BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

*Propagation and interaction of singularities in nonlinear hyperbolic problems*, by Michael Beals. Birkhäuser, Basel, 1989, 143 pp., $29.00. ISBN 3-8176-3449-5

Why should you be interested in the propagation of singularities? The answer is tied up with two other fundamental questions: What is a wave and how do waves serve to send signals? To address all three, begin by considering the classical wave equation $u_{tt} = c^2 u_{xx}$ with $t, x \in \mathbf{R} \times \mathbf{R}$ and $c \in ]0, \infty[$. D'Alembert observed that the general solution is $\varphi(x + ct) + \psi(x - ct)$ with $\varphi$ and $\psi$ arbitrary functions of one variable. The function $\psi(x-ct)$ represents a wave of arbitrary cross-section propagating rigidly at speed $c$. Similarly $\varphi(x + ct)$ is a wave moving with speed $-c$. Thus the wave equation has two possible modes of propagation, leftward and rightward at speed $c$.

If one wants to send a message, one could for example represent dot by an upward bump of height 1 and width $w$ and dash by a two humped camel-shaped bump of height 1 and width $w$ and then use the wave equation and Morse code. A similar strategy could be achieved with any equation which transmits two distinct localized waveforms. This is a sort of digital messaging; it is such transmission of information that takes place along axons (the wires in the nervous system) and in computers.

We are miseducated in elementary science courses to believe that sine waves of the form $\sin(k(x \pm ct))$ are what one should think of as waves. Imagine trying to communicate using these. One bump is like any other; there is no beginning and no end. You would like to place a marker on a particular bump and say something like: "A bump with a marker is a dot and one without is a dash." This works, but you can see that the sine wave plays no role; it is just the marker which sends the signal. In summary, for the sake of communication what is useful are localized disturbances which remain recognizable after propagation, and which have understandable laws of motion. Many partial differential equations possess such solutions and may be used in communication.

What distinguishes wave equations, or hyperbolic equations is that they possess an infinite variety of such signals, and in particular signals localized in arbitrarily small regions of space. Thus one need not encode messages digitally, and the density of information is, in principal, unlimited.

For the simple D'Alembert wave equation, the infinity of possible wave forms is evident from the explicit solution. For more complicated equations, for example the wave equation with variable sound speed $u_{tt} = c^2(t, x)u_{xx}$ with $c \in c^\infty(\mathbf{R}_t \times \mathbf{R}_x : \mathbf{R}^+)$, this is not so clear. Riemann [7] and then Hadamard [4] realized that one way to demonstrate the existence of arbitrarily short signals, is to use singular solutions as follows.

Solve the initial value problem with

$$u(0, x) = f(x) \qquad u_t(0, x) = 0,$$

where $f \in C^2(\mathbf{R})$ is piecewise smooth with respect to the partition $-\infty < a < +\infty$ of $\mathbf{R}$. Suppose in addition that the third derivative of $f$ jumps at $a$, that is,

$$f'''(a+) - f'''(a-) \neq 0.$$

Let $\Gamma^a_\pm$ ($\equiv$ *characteristic curves*) be the integral curves of the vector fields $\partial_t \pm c(t, x)\partial_x$ which pass through the point $(0, a)$. Suppose, for example, that $\nabla_{t,x} c$ is bounded so that these integral curves exist globally and partition $\mathbf{R}_t \times \mathbf{R}_x$ into four "quadrants." Riemann showed that the unique solution of the initial value problem is piecewise smooth on space-time in the sense that $u$ extends from the interior of each quadrant to a smooth function on each closed quadrant. The solution is globally $C^2$ and the third derivatives $u_{xxx}$ and $u_{ttt}$ jump across the boundaries of the quadrants. The points of discontinuity are infinitely localized signals and their speeds of propagation are $\pm c(t, x)$. This is the most direct way to show that the values of $c$ represent speeds of propagation for the problem. The usual uniqueness proofs only show that they are upper bounds for speeds of propagation.

