$ and σ is the Lebesgue measure, then a local gradient can be introduced (Carlen and Pardoux 1990; Elliott and Tsoi 1993; Privault 1994a). let $(T_k)_{k\geqslant 1}$ denote the jump times of the canonical Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and let $\tau_k = T_k - T_{k-1}$, $k \geqslant 1$, denote its interjump times, with $T_0 = 0$. Let $\mathcal{S}$ denote the set of smooth random functionals F of the form

$$F = f(\tau_1, \ldots, \tau_n), \qquad n \geqslant 1,$$

where f is of class $\mathcal{C}^1$ on $\mathbb{R}^n$ and has compact support. Let $\tilde{D}$ denote the closable gradient defined as

$$\tilde{D}_t F = -\sum_{k=1}^{n} \mathbf{1}_{[T_k, T_{i+1}[}(t) \partial_k f(\tau_1, \ldots, \tau_n), \qquad t \in \mathbb{R}_+,\ F \in \mathcal{S}.$$

Then the relation $\mathrm{E}[D_t F \,|\, \mathcal{F}_t] = \mathrm{E}[\tilde{D}_t F \,|\, \mathcal{F}_t]$, holds $t \in \mathbb{R}_+$, and the Clark formula can be written for $F \in \mathrm{Dom}(\tilde{D})$ as

$$F = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[\tilde{D}_t F \,|\, \mathcal{F}_t] \mathrm{d}(N_t - t); \tag{6.1}$$

see Theorem 1 in Privault (1994a).

Corollary 6.1. *Let* $F \in \mathrm{Dom}(\tilde{D})$. *We have*

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{2\|\tilde{D}F\|^2_{L^2(\mathbb{R}_+, L^\infty(\Omega))}} \right), \tag{6.2}$$

and

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{4\|\tilde{D}F\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right). \tag{6.3}$$

Proof. For (6.2) we note that the Wiener space proof of Corollary 4.4 is valid on Poisson space since $\tilde{D}$ satisfies the chain rule of derivation and the Clark formula (6.1).

Concerning (6.3), we construct the exponential random variables $(\tau_k)_{k \geqslant 1}$ as half sums of squared independent Gaussian random variables. Let $F = f(\tau_1, \ldots, \tau_n)$, and consider the Wiener functional ΘF given by

$$\Theta F = f\left(\frac{x_1^2 + y_1^2}{2}, \ldots, \frac{x_n^2 + y_n^2}{2} \right),$$

where $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ denote two independent collections of normal random variables that may be constructed as Brownian single stochastic integrals. Using the fact that F and ΘF have same law, and the relation

$$2\Theta |\tilde{D}F|^2_{L^2(\mathbb{R}_+)} = |\hat{D}\Theta F|^2_{L^2(\mathbb{R}_+)} \tag{6.4}$$

(see Lemma 1 in Privault 1994b), the application on Wiener space of Corollary 4.3 to ΘF yields (6.3). $\square$

The bounds (6.2) and (6.3) imply the exponential integrability $\mathrm{E}[\mathrm{e}^{\alpha F^2}] < \infty$ for all $\alpha < (2\|\tilde{D}F\|^2_{L^2(\mathbb{R}_+, L^\infty(\Omega))})^{-1}$ and $\alpha < (4\|\tilde{D}F\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))})^{-1}$, respectively. The above results can be obtained from logarithmic Sobolev inequalities, i.e. by application of Corollary 2.5 Ledoux (1999) to Theorem 0.7 in Ané (2000); see (4.4) in Ledoux (1999) for a formulation in terms of exponential random variables.

7. Discrete settings

The covariance representations (2.3) and (2.4) which lead to the concentration and deviation inequalities of the previous sections have versions in discrete settings. Our purpose is now to explore the consequences of such representations. We consider the discrete structure equation

$$Y_k^2 = 1 + \varphi_k Y_k, \qquad k \in \mathbb{N}, \tag{7.1}$$

i.e. $(\varphi_k)_{k\in\mathbb{N}}$ is a deterministic sequence of real numbers and $(Y_k)_{k\geqslant 1}$ is a sequence of centred independent random variables. Since (7.1) is a second-order equation, there is a family $(X_k)_{k\geqslant 1}$ of independent Bernoulli $\{-1, 1\}$-valued random variables such that

$$Y_k = \frac{\varphi_k + X_k\sqrt{\varphi_k^2 + 4}}{2}, \qquad k \geqslant 1.$$

The family $(X_k)_{k\in\mathbb{N}}$ is constructed as a family of canonical projections on $\Omega = \{-1, 1\}^{\mathbb{N}}$, under the measure P determined by condition (7.1) and the fact that $\mathrm{E}[Y_k] = 0$ (which implies that $\mathrm{E}[Y_k^2] = 1$), i.e.

