

## ON $so_8$ AND THE TENSOR OPERATORS OF $sl_3$

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Physicists motivated by problems in high energy physics have for many years been groping their way toward a theory of tensor operators of simple lie algebras. Their work, calculations, insights, theorems, and guesses have appeared in the physics literature in a form not easily understood by either mathematicians or physicists [2, 3]. Extensive conversation with L. C. Biedenharn, a chief worker in this field, has convinced me that it is a mathematical gold mine. It has inspired the work described here.

In this announcement the tensor operators of  $sl_3$  are analyzed in terms of a beautiful algebraic structure involving  $so_8$ , whose existence was previously unsuspected even by physicists.

Proofs will appear in [1 and 5].

The fundamental problem is the *explicit* decomposition of all finite dimensional tensor product representations  $V \otimes W$  of a simple lie algebra  $g$ . For  $g = sl_2$ , the famous Clebsch-Gordan coefficients provide a complete solution.

We shall study the equivalent problem of decomposing all the  $\text{Hom}_{\mathbb{C}}(V, W)$ , the spaces of "tensor operators".

A tensor operator is an irreducible subrepresentation of  $\text{Hom}_{\mathbb{C}}(V, W)$  where  $V$  and  $W$  are irreducible  $g$ -representations.

Irreducible representations of  $g$  are labelled by their highest weights. To a tensor operator one can assign two weights: (1) the highest weight of the tensor operator as an irreducible  $g$ -representation; (2) the weight which is the difference of the highest weights of  $V$  and  $W$ , the "shift in representations" effected by the tensor operator. Tensor operators with the same highest weights but different "shift weights" map between different spaces and would therefore seem to be unrelated. But the fact that the shift weight of a tensor operator is necessarily an actual weight of the tensor operator as a representation suggests otherwise. Much work, including my own, can be described as the pursuit of the analogy between the pairs "actual weight and highest weight" and "shift weight and highest weight." The principal difficulty is that for  $g$  other than  $sl_2$  irreducible representations may have weights with multiplicity greater than one.

Denote by  $V_\lambda$  a (finite dimensional) irreducible representation of  $g$  with highest weight  $\lambda$ .

The multiplicity of  $V_\lambda$  in  $\text{Hom}_{\mathbb{C}}(V_\alpha, V_{\alpha+\mu})$  is bounded by the multiplicity of the weight  $\mu$  in  $V_\lambda$ , with equality of multiplicities for generic  $\alpha$  (i.e., for  $\alpha$  far from Weyl chamber walls).

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The multiplicity of  $V_\lambda$  in  $\text{Hom}_{\mathbf{C}}(V_\alpha, V_{\alpha+\mu})$  can thus be thought of as “independent” of  $\alpha$ . This suggests that there may be a construction of  $V_\lambda$ -operators in  $\text{Hom}_{\mathbf{C}}(V_\alpha, V_{\alpha+\mu})$  which is in some sense independent of  $\alpha$ . The desire for such a construction motivates all that follows.

Elements of the universal enveloping algebra  $\mathcal{U}$  of  $\mathfrak{g}$  provide examples of what we want for  $\mu = 0$ . The algebra  $\mathcal{U}$  can be viewed as a subalgebra of  $\mathcal{E} = \text{End}_{\mathbf{C}}(\bigoplus V_\alpha)$ , where the sum includes exactly one finite dimensional irrep of  $\mathfrak{g}$  from each isomorphism class. Each element of  $\mathcal{U}$  maps each  $V_\alpha$  into itself. What we seek is a suitable enlargement  $\mathcal{A}$  of  $\mathcal{U}$  within  $\mathcal{E}$  which contains transformations mapping each  $V_\alpha$  to each  $V_\beta$ .

A theory for general  $\mathfrak{g}$  does not yet exist. One must now begin with a concrete realization of  $\bigoplus V_\alpha$  and describe the algebra  $\mathcal{A}$  by giving explicit generators. So far,  $\mathcal{A}$  has been constructed only for  $\mathfrak{g}$  equal to  $\mathfrak{sl}_2$  or  $\mathfrak{sl}_3$ .