The second classical occurrence of singular solutions again dates to Riemann. He showed that the value of the solution $u$ to the initial value problem solved above is a linear functional of the form

$$u(t, x) = \int R(t, x, 0, y) f(y)\, dy.$$

The bounded function $R$ is the initial-value problem analogue of *Green's function* and is called the *Riemann function.* With the wisdom of Distribution Theory it is easy to see that $R(t, x, s, y)$ is determined by the initial-value problem in $s$, $y$ with initial time $s = t$:

$$R_{ss} - (c^2(s, y)R)_{yy} = 0,$$
$$R(t, x, t, y) = 0, \qquad R_t(t, x, t, y) = \delta(x - y).$$

Where $R$ is smooth, the solution depends "mildly" on $f$. It is at the singularities of $R$ that the values of $f$ are especially important. Riemann showed that $R(t, x, \cdot, \cdot)$ is smooth except at the characteristic curves through the point $t$, $x$.

The generalization of these ideas to higher dimensions motivated much of the early work on propagation of singularities for linear partial differential equations. To study the singularities of solutions with initial data equal to $\delta(x)$, Lax [5], used the Fourier decomposition

$$\delta(x) = \text{const.} \int e^{ix\xi}\, d\xi$$

as a superposition of planar oscillations. The contributions from $\xi$ large govern the singularities. Lax then solved the initial value problem with data $e^{ix\xi}$ for $\xi$ large and superposed the results. This strategy lead to the creation of what are now called Fourier Integral Operators. An alternate strategy [3] is to use Radon's decomposition

$$\delta(x) = \text{const.} \int e^{irs} r^{d-1} \delta(x \cdot \omega - s)\, d\omega\, dr\, ds$$

as a superposition of $\delta$ functions singular across the planes $x \cdot \omega = s$. The solution of the initial value problem with such initial data are singular across the characteristic hypersurfaces in space-time which pass through the initial discontinuity surface $\{x \cdot \omega = s\}$. They represent travelling waves of singularities.

The subject of the current book is the nonlinear versions of the themes described above. In the late seventies, it was discovered that nonlinear interaction of singularities can produce singularities in places where they would have been absent for linear wave equations. At the same time it was observed that these singularities were normally in higher derivatives than the singularities that produced them so that if one were blind to derivatives beyond a certain order, the nonlinear case would resemble the linear one. The elucidation of these phenomena has been the work of many workers during the intervening time, Beals himself being a major contributor. Beals' book describes both the "classical" results and the state of the art on some still unresolved questions.

Why is the nonlinear case different? The key observation comes from the microlocal analysis of Hörmander. If $u$ is a function or distribution and $\mathbf{x}$ is a point in the interior of the domain of definition then one can study the local regularity of $u$ near $\mathbf{x}$ by studying the global regularity of $\varphi u$ where $\varphi$ is a smooth function with $\varphi(\mathbf{x}) \neq 0$ and $\operatorname{supp}(\varphi)$ contained in a very small neighborhood of $\mathbf{x}$. This global regularity can be examined by studying the asymptotic behavior of the Fourier transform $\mathscr{F}(\varphi u)(\xi)$ as $|\xi| \to \infty$. In the same way, the microlocal regularity of $u$ at $\mathbf{x}$, $\boldsymbol{\xi}$ with $\boldsymbol{\xi} \neq 0$ is examined by studying the global regularity of $\psi(\xi)\mathscr{F}(\varphi u)$ where $\psi$ is a smooth cutoff function on $\mathbf{R}^d_\xi \backslash 0$ homogeneous of degree zero, supported in a small conic neighborhood of $\boldsymbol{\xi}$, and, nonzero at $\boldsymbol{\xi}$. For example $u$ is said to be in the Sobolev space $H^s$ microlocally at $\mathbf{x}$, $\boldsymbol{\xi}$, written $u \in H^s(\mathbf{x}, \boldsymbol{\xi})$ if there are functions $\varphi$, $\psi$ as above with $\langle\xi\rangle^s\psi(\xi)\mathscr{F}(\varphi u) \in L^2(\mathbf{R}^d)$. It is not hard to show that $m$th order differential operators, $P(x, D)$, with smooth coefficients preserve such regularity in the sense that

$$u \in H^s(x, \xi) \text{ implies } Pu \in H^{s-m}(x, \xi).$$

This property, called *microlocality*, is not shared by nonlinear operations. It is not true that if $u \in H^s(x, \xi)$ then $u^2$ or $\sin(u)$ belongs to $H^s(x, \xi)$. For example, in the Fourier transform variables squaring corresponds to convolution, and slow decay in directions other than $\eta$ can spread to slow decay of the convolution in direction $\eta$. The smoother is $u$, the more rapid is the decay of the Fourier transform and the weaker are the effects of spreading. The consequence is that the smoother solutions are, the more propagation of singularities resembles the linear case. The quantification of this phenomenon and the construction of examples demonstrating that the positive results are sharp is the content of Chapters 1 and 2 of Beals' book.