$$p_k = P(X_k = 1) = P\left(Y_k = \sqrt{\frac{q_k}{p_k}}\right) = \frac{1}{2} - \frac{\varphi_k}{2\sqrt{\varphi_k^2 + 4}}, \qquad k \in \mathbb{N},$$

and

$$q_k = P(X_k = -1) = P\left(Y_k = -\sqrt{\frac{p_k}{q_k}}\right) = \frac{1}{2} + \frac{\varphi_k}{2\sqrt{\varphi_k^2 + 4}}, \qquad k \in \mathbb{N}.$$

Let $J_n(f_n)$ denote the multiple stochastic integral of $f_n \in \ell^2(\mathbb{N})^{\circ n}$ (the space of square-summable symmetric functions on $\mathbb{N}^n$), defined as

$$J_n(f_n) = \sum_{(k_1,\ldots,k_n)\in\Delta_n} f_n(k_1, \ldots, k_n) Y_{k_1} \ldots Y_{k_n},$$

where

$$\Delta_n = \{(k_1, \ldots, k_n) \in \mathbb{N}^n : k_i \neq k_j, 1 \leqslant i < j \leqslant n\},$$

with the isometry

$$\mathrm{E}[J_n(f_n)J_m(g_m)] = n!\mathbf{1}_{\{n=m\}}\langle \mathbf{1}_{\Delta_n} f_n, g_m\rangle_{\ell^2(\mathbb{N})^{\otimes n}}.$$

We have

$$J_n(f_n) = n! \sum_{k_n=0}^{\infty} \sum_{0\leqslant k_{n-1}<k_n} \cdots \sum_{0\leqslant k_1<k_2} f_n(k_1, \ldots, k_n) Y_{k_1} \cdots Y_{k_2}. \tag{7.2}$$

Let $S_n = \sum_{k=0}^{k=n} (X_k + 1)/2$ be the random walk associated with $(X_k)_{k\geqslant 0}$ (see also Holden *et al.* 1992; Leitz-Martini 2000). If $p_k = p$ and $q_k = q$, $k \in \mathbb{N}$, then $J_n(\mathbf{1}_{[0,N]}{}^n)$ is the

Krawtchouk polynomial $K_n(S_N; N+1, p)$ of order n, with parameter $(N+1, p)$ (see Privault and Schoutens 2002). The set $\mathcal{P}$ of polynomials in $X_1, X_2, X, \ldots$ is dense in $L^2(\Omega, P)$, hence any $F \in L^2(\Omega, P)$ can be represented as a series of multiple stochastic integrals:

$$F = \sum_{n=0}^{\infty} J_n(f_n), \qquad f_k \in \ell^2(\mathbb{N})^{\circ k}, \quad k \geqslant 0, \qquad J_0(f_0) = \mathrm{E}[F].$$

Definition 7.1. *We densely define the linear gradient operator* $D: L^2(\Omega) \to L^2(\Omega \times \mathbb{N})$ *as*

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k)\mathbf{1}_{\Delta_n}(*, k)), \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}, \; n \in \mathbb{N}.$$

We have, for $(k_1, \ldots, k_n) \in \Delta_n$,

$$D_k\left(\prod_{i=1}^{n} Y_{k_i}\right) = \mathbf{1}_{\{l \in \{k_1, \ldots, k_n\}\}} \prod_{\substack{i=1 \\ k_i \neq k}}^{n} Y_{k_i},$$

hence the probabilistic interpretation of D_k is

$$D_k F(S.) = \sqrt{p_k q_k}\big(F\big(S. + \mathbf{1}_{\{X_i=-1\}}\mathbf{1}_{\{k \leqslant \cdot\}}\big) - F\big(S. - \mathbf{1}_{\{X_i=1\}}\mathbf{1}_{\{k \leqslant \cdot\}}\big)\big).$$

When restricted to cylindrical functionals of the form $F = f(X_1, \ldots, X_n)$, the gradient D is the finite difference operator

$$D_k F = \sqrt{p_k q_k}(f(X_1, \ldots, X_{k-1}, +1, X_{k+1}, \ldots, X_n) - f(X_1, \ldots, X_{k-1}, -1, X_{k+1}, \ldots, X_n)),$$

which (in the symmetric case $p_k = q_k = \frac{1}{2}$, $k \in \mathbb{N}$) is the operator considered in Bobkov *et al.* (2001a). The operator D does not satisfy the same product rules as in the continuous-time case (2.7); instead we have the following:

