BOOK REVIEWS

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Brauer trees of sporadic groups, by G. Hiss and K. Lux. Clarendon Press, Oxford, 1989, 525 pp., $75.00. ISBN 0-19-853381-0

Let G be a finite group. Let p be a prime, let K be a finite extension of the field of p-adic numbers, let R be the ring of integers in K, let π be a prime in R and let $\overline{R} = R/\pi R$ be the residue field of K. The theory of modular representations consists of the study of the group rings $R[G]$, $\overline{R}[G]$ and their relations with each other and with $K[G]$. It is usually convenient to take K sufficiently large so that every irreducible $K[G]$ or $\overline{R}[G]$ module is absolutely irreducible, and throughout this review it will be assumed that K has been so chosen. Since K has characteristic 0, the study of $K[G]$ is essentially equivalent to the study of $\mathbf{C}[G]$, which is due to G. Frobenius and I. Schur and is considered classical.

If V is an R free $R[G]$ module, then $V_K = V \otimes K$ is a $K[G]$ module and $\overline{V} = V \otimes \overline{R}$ is an $\overline{R}[G]$ module. This procedure can be partially reversed as follows. If X is a $K[G]$ module, then $X = V_K$ for some R-free $R[G]$ module V. The module V is far from unique; however, the composition factors of $\overline{V}$, with multiplicities, are completely determined by X. If U_1 and U_2 are $\overline{R}[G]$ modules, write $U_1 \leftrightarrow U_2$ if they have the same composition factors with multiplicities (equivalently if they are equal in the Grothendiek group defined by short exact sequences). Let $\{X_i\}$, $\{Y_u\}$ be a complete set of representatives of isomorphism classes of irreducible $K[G]$ modules, $\overline{R}[G]$ modules respectively. For each i, choose an R-free $R[G]$ module V_i with $V_i \otimes K \simeq X_i$. Let $\overline{V}_i \leftrightarrow \oplus\Sigma d_{ui} Y_u$. The remark above asserts that the d_{ui}

are determined by X_i. If $d_{ui} \neq 0$, Y_u is called an irreducible constituent of X_i. It is usually more convenient to state these facts in terms of characters and Brauer characters but for brevity we will stay with this language.

These results and others were established by Richard Brauer in the 1930s and 1940s and can be found in any book on the subject. If p does not divide the order of G, the theory gives nothing new. The next case arises if $p \mid |G|$ but $p^2 \nmid |G|$. Brauer investigated this situation in a series of papers climaxing with [Br, 39, BR, 40], where he also proved some applications of his results. Actually he considered the more general situation of blocks of defect 1. Looking at these papers with hindsight, it seems likely that this work led him to the definition of a defect group. Some of his results will be described below. This is not the place to go into technical definitions so that the reader who is not familiar with the theory may substitute "Sylow p-group" for "defect group" whenever the latter term is used. This special case is the one which occurs most frequently.

It seemed evident at the time that similar results should be true for cyclic defect groups in general (not only groups of order p). However, it took almost a quarter of a century before this generalization was accomplished. By using J. A. Green's reformulation of modular representation theory, J. G. Thompson was able to handle the case of a cyclic Sylow group under several additional assumptions [T]. E. C. Dade very quickly realized that this approach supplied the missing step he needed to handle cyclic defect groups in general [D]. (Due to different waiting periods at different journals, these papers appeared in reverse chronological order.)

Let $R[G] = \oplus\Sigma A_j$, where each $A_j \neq 0$ is an indecomposable 2-sided ideal of $R[G]$. The block B_j corresponding to A_j is the set of all (ismorphism classes) of $R[G]$ modules which annihilate A_k for all $k \neq j$. A $K[G]$ module X is in the block B_j if $X \simeq V \otimes K$ for an $R[G]$ module V in B_j. It is easily seen that every nonzero indecomposable $R[G]$, $\overline{R}[G]$ or $K[G]$ module is in exactly one block. Furthermore, if X_i and Y_u are in distinct blocks, then $d_{iu} = 0$.

Using these definitions, we can now state some results.

Let P be a cyclic defect group of a block B with $|P| = p^d \neq 1$. Let P_0 be the unique subgroup of order p in P. Then $\mathbf{N}_G(P_0)/\mathbf{C}_G(P_0)$ is a group of automorphisms of P_0, and so is cyclic of order e with $e \mid (p-1)$.

Theorem. (i) *There are exactly* e *irreducible* $\overline{R}[G]$ *modules* $Y_1, \ldots, Y_e$ *up to isomorphism in* B.

(ii) B *contains exactly* $e+(p^d-1)/e$ *irreducible* $K[G]$ *modules* $X_1, \ldots, X_e, Z_1, \ldots, Z_{(p^d-1)/e}$ *up to isomorphism. The modules* Z_i *are called exceptional and all have the same irreducible constituents (with the same multiplicity). Let* X_{e+1} *be one of these.*

(iii) *Each decomposition number in* B *is* 0 *or* 1.

