as a discussion of the special case of log convex kernels.

The book does what it says it will do. It provides a coherent presentation. It is a demanding but rewarding book on a subject which has been extensively explored over the past thirty years. The patient reader will come away with a real sense of the accomplishments of these explorations.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 1, July 1991

0273-0979/91 $1.00 + $.25 per page

*Fractals everywhere*, by Michael F. Barnsley. Academic Press, New York, 1988, 394 pp., $39.95. ISBN 0-12-079062-9

Fractal books have always been blockbusters, at least in terms of sales, and the book under review (aimed at undergraduates) seems to be no exception. Filled with color illustrations, lots of figures, examples, exercises, and, above all, a style of writing more closely associated with the advertising industry than with mathematical work, this book is superficially a very attractive buy for a student wanting to know about fractal geometry. In fact the book covers only one branch of fractal geometry known as 'iterated function systems,' but does cover this thoroughly, and in a highly unusual way.

I will try to judge what this book achieves for the student reader and for the professional mathematician, as well as the impression it makes for fractal geometry within mathematics.

First I want to give some background and a summary of the contents of the book. Fractal sets with self-similarity properties (for example the middle-third Cantor set) have been studied for many years. In an important paper in 1981, Hutchinson gave a formal definition of a self-similar set. A compact set $A$ in $\mathbf{R}^n$ (or more generally in a complete metric space) is *self-similar* if there exist contraction maps $w_i\colon \mathbf{R}^n \to \mathbf{R}^n$ $(i = 1, \dots, m)$ such that

$$A = \bigcup_{i=1}^{m} w_i(A).$$

Hutchinson showed that given any set of contractions $w_1, \dots, w_m$, there is a unique nonempty compact self-similar set $A$ satisfying the above equation. Hutchinson calculated the Hausdorff dimension of certain self-similar sets by introducing a family of Borel measures supported on $A$. Self-similar sets are (usually) extremely complicated geometrical objects. On the other hand, since they are completely determined by the contraction mappings $w_1, \dots, w_m$ they can be described in a very simple way. Barnsley (together with co-workers) has adopted self-similar sets. He has given them a new name: the collection of maps $w_1, \dots, w_m$ is an *iterated function system* (i.f.s.), and the self-similar set $A$ is the *attractor* of the i.f.s. Most importantly, he has proposed the use of i.f.s.'s for data compression by approximating complex images with self-similar sets. By storing the $w_1, \dots, w_m$, instead of a bit pattern for the image, one achieves data compression. One has first to find the $w_1, \dots, w_m$ of course, but no good algorithms are available (at least in public). This is the background to much of the book and explains the choice of topics. The author is now on leave from his job at Georgia Tech to head a commercial company attempting to implement this idea. The great secrecy with which his company operates and the extraordinary claims that he has made as to the potential of his techniques have lead to great interest in iterated function systems (Barnsley claimed in *Byte* magazine that his methods could achieve a data compression ratio of 10,000 to 1; current techniques achieve ratios of between 5 and 20 to 1). Sadly, anyone searching for clues about data compression is better advised to look elsewhere as Barnsley is giving nothing away about his progress in this direction.

In the book under review, Barnsley explains the idea of an i.f.s. in the setting of a general compact metric space. Before doing this, prerequisites in the theory of metric spaces and elementary analysis are covered so that virtually no mathematics beyond näive set theory is required. A large amount of space is devoted to describing the geometry of various sorts of mappings on $\mathbf{R}^n$ and $\mathbf{C}$ (including nine pages on affine maps of $\mathbf{R}^2$ !).

In Chapter four the analogies between iterated function systems and dynamical systems are dicussed. Chapter five defines the fractal dimension (often called box-dimension or capacity) and Hausdorff measure and dimension (but without any further discussion of the mathematical properties of Hausdorff dimension). In Chapter six we are shown how to construct nowhere differentiable functions which interpolate given data points, and how to calculate the fractal dimension of certain graphs.

Chapter seven starts by discussing the so-called 'escape-time' algorithm for generating the attractor of an i.f.s. This algorithm is truly appalling in its efficiency—so one can only conclude that it was introduced because a similar algorithm can be used to make pictures of Julia sets, and that this gives an excuse to discuss Julia sets (whose theory would otherwise not be close enough to i.f.s. theory to justify inclusion). Chapter eight introduces parameter spaces (and begins with a very nice explanation of the idea of a parameter space), and discusses certain one- and two-parameter families of i.f.s.'s. Finally, Chapter nine aims to explain the so-called 'random-iteration' algorithm for generating a self-similar set. This involves the use of measure theory, and so most of Chapter nine consists of a rapid introduction to $\sigma$-algebras and measures (but not rapid enough to cover measurable functions).

The book is apparently based on an undergraduate course on fractal geometry given at Georgia Tech., and covers plenty of topics at a level accessible to undergraduates.