Proposition 7.2. *Let* $F, G: \Omega \to \mathbb{R}$. *Then*

$$D_k(FG) = F D_k G + G D_k F - \frac{X_k}{\sqrt{p_k q_k}} D_k F D_k G, \qquad k \geqslant 0,$$

and

$$D_k \mathrm{e}^F = -X_k \sqrt{p_k q_k}\, \mathrm{e}^F \left(\exp - \frac{X_k}{\sqrt{p_k q_k}} D_k F - 1\right). \tag{7.3}$$

Proof. Let $F_+^k = F(S. + \mathbf{1}_{\{X_k=-1\}}\mathbf{1}_{\{k \leqslant \cdot\}})$ and $F_k^- = F(S. - \mathbf{1}_{\{X_k=1\}}\mathbf{1}_{\{k \leqslant \cdot\}})$, $k \geqslant 0$. We have

$$
\begin{aligned}
D_k(FG) &= \sqrt{p_k q_k}(F_k^+ G_k^+ - F_k^- G_k^-) \\
&= \mathbf{1}_{\{X_k=-1\}}\sqrt{p_k q_k}(F(G_k^+ - G) + G(F_k^+ - F) + (F_k^+ - F)(G_k^+ - G)) \\
&\quad + \mathbf{1}_{\{X_k=1\}}\sqrt{p_k q_k}(F(G - G_k^-) + G(F - F_k^-) - (F - F_k^-)(G - G_k^-)) \\
&= \mathbf{1}_{\{X_k=-1\}}\left(FD_k G + GD_k F + \frac{1}{\sqrt{p_k q_k}} D_k F D_k G \right) \\
&\quad + \mathbf{1}_{\{X_k=1\}}\left(FD_k G + GD_k F - \frac{1}{\sqrt{p_k q_k}} D_k F D_k G \right).
\end{aligned}
$$

We have

$$
\begin{aligned}
D_k \mathrm{e}^F &= \mathbf{1}_{\{X_k=1\}}\sqrt{p_k q_k}(\mathrm{e}^F - \mathrm{e}^{F_k^-}) + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_k q_k}(\mathrm{e}^{F_k^+} - \mathrm{e}^F) \\
&= \mathbf{1}_{\{X_k=1\}}\sqrt{p_k q_k}\ \mathrm{e}^F \left(1 - \exp - \frac{1}{\sqrt{p_k q_k}} D_k F \right) \\
&\quad + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_k q_k}\, \mathrm{e}^F \left(\exp \frac{1}{\sqrt{p_k q_k}} D_k F - 1 \right) \\
&= -X_k \sqrt{p_k q_k}\, \mathrm{e}^F \left(\exp - \frac{X_k}{\sqrt{p_k q_k}} D_k F - 1 \right).
\end{aligned}
$$

□

The next result is the predictable representation of the functionals of $(S_n)_{n\geqslant 0}$. Let $\mathcal{F}_N = \sigma(X_0, \ldots, X_N)$, $N \in \mathbb{N}$.

Proposition 7.3. *We have the Clark formula*

$$
F = \mathrm{E}[F] + \sum_{k=1}^{\infty} \mathrm{E}[D_k F \,|\mathcal{F}_{k-1}] Y_k, \qquad F \in L^2(\Omega).
$$

Proof. For $F = J_n(f_n)$ we have, using (7.2) (see Privault and Schoutens 2002),

$$
\begin{aligned}
F = J_n(f_n) = n! J_n(f_n \mathbf{1}_{\Delta_n}) &= n \sum_{k=1}^{\infty} J_{n-1}(f_n(\cdot, k)\mathbf{1}_{[1,k-1]^{n-1}}(\cdot)) Y_k \\
&= \sum_{k=1}^{\infty} \mathrm{E}[D_k J_n(f_n) \,|\, \mathcal{F}_{k-1}] Y_k.
\end{aligned}
$$