Chapters 3 and 4 are devoted to the study of solutions with conormal singularities. The solutions singular aross characteristic hypersurfaces which enter in the description of the Riemann function are such distributions. If $\Sigma$ is a smooth hypersurface in $\Omega$ and $u$ is a distribution defined on $\Omega$, then $u$ is $H^s$-conormal with respect to $\Sigma$ if for any finite family of compactly supported smooth vector fields $V_1, \dots, V_N$ on $\Omega$ each tangent to $\Sigma$ one has $V_1 \cdots V_N u \in H^s(\mathbf{R}^d)$. The singular support of such a distribution is contained in $\Sigma$. Furthermore, if $x$ belongs to $\Sigma$ and $\xi$ is not orthogonal to the tangent space to $\Sigma$ at $x$ then it is easy to show that $u \in H^k(x, \xi)$ for all $k$. Thus microlocally, the singularities of such distributions belong to the orthogonal of the tangent space to $\Sigma$. If $s > d/2$, then conormal distributions are preserved by smooth nonlinear maps, that is, if $u$ is $H^s$-conormal with respect to $\Sigma$ and $s > d/2$, then $f(u)$ is $H^s$-conormal too. In particular the singularities cannot spread from $T(\Sigma)^\perp$. This stability is the key to the fact that solutions with conormal singularities can be described in much greater detail.

The final two chapters describe singularities and conormal solutions in the presence of boundaries where in addition to propagation in the interior, reflection and diffraction at the boundary must be described.

The book of Beals is engagingly written and does a superb job of extracting the essential ideas in a broad variety of results and methods. In addition, many published proofs are substantially shortened and improved. The exposition is complemented by an excellent collection of figures. Unhappily, there are an appreciable number of typographical errors. Somewhat disconcertingly, one occurs in the first proof of the text on page 6 where integration by parts in the integral $I_N$ does not yield an expression depending on $M$ as advertised. The book is written for people already familiar with the classical calculus of pseudodifferential operators, and such persons should be able to repair the argument. A graduate student with a brief encounter with Partial Differential Equations will likely have difficulty.

An excellent short exposition with detailed statements is Bony's article [1] reviewing work through 1982. The article [6] gives a detailed treatment of the one dimensional case (see [2] for the shockless quasilinear version). If you want to learn the details of the multidimensional theory, I think that Beals' book is the best place to look.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

*Distributions and analytic functions*, by Richard D. Carmichael and Dragiša Mitrović. Pitman Research Notes in Mathematics Series, vol. 206, Longman Scientific & Technical, Harlow, England. ISBN 0-582-01856-0; John Wiley & Sons, New York, (USA only), 1989, xii+347 pp., $42.00. ISBN 0-470-21398-5

The ties between holomorphic functions and the distributions of Laurent Schwartz originated long before distributions were even discovered. According to Daniele Struppa [15] it was Francesco Severi who in 1924 suggested to Luigi Fantappiè to study the functional which associates with a function its derivative at some point, i.e., what we call now the distributional derivative of the Dirac measure.

Inspired by this suggestion, Fantappiè created the theory of analytic functionals [3]. He considers holomorphic functions $f$, each having as its domain of definition $M$ a region of the Riemann sphere $\mathbf{P}_1(\mathbf{C})$. It is assumed that $M \neq \mathbf{P}_1(\mathbf{C})$, and if the point at infinity $\omega$ belongs to $M$, then $f(\omega) = 0$. An analytic line is a function $y(t, z)$ of two variables, holomorphic in each variable. An analytic functional is a map $F$ which associates with each $f$ a scalar $F[f]$ such that if $F$ acts on the analytic line $y(t, z)$ considered as a function of $t$, the resulting function $F[y(\cdot, z)]$ shall be holomorphic in $z$. A particular analytic line is given by $\frac{1}{2\pi i}\frac{1}{z-t}$, and $F_t[\frac{1}{2\pi i}\frac{1}{z-t}]$ is called the Fantappiè indicatrix of $F$.

The Portuguese mathematician José Sebastião e Silva, who studied in Rome during several years, made the first attempt to apply