(iv) *Each* Y_u *is a constituent of exactly two of the* X_i*'s.*

In view of these results one can define a graph τ as follows.

Each Y_u corresponds to an edge of τ. The two vertices of Y_u correspond to the two modules X_i, X_j with $d_{ui} = d_{uj} = 1$.

Theorem. τ *is a tree.*

τ is the *Brauer tree* of the block B.

There are various results which give information about τ. For instance, the vertices and edges of τ corresponding to self-dual modules form a straight line (also called an open polygon). This is called the *real stem* of τ. (It consists of those modules which afford real characters, with a special proviso for the exceptional vertex.)

Somewhat like other mathematical icons such as Dynkin diagrams or Young tableaux, Brauer trees can be studied on several levels. These are very lucidly described in Chapters 1 and 2 of this book.

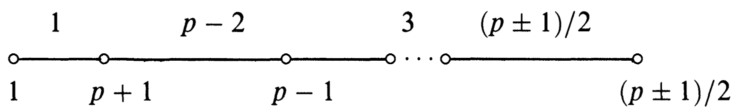
The most basic problem is to describe the shape of the tree. Next one can ask which vertex or edge corresponds to which irreducible module. There is also a natural embedding of the tree in the plane, which determines the structure of the indecomposable projective $\overline{R}[G]$ modules in B. These questions are of different levels of difficulty. For instance, the vertices on the real stem can easily be read off from the character table of G. However, it can be an extremely difficult problem to decide which vertex corresponds to which module.

A complete answer to all these questions yields a complete description of all indecomposable $\overline{R}[G]$ modules in B. This is due to the work of J. A. Green [G], G. J. Janusz [J] and H. Kupisch [K]. A result of M. Benard [B] makes it possible to compute Schur indices over $\mathbf{Q}_p$. These facts are all described in [F1] for instance.

Since the Brauer tree contains so much information about the group G, its structure will necessarily depend heavily on G. Here are a few examples.

If G is p-solvable, then τ is a star with the exceptional vertex at the center. In general, if τ is a star, it may be extremely difficult to determine the planar embedding of τ.

If $G = PSL_2(p)$ and B is the principal p-block, then τ is



The numbers are the dimensions of the corresponding modules. Since there is at most one irreducible $\overline{R}[G]$ of a given dimension, it is not difficult to determine which $K[G]$ module corresponds to which vertex. Everything is known about this tree, and one can thus for instance describe all indecomposable $\overline{R}[G]$ modules of G.

These examples show that there are infinitely many open polygons which are shapes of Brauer trees. By using the classification of finite simple groups it can be shown that up to a natural equivalence relation, only finitely many trees which are not open polygons can occur as shapes of Brauer trees [F2]. This raises the question of finding the Brauer trees which are not open polygons. For instance, Figure 1 is an example: it corresponds to the faithful 13-block of $2Ru$; see [F1, page 148].

The object of this book is to describe information concerning the Brauer trees of the sporadic groups and their covering groups. At the beginning there are five chapters which describe the methods used and how to read the trees and tables. The bulk of the book, pages 64–519, consists of trees and related information.

The methods for finding the trees are highly nontrivial and rely heavily on the computer. They are necessarily of an ad hoc nature, since each group and each block must be treated individually. However, there are some general approaches, described in the early part of the book, which can be used. Among the deepest are those due to Richard Parker. He had previously computed some of these trees and his results appear here for the first time in print.

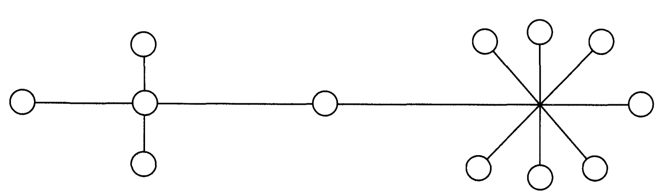


FIGURE 1

This book is a valuable, and indeed unique, addition to the literature and will be a standard reference. The introductory chapters are very clearly written. The authors derive many previously known results by alternative methods. This gives an independent check of these results. I know of no errors in the trees or tables and have great confidence in the general accuracy of the material presented here.

This book will be an essential part of every mathematical library.

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Elements of differentiable dynamics and bifurcation theory, by David Ruelle. Academic Press, New York, 1989, 187 pp., $27.50. ISBN 0-12-601710-7

About a year and a half ago at the Thom Symposium in Paris many of the talks traced their genesis to René Thom's seminar in the Bois Ste. Marie at the Institut des Hautes Etudes Scientifiques. What a wonderful seminar I thought, and recalled my own stay at the I.H.E.S. in 1969–70 and Thom's seminar that year (which was not one of the ones mentioned). Steve Smale,