This identity also shows that $F \mapsto \mathrm{E}[D.F \,|\, \mathcal{F}_{\cdot-1}]$ has norm equal to one as an operator from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{N})$:

$$
\|\mathrm{E}[D.F \,|\, \mathcal{F}_{\cdot-1}]\|^2_{L^2(\Omega\times\mathbb{N})} = \|F - \mathrm{E}[F]\|^2_{L^2(\Omega)} \leqslant \|F - \mathrm{E}[F]\|^2_{L^2(\Omega)} + \mathrm{E}[F]^2 \leqslant \|F\|^2_{L^2(\Omega)},
$$

hence the Clark formula extends to $F \in L^2(\Omega)$. $\square$

The Clark formula implies the covariance identity

$$\operatorname{cov}(F, G) = \mathrm{E}\left[\sum_{k=1}^{\infty} D_k F \,\mathrm{E}[D_k G \mid \mathcal{F}_{k-1}]\right], \tag{7.4}$$

and we also have, as in the continuous-time case,

$$\operatorname{cov}(F, G) = \mathrm{E}\left[\sum_{k=1}^{\infty} \int_0^{\infty} \mathrm{e}^{-s} D_k F P_s D_k G \mathrm{d}s\right], \tag{7.5}$$

where $(P_t)_{t\in\mathbb{R}_+}$ denotes the semigroup

$$P_t F = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} J_n(f_n), \qquad t \in \mathbb{R}_+, \qquad F = \sum_{n=0}^{\infty} J_n(f_n).$$

The next result shows that the semigroup $(P_t)_{t\in\mathbb{R}_+}$ admits a representation by a probability kernel and an Ornstein–Uhlenbeck type process which (in the symmetric case $p_k = q_k = \frac{1}{2}$, $k \in \mathbb{N}$) is in fact the Brownian motion on $\{-1, 1\}^N$ considered in Ané and Ledoux (2000).

Proposition 7.4. *For $F \in L^2(\Omega, \mathcal{F}_N)$,*

$$P_t F(\omega') = \int_\Omega F(\omega) q_t^N(\omega, \omega') \mathrm{d}P(\omega), \qquad \omega, \omega' \in \Omega, \tag{7.6}$$

where $q_t^N(\omega, \omega')$ is the kernel

$$q_t^N(\omega, \omega') = \prod_{i=1}^{i=N} (1 + \mathrm{e}^{-t} Y_i(\omega) Y_i(\omega')), \qquad \omega, \omega' \in \Omega.$$

Proof. Since $L^2(\Omega, \mathcal{F}_N)$ is finite (2^{N+1}-dimensional) it suffices to consider the functional $Y_{k_1} \ldots Y_{k_n}$ with $(k_1, \ldots, k_n) \in \Delta_n$. We have, for $\omega' \in \Omega$, $k \in \mathbb{N}$,

$$\begin{aligned} &\mathrm{E}[Y_k(\cdot)(1 + \mathrm{e}^{-t} Y_k(\cdot) Y_k(\omega'))] \\ &\quad = p_k \sqrt{\frac{q_k}{p_k}} \left(1 + \mathrm{e}^{-t} \sqrt{\frac{q_k}{p_k}} Y_k(\omega')\right) - q_k \sqrt{\frac{p_k}{q_k}} \left(1 - \mathrm{e}^{-t} \sqrt{\frac{p_k}{q_k}} Y_k(\omega')\right) = \mathrm{e}^{-t} Y_k(\omega'), \end{aligned}$$

which implies, by independence of $(X_k)_{k\in\mathbb{N}}$,

$$P_t(Y_{k_1} \ldots Y_{k_n})(\omega') = \mathrm{e}^{-nt} Y_{k_1}(\omega') \ldots Y_{k_n}(\omega') = \mathrm{E}[Y_{k_1} \ldots Y_{k_n} q_t^N(\cdot, \omega')], \qquad \omega' \in \Omega.$$

$\square$

The Ornstein–Uhlenbeck process $((X_k^t)_{k\in\mathbb{N}})_{t\in\mathbb{R}_+}$ associated with $(P_t)_{t\in\mathbb{R}_+}$ satisfies

$$P(X_k^t = 1 \mid X_k^0 = 1) = p_k + \mathrm{e}^{-t} q_k, \qquad P(X_k^t = -1 \mid X_k^0 = 1) = q_k(1 - \mathrm{e}^{-t}),$$

$$P(X_k^t = 1 \mid X_k^0 = -1) = p_k(1 - \mathrm{e}^{-t}), \qquad P(X_k^t = -1 \mid X_k^0 = -1) = q_k + \mathrm{e}^{-t} p_k, \quad k \in \mathbb{N}.$$