The most remarkable aspect of the book is the style in which it has been written. The presentation dominates to such an extent that it is impossible to remain neutral. You either love the style or you hate it. I suspect that most mathematicians will hate it (more about this later). The 'gee-whiz' approach to mathematical writing uses words that do not often appear in the mathematician's vocabulary. "You risk the loss of your childhood vision... ." proclaims the introduction. Fair enough—perhaps more math books should carry such a health-warning. In his description of the contents, the authors uses the following words: delight; compelling; beautiful; surprises; exciting; wind blowing through a fractal tree; visual; rottenly inaccurate; and, secret agent. We learn in the book that "the fixed points of a transformation restrict the motion of the space under non-violent, non-ripping transformations of bounded deformation." Later we are told that the Julia set is the "closure of the set of points whose orbits wander, hopelessly, forever." The temptation for a reviewer to quote large chunks of text is great, but I shall resist.

What can be said about the substance of the book? I have two major criticisms. First, the selection of material covered is almost exclusively the work of Barnsley and his co-workers at Georgia Tech; the subject of fractals is much wider than iterated function systems, but one would not guess this from reading the book (a new undergraduate book on fractals by Falconer, [F], has a nicer balance of topics). My second criticism is that the connections between subjects in the book and the mathematical world outside are not explained. Probability theory, approximation theory, dynamical systems theory, and more are alluded to without making connections to things which students could be expected to know. One often gets the impression that so much effort has been made to describe the intuitive ideas that the need for precision, detail, and good references has been forgotten in the process. The worst offender is the chapter on "chaotic dynamics on fractals." Many of the ideas in this chapter have been taken over, rather clumsily, from the dynamical systems literature. Definitions are often badly formulated, such as that of an attractive fixed point (under which the linear map of $\mathbf{R}^2$ induced by the matrix

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

would not have an attracting fixed point in the Euclidean metric). The section on the 'shadowing theorem' is thoroughly misleading. The author claims that his result is useful to establish the meaningfulness of inaccurately computed orbits. The theorem however assumed that his inaccurately computed orbit magically stays on the attractor $A$ (numerical errors are rarely so favorable), and tells us nothing about the behavior of a genuine inaccurately computed orbit (which will diverge from $A$). Moreover the theorem here is a reworking of Bowen's pseudo-orbit tracing property for hyperbolic dynamical systems, but no reference to Bowen is made here (or indeed to any other dynamical systems sources in the whole chapter).

There are lots of figures and exercises in the book—but you can have too much of a good thing. It might seem natural to have many figures in a book on fractals, but many of the figures here are frivolous. The examples and exercises seem mostly okay, but there are many idiosyncrasies and several mistakes. In an exercise on parameter spaces, the reader is asked to study maps colored according to criteria such as rainfall, population density, vegetation, and elevation, and to discuss what these maps tell you about the local geometry of the earth. In an example discussing the Hausdorff measure of a Sierpiński gasket the author makes a wholly wrong assumption about the way in which one economically covers the set so that he makes an incorrect calculation of Hausdorff measure. Anyone using this book to support a course should carefully check material beforehand.

As I have said above, the treatment of i.f.s. theory is quite thorough. The only part of the theory that is missing is that of "recurrent' i.f.s.'s. That is a pity because they enable you to generate a much greater variety of sets than the usual i.f.s.'s. (I have to declare an interest—I have worked on recurrent i.f.s.'s.)

Despite my criticisms though, I do think that the book could be used to support a nice course on fractals for final year undergraduates. Suprisingly perhaps, I believe that the book is even more attractive for nonmathematicians. Almost extraordinary care is taken to explain many things in an intuitive way, and the general style of the author is to suggest continually that everything is fun and easy to do. This makes the book very accessible for people with a weak mathematical background.

I seem to be back to the question of style, so this may be a good moment to say why I think that most mathematicians will hate the book. We all go through a heavy process of cultural conditioning when we go through the years of study needed to get Ph.D.'s. Barnsley's often frivolous approach does not easily fit in with our idea of a serious mathematical textbook. In this respect I think we are unnecessarily Calvinistic, which is a pity because nonmathematicians (including the general public) will probably respond much more positively.

The style of this book also makes it a bad advert for those mathematicians (such as Krantz—see his review of other colorful fractal books, [K]) who seem to believe that fractal geometry is a triumph of style over substance.

There are two issues at stake here: whether fractal geometry does have substance; and whether it is possible to present mathematics in a popular way. I will take the second issue first. Some people seem to think that the general public is incapable of understanding any deep mathematical concepts. The conclusion is then that anything (such as fractal geometry) which is appreciated cannot contain interesting ideas. I find this attitude difficult to understand. After all, if physicists, biologists, and chemists can popularize their work, why not mathematicians? On the question of the content of fractal geometry I can not help feeling that Krantz's arguments are based on tautology. Anybody carrying out interesting work is classified as "not a fractal geometer" and his/her work excluded from consideration; what remains is (unsurprisingly) not interesting.

What certainly *is* true is that there are plenty of people (publishers amongst them) who get carried away with the visual beauty of fractals and try to express their feelings in print. This is almost invariably a bad idea. Barnsley's book in particular suffers from this problem. On the other hand his energy and enthusiasm for the subject shine through, and many of his readers will catch both of these too.

At any rate, Barnsley's book gives insight (in a rather idiosyncratic way) into topics not existing in the standard mathematics undergraduate course which are attractive and comprehensible to students. My personal feeling about it is positive, but with a number of strong reservations. If I were running a course on fractals I would probably use parts of Barnley's book in conjunction with one of the other undergraduate texts available.

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