For  $\mathfrak{g} = \mathfrak{sl}_2$ , one takes  $\bigoplus V_\alpha = \mathbf{C}[x, y]$ , the ring of polynomials in two variables. The action of  $\mathfrak{sl}_2$  is given by  $E_{12} = x\partial_y$ ,  $E_{21} = y\partial_x$ . We take for  $\mathcal{A}$  the Weyl algebra of all polynomial differential operators on  $\mathbf{C}[x, y]$ . Through the adjoint action,  $\mathcal{A}$  becomes an  $\mathfrak{sl}_2$ -representation which is easily decomposed. That decomposition yields the Clebsch-Gordan coefficients, as is well known [4].

For  $\mathfrak{g} = \mathfrak{sl}_3$ , we begin with  $W = \text{Sym}(U_1 \oplus U_2)$ , where  $U_1$  and  $U_2$  are the two fundamental representations. Choosing bases appropriately,  $W = \mathbf{C}[a_1, a_2, a_3, b_1, b_2, b_3]$ , a polynomial ring in six variables, with  $\mathfrak{sl}_3$  action given by  $E_{ij} = a_i\partial_{a_j} - b_j\partial_{b_i}$ . The subalgebra of all highest weight vectors in  $W$ , namely those which are annihilated by  $\mathfrak{n}^+ = \text{span}\{E_{12}, E_{23}, E_{13}\}$ , is generated by the three elements  $a_1$ ,  $b_3$ , and  $M = a_1b_1 + a_2b_2 + a_3b_3$ . From this it follows easily that the  $\mathfrak{sl}_3$ -subrepresentation  $V = \text{kernel}(\Delta)$  of  $W$ , with  $\Delta = \sum \partial_{a_i}\partial_{b_i}$ , decomposes as a multiplicity free sum of finite dimensional irreducible representations containing a representative from each isomorphism class.

The algebra  $\mathcal{A}$  will be a subalgebra of  $\text{End}_{\mathbf{C}}(V)$ .

We pause for some notation. Let  $\mathfrak{h}$  be the Cartan subalgebra of diagonal matrices in  $\mathfrak{sl}_3$ , the lie algebra of all  $3 \times 3$  traceless complex matrices. A vector  $\lambda = (x_1, x_2, x_3) \in \mathbf{Z}^3$  will be identified with a weight of  $\mathfrak{sl}_3$  by the formula  $\lambda(H) = \sum x_i y_i$  for  $H = \sum y_i E_{ii} \in \mathfrak{h}$ .

For weights  $\omega$  and  $\mu$  which are permutations of  $(100)$  we will write  $\binom{\mu}{\omega}$  for a specific transformation (to be defined shortly) in  $\text{End}_{\mathbf{C}}(V)$  which has the following two properties:

- (i)  $\binom{\mu}{\omega}(V_\alpha) \subset V_{\alpha+\mu}$  for each irrep  $V_\alpha \subset V$ .
- (ii) Upon commutation with  $\mathfrak{sl}_3$ ,  $\binom{\mu}{\omega}$  transforms as a vector of weight  $\omega$  in a representation isomorphic to  $V_{(100)}$ .

The nine  $\binom{\mu}{\omega}$  are called *elementary operators*. By definition,  $\mathcal{A}$  is the subalgebra of  $\text{End}_{\mathbf{C}}(V)$  generated by the nine elementary operators.

Now for the formulas.

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} &= \partial_{b_1}, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} &= b_3 \partial_{a_2} - b_2 \partial_{a_3} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} &= a_1 \left( 2 + \sum a_i \partial_{a_i} + \sum b_i \partial_{b_i} \right) - \left( \sum a_i b_i \right) \partial_{b_1}. \end{aligned}$$

The other six elementary operators are generated from these by commutation with  $\mathfrak{sl}_3$ .

We comment only on the formula for  $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ , for it is the only one which may require more than a moment of fiddling to discover. A first guess for  $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$  would be simply  $a_1$ , but  $a_1$  does not map  $V$  into  $V$ ; that is,  $\Delta a_1$  does not vanish on  $V$ . But it nearly does so in the sense that  $\Delta^2 a_1$  vanishes on  $V$ . Notice that  $\Delta$ ,  $M$ , and  $[\Delta, M]$  span a lie algebra isomorphic to  $\mathfrak{sl}_2$ . The representation theory of  $\mathfrak{sl}_2$  suggests that one modify the formula by adding to  $a_1$  a scalar multiple (depending on  $V_\alpha$ ) of  $M\Delta a_1$ . This idea leads to a correct formula for  $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ .

The algebra  $\mathcal{A}$  is large enough for our purposes. It contains the universal enveloping algebra  $\mathcal{U}$  of  $\mathfrak{sl}_3$ ; and every  $T$  in every  $\text{Hom}(V_\alpha, V_\beta)$  is the restriction of some element of  $\mathcal{A}$ .

**THEOREM 1.** *The lie algebra (under commutation) generated by the nine elementary operators is isomorphic to the twenty-eight dimensional lie algebra  $\mathfrak{so}_8$ . The dimension is accounted for by the nine elementary operators, nine conjugates to them, the eight-dimensional  $\mathfrak{sl}_3$ , and two  $\mathfrak{sl}_3$ -invariant operators.*

Thus  $\mathcal{A}$  is isomorphic to a quotient of the universal enveloping algebra of  $\mathfrak{so}_8$ . Using this fact as a starting point, the representation of  $\mathfrak{sl}_3$  on  $\mathcal{A}$  via the adjoint action can be decomposed explicitly.

Let  $\mathcal{B}$  denote the commutant of the raising operators  $\{E_{12}, E_{23}\}$  in  $\mathcal{A}$ . For weights  $\mu, \lambda$  of  $\mathfrak{sl}_3$  let  $\mathcal{B}(\mu)_\lambda$  be the space of all  $T \in \mathcal{B}$  such that (i)  $T(V_\alpha) \subset V_{\alpha+\mu}$  for all  $\alpha$ , and (ii)  $T$  is of  $\mathfrak{sl}_3$ -weight  $\lambda$ .

Our main result is

**THEOREM 2.**

- (i)  $\mathcal{B}(\mathfrak{0}) = \mathbb{C}[R, S]$  with  $R = \sum a_i \partial_{a_i}$  and  $S = \sum b_i \partial_{b_i}$ .
- (ii)  $\mathcal{B}(\mu)_\lambda$  is a free  $\mathcal{B}(\mathfrak{0})$ -module of rank equal to the multiplicity of the weight  $\mu$  in the representation  $V_\lambda$ . An explicit  $\mathcal{B}(\mathfrak{0})$ -basis for  $\mathcal{B}(\mu)_\lambda$  can be given.

The basis of  $\mathcal{B}(\mu)_\lambda$  referred to in (ii) constitutes the  $\alpha$ -independent construction of  $V$ -operators in  $\text{Hom}(V_\alpha, V_{\alpha+\mu})$  originally sought.

Biedenharn and Louck [2] earlier constructed some operators on  $V$  by giving matrix entries. Their operators can be obtained by doing Gram-Schmidt orthonormalizations on the above bases for the  $\mathcal{B}(\mu)_\lambda$ . We next describe the inner product with respect to which the orthonormalization is taken.

The algebra  $\mathcal{A}$  contains  $\mathfrak{so}_8$  which acts on  $V$ . Thus  $V$  is an irreducible infinite dimensional  $\mathfrak{so}_8$ -representation with a highest weight. Results of Enright, Howe and Wallach [6] imply the unitarizability of  $V$  as a representation of an explicitly given real form of  $\mathfrak{so}_8$  isomorphic to  $\mathfrak{so}(6, 2)$ . The

inner product on  $V$  induces a Hilbert-Schmidt inner product on each

$$\text{Hom}(V_\alpha, V_{\alpha+\mu}).$$

Via the adjoint action,  $\mathcal{A}$  is itself an  $\mathfrak{so}_8$ -representation.

**THEOREM 3.** *The  $\mathfrak{so}_8$ -representation  $\mathcal{A}$  decomposes as a multiplicity free sum of finite dimensional irreducible representations. The  $\mathfrak{so}_8$ -highest weight operators in  $\mathcal{A}$  are polynomials in a single operator.*

The two-sided ideals of  $\mathcal{A}$  are  $\mathfrak{so}_8$ -subrepresentations. By Theorem 3 there cannot be many.

**THEOREM 4.**  *$\mathcal{A}$  contains no nonzero proper two-sided ideal.*

We note that the Weyl algebra  $\mathcal{A}$  associated to  $\mathfrak{sl}_2$  also has the property of Theorem 4.

Does the theory described here generalize to other simple lie algebras? The case of  $\mathfrak{sl}_4$  is under active investigation and seems significantly more complex.

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