In other words, the hitting time $\tau_{1,-1} \in \mathbb{R}_+ \cup \{+\infty\}$ of -1 starting from +1 has distribution

$$P(\tau_{1,-1} < t) = q_k(1 - \mathrm{e}^{-t}), \qquad t \in \mathbb{R}_+,$$

while the hitting time $\tau_{-1,1}$ of +1 starting from -1 has distribution

$$P(\tau_{-1,1} < t) = p_k(1 - \mathrm{e}^{-t}), \qquad t \in \mathbb{R}_+.$$

The covariance identity (7.5) and the representation (7.6) imply the inequality

$$\|P_s DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leqslant \|P_s |DF|_{\ell^2(\mathbb{N})}\|_{L^\infty(\Omega)} \leqslant \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \qquad s \in \mathbb{R}_+,$$

for $F \in \mathrm{Dom}(D)$, hence the next proposition can be proved in a way similar to Proposition 3.3.

Proposition 7.5. *Let* $F \in \mathrm{Dom}(D)$. *Then*

$$\mathrm{E}[\mathrm{e}^{t(F - E[F])}] \leqslant \exp\left(\int_0^t h(s)\mathrm{d}s\right), \qquad 0 \leqslant t \leqslant t_0, \tag{7.7}$$

where h *is any of the following functions:*

$$h(s) = \sum_{k=0}^{\infty} \|D_k F\|_\infty \|\mathrm{e}^{-sF} D_k \mathrm{e}^{sF}\|_\infty, \tag{7.8}$$

$$h(s) = \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \|\mathrm{e}^{-sF} D\mathrm{e}^{sF}\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \tag{7.9}$$

$$h(s) = \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \qquad s \in [0,\ t_0]. \tag{7.10}$$

Although D does not satisfy the same product rule as in the continuous case, from (7.3) we still have the bound

$$|\mathrm{e}^{-sF} D_k \mathrm{e}^{sF}| \leqslant \sqrt{p_k q_k}\left(\exp\left(\frac{s}{\sqrt{p_k q_k}} |D_k F|\right) - 1\right), \qquad k \in \mathbb{N}, \tag{7.11}$$

which gives the following corollary to Proposition 7.5.

Corollary 7.6. *Let* $F \in \mathrm{Dom}(D)$. *Then*

$$\mathrm{E}[\mathrm{e}^{t(F - \mathrm{E}[F])}] \leqslant \exp\left(\int_0^t h(s)\mathrm{d}s\right), \qquad 0 \leqslant t \leqslant t_0, \tag{7.12}$$

where h *is any of the following functions:*

$$h(s) = \sum_{k=0}^{\infty} \|D_k F\|_{\infty} \left\| \sqrt{p_k q_k} \left(\exp\left(\frac{s}{\sqrt{p_k q_k}} |D_k F| \right) - 1 \right) \right\|_{\infty}, \tag{7.13}$$

$$h(s) = \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \left\| \sqrt{p_\cdot q_\cdot} \left(\exp\left(\frac{s}{\sqrt{p_\cdot q_\cdot}} |D_\cdot F| \right) - 1 \right) \right\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \tag{7.14}$$

$$h(s) = \left\| \sqrt{p_k q_k} \frac{1}{D_k F} \left(\exp\left(\frac{s}{\sqrt{p_k q_k}} |D_k F| \right) - 1 \right) \right\|_{\infty} \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \qquad s \in [0,\ t_0]. \tag{7.15}$$

Again, the inequalities given by (7.13), (7.14) and (7.15) are not comparable. The bound

$$\sqrt{p_k q_k} \left(\exp\left(\frac{s}{\sqrt{p_k q_k}} |D_k F| \right) - 1 \right) \leqslant s|D_k F| \exp\left(\frac{s}{\sqrt{p_k q_k}} |D_k F| \right), \qquad k \in \mathbb{N},$$

also shows that Corollary 7.6 holds with

$$h(s) = s \sum_{k=0}^{\infty} \|D_k F\|^2_{\infty} \left\| \exp \frac{s}{\sqrt{p_k q_k}} |D_k F| \right\|_{\infty},$$

and

$$h(s) = s \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \| \exp \frac{s}{\sqrt{p_\cdot q_\cdot}} |D_\cdot F| . |D_\cdot F\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \qquad s \in [0,\ t_0].$$

The following corollary is obtained with the same proof as in the Poisson space.

Corollary 7.7. *Let* $F \in \mathrm{Dom}(D)$ *be such that* $(1/\sqrt{p_k q_k})|D_k F| \leqslant K$, $k \in \mathbb{N}$, *for some* $K \geqslant 0$, *and* $\|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} < \infty$. *Then*

$$\begin{aligned} P(F - \mathrm{E}[F] \geqslant x) &\leqslant \exp\left(- \frac{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}} \right) \right) \\ &\leqslant \exp\left(- \frac{x}{2K} \log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}} \right) \right), \end{aligned}$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geqslant 0$.

Proof. Use the inequality

$$0 \leqslant \frac{\mathrm{e}^{-sF} D_k \mathrm{e}^{sF}}{D_k F} = -X_k \sqrt{p_k q_k} \frac{1}{D_k F} \left(\exp\left(\frac{-sX_k}{\sqrt{p_k q_k}} D_k F \right) - 1 \right) \leqslant \frac{\mathrm{e}^{sK} - 1}{K},$$

and apply Corollary 7.6. □

If $p_k = p$ and $q_k = q$, for all $k \in \mathbb{N}$, the conditions $(1/\sqrt{pq})|D_k F| \leqslant \beta$, $k \in \mathbb{N}$, and $\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leqslant \alpha^2$ give

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{\alpha^2 pq}{\beta^2} g\left(\frac{x\beta}{\alpha^2 pq}\right)\right) \leqslant \exp\left(-\frac{x}{2\beta}\log\left(1+\frac{x\beta}{\alpha^2 pq}\right)\right),$$

which is relation (13) obtained on $\{0, 1\}^n$ in Bobkov and Ledoux (1998). In particular, if F is $\mathcal{F}_N$-measurable, then

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-Ng\left(\frac{x}{\beta N}\right)\right) \leqslant \exp\left(-\frac{x}{\beta}\left(\log\left(1+\frac{x}{\beta N}\right)-1\right)\right).$$

Finally, we obtain a Gaussian concentration inequality for functions of $(S_n)_{n\in\mathbb{N}}$, using the covariance identity (7.4). We refer to Bobkov (1995), Ledoux (1996a), Bobkov *et al.* (2001b) and Houdré and Tetali (2001) for other and better versions of this inequality obtained by different methods.

Proposition 7.8. *Let* $F : \Omega \to \mathbb{R}$ *be such that*

$$\left\| \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} |D_k F| \|D_k F\|_\infty \right\|_\infty \leqslant K^2.$$

Then

$$P(F - \mathrm{E}[F] \geqslant x) \leqslant \exp\left(-\frac{x^2}{2K^2}\right), \qquad x; \geqslant 0.$$

Proof. Using the inequality

$$|\mathrm{e}^{tx} - \mathrm{e}^{ty}| \leqslant \frac{1}{2} t|x-y|(\mathrm{e}^{tx} + \mathrm{e}^{ty}), \qquad x, y \in \mathbb{R}, \tag{7.16}$$

we have

$$\begin{aligned} |D_k \mathrm{e}^{tF}| &= \sqrt{p_k q_k}|\mathrm{e}^{tF_k^+} - \mathrm{e}^{tF_k^-}| \leqslant \frac{1}{2}\sqrt{p_k q_k}\, t|F_k^+ - F_k^-|(\mathrm{e}^{tF_k^+} + \mathrm{e}^{tF_k^-}) \\ &= \frac{1}{2} t|D_k F|(\mathrm{e}^{tF_k^+} + \mathrm{e}^{tF_k^-}) \leqslant \frac{1}{2(p_k \wedge q_k)} t|D_k F|\mathrm{E}[\mathrm{e}^{tF}|X_i, i \neq k] \\ &= \frac{1}{2(p_k \wedge q_k)} t\mathrm{E}[\mathrm{e}^{tF}|D_k F|\,|X_i, i \neq k], \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{E}[F\mathrm{e}^{tF}] &= \sum_{k=0}^{\infty} \mathrm{E}[\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k \mathrm{e}^{tF}] \leqslant \sum_{k=0}^{\infty} \|D_k F\|_\infty \mathrm{E}[|D_k \mathrm{e}^{tF}|] \\
&< t \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} \|D_k F\|_\infty \mathrm{E}[\mathrm{E}[\mathrm{e}^{tF} |D_k F| \, | X_i, \, i \neq k]] \\
&= t\mathrm{E}\left[\mathrm{e}^{tF} \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} \|D_k F\|_\infty |D_k F|\right] \\
&\leqslant t\mathrm{E}[\mathrm{e}^{tF}] \left\| \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} |D_k F| \, \|D_k F\|_\infty \right\|_\infty .
\end{aligned}
$$

We can conclude as in the proof of Corollary 4.4. □

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